Homotopy Probability Theory

Gabriel C. Drummond-Cole

CGP

Nov. 20, 2014

Overview

$\boxed{\text{Classical algebraic probability theory}}$

Overview

$$\begin{tikzcd} \boxed{\text{Classical algebraic probability theory}} \arrow[d, rightsquigarrow] \\ \boxed{L_\infty \text{ algebraic probability theory}} \end{tikzcd}$$

Overview

$$
\begin{tikzcd}
\boxed{\text{Classical algebraic probability theory}} \arrow[d, rightsquigarrow] \\
\boxed{L_\infty \text{ algebraic probability theory}} \arrow[d, rightsquigarrow] \\
\boxed{\text{Abstract homotopical algebraic probability theory}}
\end{tikzcd}
$$

Overview

$$\begin{array}{c}\boxed{\text{Classical algebraic probability theory}}\\ \downarrow \\ \boxed{L_\infty \text{ algebraic probability theory}}\\ \rightsquigarrow \\ \boxed{\text{Abstract homotopical algebraic probability theory}}\end{array}$$

The first arrow is rigorously constructed

Overview

$$\begin{tikzcd}
\boxed{\text{Classical algebraic probability theory}} \arrow[d] \\
\boxed{L_\infty \text{ algebraic probability theory}} \arrow[d, rightsquigarrow, dashed] \\
\boxed{\text{Abstract homotopical algebraic probability theory}}
\end{tikzcd}$$

The first arrow is rigorously constructed but the second arrow is still in development.

Overview

$$\begin{array}{c}\boxed{\text{Classical algebraic probability theory}}\\ \downarrow \\ \boxed{L_\infty \text{ algebraic probability theory}}\\ \rightsquigarrow \\ \boxed{\text{Abstract homotopical algebraic probability theory}}\end{array}$$

The first arrow is rigorously constructed but the second arrow is still in development.

This talk is based in part on joint work with J.-S. Park (IBS-CGP) and J. Terilla (CUNY).

Overview

$$\begin{array}{c}\boxed{\text{Classical algebraic probability theory}}\\ \downarrow \\ \boxed{L_\infty \text{ algebraic probability theory}}\\ \rightsquigarrow \\ \boxed{\text{Abstract homotopical algebraic probability theory}}\end{array}$$

The first arrow is rigorously constructed but the second arrow is still in development.

This talk is based in part on joint work with J.-S. Park (IBS-CGP) and J. Terilla (CUNY). It is intended to be nontechnical, expository, and sketchy.

Classical probability theory

Classical probability theory begins with a measure space (Ω, μ) with $\int_\Omega \mu = 1$.

Classical probability theory

Classical probability theory begins with a measure space (Ω, μ) with $\int_\Omega \mu = 1$.
We study measurable functions (usually to the ground field $\mathbb{C}$), called *random variables*.

Classical probability theory

Classical probability theory begins with a measure space (Ω, μ) with $\int_\Omega \mu = 1$.

We study measurable functions (usually to the ground field $\mathbb{C}$), called *random variables*.

Random variables come equipped with the *expectation* map to $\mathbb{C}$:

$$X \mapsto \int_\Omega X\mu.$$

Algebraic probability theory

We can axiomatize this presentation as follows:

Algebraic probability theory

We can axiomatize this presentation as follows:

Definition

An (algebraic) *classical probability space* is a unital commutative associative algebra A of *random variables*

Algebraic probability theory

We can axiomatize this presentation as follows:

Definition

An (algebraic) *classical probability space* is a unital commutative associative algebra A of *random variables* equipped with a linear unit-preserving *expectation map* E to $\mathbb{C}$.

Algebraic probability theory

We can axiomatize this presentation as follows:

Definition

An (algebraic) *classical probability space* is a unital commutative associative algebra A of *random variables* equipped with a linear unit-preserving *expectation map* E to $\mathbb{C}$.

Note that E is *not* required to be an algebra map, and in practice will not be one.

Moments and cumulants

Definition

Given a tuple of random variables $(X_1, \ldots, X_n)$, their *moment* is $E(X_1 \ldots X_n)$.

