

Homotopy Probability Theory

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CGP

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Classical algebraic probability theory

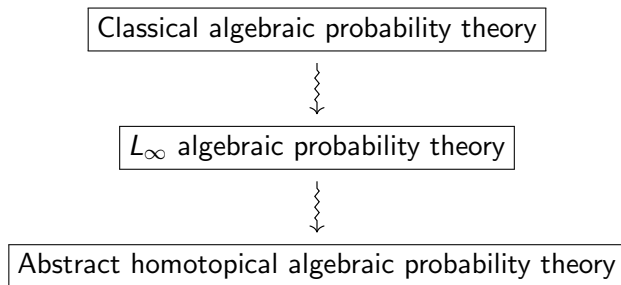
Overview

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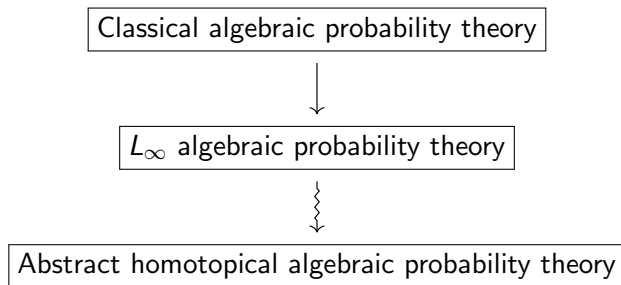


L_∞ algebraic probability theory

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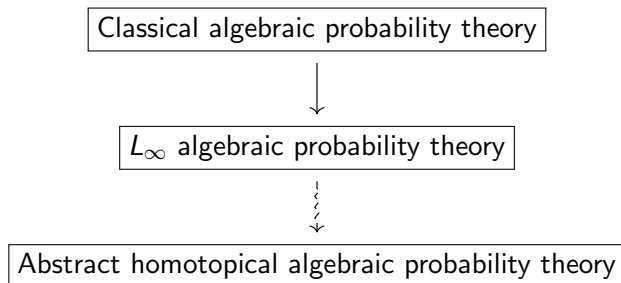


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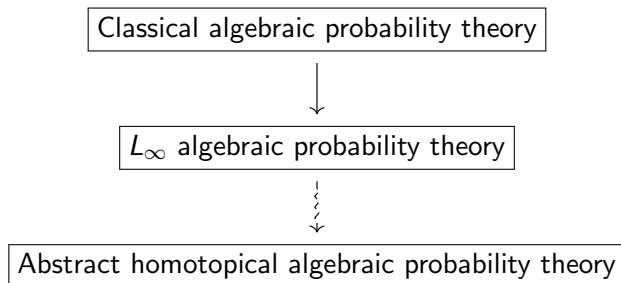
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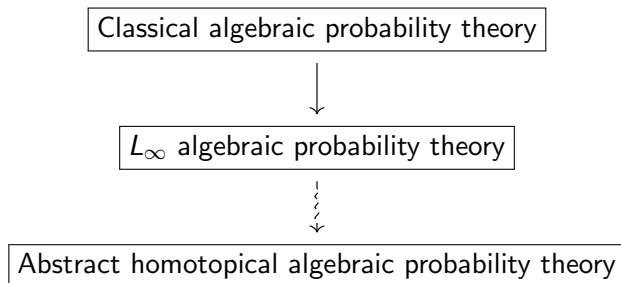
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This talk is based in part on joint work with J.-S. Park (IBS-CGP) and J. Terilla (CUNY). It is intended to be nontechnical, expository, and sketchy.

Classical probability theory

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Random variables come equipped with the *expectation* map to \mathbb{C} :

$$X \mapsto \int_{\Omega} X \mu.$$

Algebraic probability theory

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An (algebraic) *classical probability space* is a unital commutative associative algebra A of *random variables* equipped with a linear unit-preserving *expectation map* E to \mathbb{C} .

Note that E is *not* required to be an algebra map, and in practice will not be one.

Moments and cumulants

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$$E(X_1 \dots X_n) = \sum_{P=(P_1, \dots, P_k)} \kappa(P_1) \cdots \kappa(P_k)$$

where P ranges over all partitions of (X_1, \dots, X_n) .

Cumulants

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Random variables (X_1, \dots, X_n) are independent if

$$\kappa(X_{i_1}, \dots, X_{i_m}) = 0$$

whenever there are at least two distinct indices among $\{i_1, \dots, i_m\}$.

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Answer

Homotopy algebra.

L_∞ algebras

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$$SV = \bigoplus_{n=1}^{\infty} S^n V.$$