Moments and cumulants

Definition

Given a tuple of random variables $(X_1, \ldots, X_n)$, their *moment* is $E(X_1 \ldots X_n)$.

Definition

Given a tuple of random variables $(X_1, \ldots, X_n)$, their *cumulant* $\kappa(X_1, \ldots, X_n)$ is defined recursively by the equation:

Moments and cumulants

Definition

Given a tuple of random variables $(X_1, \ldots, X_n)$, their *moment* is $E(X_1 \ldots X_n)$.

Definition

Given a tuple of random variables $(X_1, \ldots, X_n)$, their *cumulant* $\kappa(X_1, \ldots, X_n)$ is defined recursively by the equation:

$$E(X_1 \ldots X_n) = \sum_{P=(P_1,\ldots,P_k)} \kappa(P_1) \cdots \kappa(P_k)$$

where P ranges over all partitions of $(X_1, \ldots, X_n)$.

Cumulants

Cumulants keep track of important information about the *independence* of random variables.

Cumulants

Cumulants keep track of important information about the *independence* of random variables.

Definition

Random variables X and Y are *independent* if

$$\kappa(\underbrace{X + tY, \ldots X + tY}_{m \text{ copies}}) = \kappa(\underbrace{X, \ldots, X}_{m \text{ copies}}) + t^m \kappa(\underbrace{Y, \ldots, Y}_{m \text{ copies}})$$

Cumulants

Cumulants keep track of important information about the *independence* of random variables.

Definition

Random variables X and Y are *independent* if

$$\kappa(\underbrace{X + tY, \ldots X + tY}_{m \text{ copies}}) = \kappa(\underbrace{X, \ldots, X}_{m \text{ copies}}) + t^m \kappa(\underbrace{Y, \ldots, Y}_{m \text{ copies}})$$

We can also phrase this in terms of "mixed cumulants."

Cumulants

Cumulants keep track of important information about the *independence* of random variables.

Definition

Random variables X and Y are *independent* if

$$\kappa(\underbrace{X + tY, \ldots X + tY}_{m \text{ copies}}) = \kappa(\underbrace{X, \ldots, X}_{m \text{ copies}}) + t^m \kappa(\underbrace{Y, \ldots, Y}_{m \text{ copies}})$$

We can also phrase this in terms of "mixed cumulants."

Definition

Random variables $(X_1, \ldots, X_n)$ are independent if

$$\kappa(X_{i_1}, \ldots, X_{i_m}) = 0$$

whenever there are at least two distinct indices among $\{i_1, \ldots, i_m\}$.

Examples of cumulants

Examples of cumulants

The first cumulant is the expectation

$$\kappa(X) = E(X).$$

Examples of cumulants

The first cumulant is the expectation

$$\kappa(X) = E(X).$$

The second cumulant is the *covariance*

$$\kappa(X, Y) = E(XY) - E(X)E(Y).$$

Examples of cumulants

The first cumulant is the expectation

$$\kappa(X) = E(X).$$

The second cumulant is the *covariance*

$$\kappa(X, Y) = E(XY) - E(X)E(Y).$$

The third cumulant is the *third central moment*

$$\kappa(X, Y, Z) =$$
$$E(XYZ) - E(XY)E(Z) - E(YZ)E(X) - E(ZX)E(Y) + 2E(X)E(Y)E(Z).$$

Examples of cumulants

The first cumulant is the expectation

$$\kappa(X) = E(X).$$

The second cumulant is the *covariance*

$$\kappa(X, Y) = E(XY) - E(X)E(Y).$$

The third cumulant is the *third central moment*

$$\kappa(X, Y, Z) =$$

$$E(XYZ) - E(XY)E(Z) - E(YZ)E(X) - E(ZX)E(Y) + 2E(X)E(Y)E(Z).$$

It keeps track of the *skewness* of X via the formula

$$\gamma_1 = \frac{\kappa(X, X, X)}{\kappa(X, X)^{\frac{3}{2}}}.$$

Examples of cumulants

The first cumulant is the expectation

$$\kappa(X) = E(X).$$

The second cumulant is the *covariance*

$$\kappa(X, Y) = E(XY) - E(X)E(Y).$$

The third cumulant is the *third central moment*

$$\kappa(X, Y, Z) =$$

$$E(XYZ) - E(XY)E(Z) - E(YZ)E(X) - E(ZX)E(Y) + 2E(X)E(Y)E(Z).$$

It keeps track of the *skewness* of X via the formula

$$\gamma_1 = \frac{\kappa(X, X, X)}{\kappa(X, X)^{\frac{3}{2}}}.$$

The fourth cumulant keeps track of excess kurtosis:

$$\gamma_2 = \frac{\kappa(X, X, X, X)}{\kappa(X, X)^2}.$$

Examples of cumulants

The first cumulant is the expectation

$$\kappa(X) = E(X).$$

The second cumulant is the *covariance*

$$\kappa(X, Y) = E(XY) - E(X)E(Y).$$

The third cumulant is the *third central moment*

$$\kappa(X, Y, Z) =$$

$$E(XYZ) - E(XY)E(Z) - E(YZ)E(X) - E(ZX)E(Y) + 2E(X)E(Y)E(Z).$$

It keeps track of the *skewness* of X via the formula

$$\gamma_1 = \frac{\kappa(X, X, X)}{\kappa(X, X)^{\frac{3}{2}}}.$$

The fourth cumulant keeps track of excess kurtosis:

$$\gamma_2 = \frac{\kappa(X, X, X, X)}{\kappa(X, X)^2}.$$

Cumulants

Moral

Cumulants are a useful repackaging of the interaction between the expectation and the product.

Cumulants

Moral

Cumulants are a useful repackaging of the interaction between the expectation and the product.

Question

How can we justify the cumulant formula in a way that leads to good generalizations?

Cumulants

Moral

Cumulants are a useful repackaging of the interaction between the expectation and the product.

Question

How can we justify the cumulant formula in a way that leads to good generalizations?

Answer

Homotopy algebra.

L_∞ algebras

Definition

An L_∞ *algebra* structure on V is a square zero differential on the symmetric coalgebra

$$SV = \bigoplus_{n=1}^{\infty} S^n V.$$

L_∞ algebras

Definition

An L_∞ *algebra* structure on V is a square zero differential on the symmetric coalgebra

$$SV = \bigoplus_{n=1}^{\infty} S^n V.$$

Fact

The data of an L_∞ algebra can be specified by a map $\ell : SV \to V$ satisfying certain relations.

L_∞ algebras

Definition

An L_∞ *algebra* structure on V is a square zero differential on the symmetric coalgebra

$$SV = \bigoplus_{n=1}^{\infty} S^n V.$$

Fact

The data of an L_∞ algebra can be specified by a map $\ell : SV \to V$ satisfying certain relations.

The map ℓ has components $\ell_1, \ell_2, \ldots$, which are

L_∞ algebras

Definition

An L_∞ *algebra* structure on V is a square zero differential on the symmetric coalgebra

$$SV = \bigoplus_{n=1}^{\infty} S^n V.$$

Fact

The data of an L_∞ algebra can be specified by a map $\ell : SV \to V$ satisfying certain relations.

The map ℓ has components $\ell_1, \ell_2, \ldots$, which are a differential,

L_∞ algebras

Definition

An L_∞ *algebra* structure on V is a square zero differential on the symmetric coalgebra

$$SV = \bigoplus_{n=1}^{\infty} S^n V.$$

Fact

The data of an L_∞ algebra can be specified by a map $\ell : SV \to V$ satisfying certain relations.

The map ℓ has components $\ell_1, \ell_2, \ldots$, which are a differential, a bracket which satisfies the Jacobi relation up to homotopy,

L_∞ algebras

Definition

An L_∞ *algebra* structure on V is a square zero differential on the symmetric coalgebra

$$SV = \bigoplus_{n=1}^{\infty} S^n V.$$

Fact

The data of an L_∞ algebra can be specified by a map $\ell : SV \to V$ satisfying certain relations.

The map ℓ has components $\ell_1, \ell_2, \ldots$, which are a differential, a bracket which satisfies the Jacobi relation up to homotopy, a homotopy for the Jacobi relation,

L_∞ algebras

Definition

An L_∞ *algebra* structure on V is a square zero differential on the symmetric coalgebra

$$SV = \bigoplus_{n=1}^{\infty} S^n V.$$

Fact

The data of an L_∞ algebra can be specified by a map $\ell : SV \to V$ satisfying certain relations.

The map ℓ has components $\ell_1, \ell_2, \ldots$, which are a differential, a bracket which satisfies the Jacobi relation up to homotopy, a homotopy for the Jacobi relation, and higher coherent homotopies.

L_∞ algebras

Definition

An L_∞ *algebra* structure on V is a square zero differential on the symmetric coalgebra
$$SV = \bigoplus_{n=1}^{\infty} S^n V.$$

Fact

The data of an L_∞ algebra can be specified by a map $\ell : SV \to V$ satisfying certain relations.

The map ℓ has components $\ell_1, \ell_2, \ldots$, which are a differential, a bracket which satisfies the Jacobi relation up to homotopy, a homotopy for the Jacobi relation, and higher coherent homotopies.
Don't worry about the relations because in this talk I'm mainly talking about L_∞ algebras where $\ell_i = 0$ for all i.

L_∞ morphisms

Definition

An L_∞ *morphism* $V \to W$ is a coalgebra map $SV \to SW$ commuting with the differentials.

L_∞ morphisms

Definition

An L_∞ *morphism* $V \to W$ is a coalgebra map $SV \to SW$ commuting with the differentials.

Fact

The data of an L_∞ morphism can be specified by a map $f : SV \to W$ satisfying certain (related but different) relations.

L_∞ morphisms

Definition

An L_∞ *morphism* $V \to W$ is a coalgebra map $SV \to SW$ commuting with the differentials.

Fact

The data of an L_∞ morphism can be specified by a map $f : SV \to W$ satisfying certain (related but different) relations.

The map f has components $f_1, f_2, \ldots$, which are a chain map which respects brackets up to homotopy, a homotopy for f_1, and higher coherent homotopies.

L_∞ morphisms

Definition

An L_∞ *morphism* $V \to W$ is a coalgebra map $SV \to SW$ commuting with the differentials.

Fact

The data of an L_∞ morphism can be specified by a map $f : SV \to W$ satisfying certain (related but different) relations.

The map f has components $f_1, f_2, \ldots$, which are a chain map which respects brackets up to homotopy, a homotopy for f_1, and higher coherent homotopies.

The map f is an isomorphism if and only if f_1 is an isomorphism $V \to W$.

Reformulation of cumulants

Construction

Consider a classical probability space A as an L_∞ algebra with trivial (zero) L_∞ structure. Do the same for $\mathbb{C}$.

$$(SA, 0) \qquad\qquad (S\mathbb{C}, 0)$$

Reformulation of cumulants

Construction

Consider a classical probability space A as an L_∞ algebra with trivial (zero) L_∞ structure. Do the same for $\mathbb{C}$.

Let E be the L_∞ map $A \to \mathbb{C}$ where $S^n A \to \mathbb{C}$ is E if $n = 1$.

$$(SA, 0) \xrightarrow[E]{} (S\mathbb{C}, 0)$$

Reformulation of cumulants

Construction

Consider a classical probability space A as an L_∞ algebra with trivial (zero) L_∞ structure. Do the same for $\mathbb{C}$.

Let E be the L_∞ map $A \to \mathbb{C}$ where $S^n A \to \mathbb{C}$ is E if $n = 1$.

Let M_A be the L_∞ map $A \to A$ where $S^n A \to A$ is multiplication using the product in A,

$$\begin{array}{ccc} (SA, 0) & & \\ {\scriptstyle M_A}\big\downarrow & & \\ (SA, 0) & \xrightarrow[E]{} & (S\mathbb{C}, 0) \end{array}$$

Reformulation of cumulants

Construction

Consider a classical probability space A as an L_∞ algebra with trivial (zero) L_∞ structure. Do the same for $\mathbb{C}$.

Let E be the L_∞ map $A \to \mathbb{C}$ where $S^nA \to \mathbb{C}$ is E if $n = 1$.

Let M_A be the L_∞ map $A \to A$ where $S^nA \to A$ is multiplication using the product in A, and similarly for $\mathbb{C}$.

$$\begin{tikzcd}
(SA, 0) \arrow[d, "M_A"'] & (S\mathbb{C}, 0) \arrow[d, "M_{\mathbb{C}}"] \\
(SA, 0) \arrow[r, "E"'] & (S\mathbb{C}, 0)
\end{tikzcd}$$

Reformulation of cumulants

Construction

Consider a classical probability space A as an L_∞ algebra with trivial (zero) L_∞ structure. Do the same for $\mathbb{C}$.

Let E be the L_∞ map $A \to \mathbb{C}$ where $S^n A \to \mathbb{C}$ is E if $n = 1$.

Let M_A be the L_∞ map $A \to A$ where $S^n A \to A$ is multiplication using the product in A, and similarly for $\mathbb{C}$. This is an isomorphism since $M_1 = \mathrm{id}$.

$$\begin{tikzcd} (SA, 0) \arrow[d, "M_A"'] & (S\mathbb{C}, 0) \arrow[d, "M_{\mathbb{C}}"] \\ (SA, 0) \arrow[r, "E"'] & (S\mathbb{C}, 0) \end{tikzcd}$$

Reformulation of cumulants

Construction

Consider a classical probability space A as an L_∞ algebra with trivial (zero) L_∞ structure. Do the same for $\mathbb{C}$.

Let E be the L_∞ map $A \to \mathbb{C}$ where $S^n A \to \mathbb{C}$ is E if $n = 1$.

Let M_A be the L_∞ map $A \to A$ where $S^n A \to A$ is multiplication using the product in A, and similarly for $\mathbb{C}$. This is an isomorphism since $M_1 = \mathrm{id}$.

Let K be the L_∞ map $A \to \mathbb{C}$ given as $M_C^{-1} E M_A$.

$$\begin{tikzcd}
(SA, 0) \arrow[r, "K"] \arrow[d, "M_A"'] & (S\mathbb{C}, 0) \arrow[d, "M_\mathbb{C}"] \\
(SA, 0) \arrow[r, "E"'] & (S\mathbb{C}, 0)
\end{tikzcd}$$

Reformulating Cumulants

$$\begin{tikzcd}
(SA, 0) \arrow[r, "K"] \arrow[d, "M_A"'] & (S\mathbb{C}, 0) \arrow[d, "M_{\mathbb{C}}"] \\
(SA, 0) \arrow[r, "E"'] & (S\mathbb{C}, 0)
\end{tikzcd}$$

Fact

These maps satisfy the conditions to be L_∞ maps for a stupid reason:

Reformulating Cumulants

$$\begin{tikzcd}
(SA, 0) \arrow[r, "K"] \arrow[d, "M_A"'] & (S\mathbb{C}, 0) \arrow[d, "M_{\mathbb{C}}"] \\
(SA, 0) \arrow[r, "E"'] & (S\mathbb{C}, 0)
\end{tikzcd}$$

Fact

These maps satisfy the conditions to be L_∞ maps for a stupid reason: *any* sequence of maps is an L_∞ map between trivial L_∞ algebras.

Reformulating Cumulants

$$\begin{tikzcd}
(SA,0) \arrow[r, "K"] \arrow[d, "M_A"'] & (S\mathbb{C},0) \arrow[d, "M_{\mathbb{C}}"] \\
(SA,0) \arrow[r, "E"'] & (S\mathbb{C},0)
\end{tikzcd}$$

Fact

These maps satisfy the conditions to be L_∞ maps for a stupid reason: *any* sequence of maps is an L_∞ map between trivial L_∞ algebras.

Nevertheless, we have the following.

Reformulating Cumulants

$$\begin{tikzcd} (SA, 0) \arrow[r, "K"] \arrow[d, "M_A"'] & (S\mathbb{C}, 0) \arrow[d, "M_\mathbb{C}"] \\ (SA, 0) \arrow[r, "E"'] & (S\mathbb{C}, 0) \end{tikzcd}$$

Fact

These maps satisfy the conditions to be L_∞ maps for a stupid reason: *any* sequence of maps is an L_∞ map between trivial L_∞ algebras.

Nevertheless, we have the following.

Proposition

The L_∞ map $K = K_1, K_2, K_3, \ldots,$ recovers the cumulants of A.

L_∞ cumulants

Proposition

The L_∞ map $K = K_1, K_2, K_3, \ldots$, recovers the cumulants of A.

L_∞ cumulants

Proposition

The L_∞ map $K = K_1, K_2, K_3, \ldots$, recovers the cumulants of A.

Proof.

Extending K as a coalgebra map means splitting SA in all possible ways and then applying the appropriate version of K.

L_∞ cumulants

Proposition

The L_∞ map $K = K_1, K_2, K_3, \ldots$, recovers the cumulants of A.

Proof.

Extending K as a coalgebra map means splitting SA in all possible ways and then applying the appropriate version of K. So the composition of the top and right maps to $X_1, \ldots, X_n$ is the sum

$$\sum_{P=P_1,\ldots,P_k} K(P_1)\cdots K(P_k).$$

where the sum runs over partitions of $X_1, \ldots, X_n$.

L_∞ cumulants

Proposition

The L_∞ map $K = K_1, K_2, K_3, \ldots,$ recovers the cumulants of A.

Proof.

Extending K as a coalgebra map means splitting SA in all possible ways and then applying the appropriate version of K. So the composition of the top and right maps to $X_1, \ldots, X_n$ is the sum

$$\sum_{P=P_1,\ldots,P_k} K(P_1)\cdots K(P_k).$$

where the sum runs over partitions of $X_1, \ldots, X_n$. Since E only has E_1, applying the composition of the left and bottom maps is just $E(X_1 \cdots X_n)$.

L_∞ cumulants

Proposition

The L_∞ map $K = K_1, K_2, K_3, \ldots$, recovers the cumulants of A.

Proof.

Extending K as a coalgebra map means splitting SA in all possible ways and then applying the appropriate version of K. So the composition of the top and right maps to $X_1, \ldots, X_n$ is the sum

$$\sum_{P=P_1,\ldots,P_k} K(P_1)\cdots K(P_k).$$

where the sum runs over partitions of $X_1, \ldots, X_n$. Since E only has E_1, applying the composition of the left and bottom maps is just $E(X_1 \cdots X_n)$. This is the equation defining cumulants. □

Homotopy probability theory

This motivates a new definition:

Definition

A *homotopy probability space* is a unital commutative associative algebra A

Homotopy probability theory

This motivates a new definition:

Definition

A *homotopy probability space* is a unital commutative associative algebra A equipped with a linear unit-preserving *expectation map* E to $\mathbb{C}$ and a differential d so that $d(1) = Ed = 0$.

Homotopy probability theory

This motivates a new definition:

Definition

A *homotopy probability space* is a unital commutative associative algebra A equipped with a linear unit-preserving *expectation map* E to $\mathbb{C}$ and a differential d so that $d(1) = Ed = 0$.

Again E is *not* required to be an algebra map, and the differential is not required to be a derivation of the product.

Homotopy probability theory

This motivates a new definition:

Definition

A *homotopy probability space* is a unital commutative associative algebra A equipped with a linear unit-preserving *expectation map* E to $\mathbb{C}$ and a differential d so that $d(1) = Ed = 0$.

Again E is *not* required to be an algebra map, and the differential is not required to be a derivation of the product.

We can play the same game but we have to be a tiny bit careful. Before we were dealing with the zero L_∞ algebra on A.

Homotopy probability theory

This motivates a new definition:

Definition

A *homotopy probability space* is a unital commutative associative algebra A equipped with a linear unit-preserving *expectation map* E to $\mathbb{C}$ and a differential d so that $d(1) = Ed = 0$.

Again E is *not* required to be an algebra map, and the differential is not required to be a derivation of the product.

We can play the same game but we have to be a tiny bit careful. Before we were dealing with the zero L_∞ algebra on A. Now we can use d as ℓ_1 and have the rest of the L_∞ structure be zero.

Homotopy probability theory

This motivates a new definition:

Definition

A *homotopy probability space* is a unital commutative associative algebra A equipped with a linear unit-preserving *expectation map* E to $\mathbb{C}$ and a differential d so that $d(1) = Ed = 0$.

Again E is *not* required to be an algebra map, and the differential is not required to be a derivation of the product.

We can play the same game but we have to be a tiny bit careful. Before we were dealing with the zero L_∞ algebra on A. Now we can use d as ℓ_1 and have the rest of the L_∞ structure be zero.

But that means that M_A is an isomorphism not between (SA, d) and itself,

Homotopy probability theory

This motivates a new definition:

Definition

A *homotopy probability space* is a unital commutative associative algebra A equipped with a linear unit-preserving *expectation map* E to $\mathbb{C}$ and a differential d so that $d(1) = Ed = 0$.

Again E is *not* required to be an algebra map, and the differential is not required to be a derivation of the product.

We can play the same game but we have to be a tiny bit careful. Before we were dealing with the zero L_∞ algebra on A. Now we can use d as ℓ_1 and have the rest of the L_∞ structure be zero.

But that means that M_A is an isomorphism not between (SA, d) and itself, but rather between $(SA, M_A^{-1} d M_A)$ and (SA, d).

Homotopy random variables

Definition

A *collection of homotopy random variables* in a homotopy probability space is an L_∞ map f from a trivial L_∞ algebra to the L_∞ algebra $(A, M_A^{-1} dM_A)$.

Homotopy random variables

Definition

A *collection of homotopy random variables* in a homotopy probability space is an L_∞ map f from a trivial L_∞ algebra to the L_∞ algebra $(A, M_A^{-1} dM_A)$.

Example

Consider a trivial L_∞ algebra spanned by $(x_1, \ldots, x_n)$ and suppose $(A, d) = (A, 0)$ is an classical probability space.

Homotopy random variables

Definition

A *collection of homotopy random variables* in a homotopy probability space is an L_∞ map f from a trivial L_∞ algebra to the L_∞ algebra $(A, M_A^{-1} dM_A)$.

Example

Consider a trivial L_∞ algebra spanned by $(x_1, \ldots, x_n)$ and suppose $(A, d) = (A, 0)$ is an classical probability space. Choose random variables $X_1, \ldots, X_n$ and let $f(x_i) = X_i$.

Homotopy probability theory

Definition

The *joint moment* of a collection of homotopy random variables is the L_∞ map $EM_A f$.

Homotopy probability theory

Definition

The *joint moment* of a collection of homotopy random variables is the L_∞ map EM_Af. The *joint cumulant* of a collection of homotopy random variables is the L_∞ map $Kf = M_{\mathbb{C}}^{-1}EM_Af$.

Homotopy probability theory

Definition

The *joint moment* of a collection of homotopy random variables is the L_∞ map $EM_A f$. The *joint cumulant* of a collection of homotopy random variables is the L_∞ map $Kf = M_{\mathbb{C}}^{-1} EM_A f$.

This theory has good homotopy invariance properties.

Homotopy probability theory

Definition

The *joint moment* of a collection of homotopy random variables is the L_∞ map EM_Af. The *joint cumulant* of a collection of homotopy random variables is the L_∞ map $Kf = M_{\mathbb{C}}^{-1}EM_Af$.

This theory has good homotopy invariance properties.

Proposition

A homotopy of expectation maps induces L_∞ homotopies of moment and cumulant maps.

Homotopy probability theory

Definition

The *joint moment* of a collection of homotopy random variables is the L_∞ map $EM_A f$. The *joint cumulant* of a collection of homotopy random variables is the L_∞ map $Kf = M_{\mathbb{C}}^{-1} EM_A f$.

This theory has good homotopy invariance properties.

Proposition

A homotopy of expectation maps induces L_∞ homotopies of moment and cumulant maps.

Homotopic maps between spaces with trivial structure are equal.

Homotopy probability theory

Definition

The *joint moment* of a collection of homotopy random variables is the L_∞ map EM_Af. The *joint cumulant* of a collection of homotopy random variables is the L_∞ map $Kf = M_{\mathbb{C}}^{-1}EM_Af$.

This theory has good homotopy invariance properties.

Proposition

A homotopy of expectation maps induces L_∞ homotopies of moment and cumulant maps.

Homotopic maps between spaces with trivial structure are equal.

Corollary

Joint moments and cumulants only depend on the *homotopy class* of the expectation map and the collection of random variables.

Generalizing the theory

One would like to generalize this theory to deal with different kinds of probability space.

Generalizing the theory

One would like to generalize this theory to deal with different kinds of probability space. For example, free and Boolean probability deal with non-commutative algebras.

Generalizing the theory

One would like to generalize this theory to deal with different kinds of probability space. For example, free and Boolean probability deal with non-commutative algebras. Each kind of probability theory has its own version of independence and thus its own kind of cumulants.

Generalizing the theory

One would like to generalize this theory to deal with different kinds of probability space. For example, free and Boolean probability deal with non-commutative algebras. Each kind of probability theory has its own version of independence and thus its own kind of cumulants.

Idea

A probability space is a unital commutative algebra equipped with a unit-preserving expectation map.

Generalizing the theory

One would like to generalize this theory to deal with different kinds of probability space. For example, free and Boolean probability deal with non-commutative algebras. Each kind of probability theory has its own version of independence and thus its own kind of cumulants.

Idea

A P-probability space is a unital P-algebra equipped with a unit-preserving expectation map.

Generalizing the theory

One would like to generalize this theory to deal with different kinds of probability space. For example, free and Boolean probability deal with non-commutative algebras. Each kind of probability theory has its own version of independence and thus its own kind of cumulants.

Idea

A P-probability space is a unital P-algebra equipped with a unit-preserving expectation map.

P could be commutative, associative, or something more exotic.

Generalizing the theory

One would like to generalize this theory to deal with different kinds of probability space. For example, free and Boolean probability deal with non-commutative algebras. Each kind of probability theory has its own version of independence and thus its own kind of cumulants.

Idea

A P-probability space is a unital P-algebra equipped with a unit-preserving expectation map.

P could be commutative, associative, or something more exotic. We can also decide to get rid of the unit.

Generalizing the theory

One would like to generalize this theory to deal with different kinds of probability space. For example, free and Boolean probability deal with non-commutative algebras. Each kind of probability theory has its own version of independence and thus its own kind of cumulants.

Idea

A P-probability space is a unital P-algebra equipped with a unit-preserving expectation map.

P could be commutative, associative, or something more exotic. We can also decide to get rid of the unit.

We hope to define moments, cumulants, and collections of homotopy random variables as before. But...

Problems with the theory

There are some problems with naively following what was done for classical probability spaces.

Problems with the theory

There are some problems with naively following what was done for classical probability spaces.

Problem (minor)

We need $\mathbb{C}$ to have a $\mathcal{P}$-algebra structure. For an interesting theory, that structure should not be trivial.

Problems with the theory

There are some problems with naively following what was done for classical probability spaces.

Problem (minor)

We need $\mathbb{C}$ to have a $\mathcal{P}$-algebra structure. For an interesting theory, that structure should not be trivial.

Problem (major)

We used a non-canonical identification of the symmetric coalgebra with the symmetric algebra. That is, the map M_A, repeated multiplication, from the commutative *coalgebra* on A to itself. We would know what to do if the domain was the commutative *algebra* on A.

(Partial) solution

Solution (?)

If we require the unit to be independent of everything (have vanishing joint cumulants), then this specifies the map M_A, at least for the kinds of probability spaces that arise in practice.

(Partial) solution

Solution (?)

If we require the unit to be independent of everything (have vanishing joint cumulants), then this specifies the map M_A, at least for the kinds of probability spaces that arise in practice.

$$E(M_A(1 \cdot X_1, \dots X_n)) = K(1) \sum_{P=P_1,\dots P_k} K(P_1) \cdots K(P_k) + \text{ vanishing terms.}$$

(Partial) solution

Solution (?)

If we require the unit to be independent of everything (have vanishing joint cumulants), then this specifies the map M_A, at least for the kinds of probability spaces that arise in practice.

$$E(M_A(1 \cdot X_1, \ldots X_n)) = K(1) \sum_{P = P_1, \ldots P_k} K(P_1) \cdots K(P_k) + \text{ vanishing terms.}$$

Problem (?)

Not every kind of probability space should be unital. Furthermore, the meaning of "vanishing joint cumulants" is not clearly independent of $\mathcal{P}$.

Thank you for listening!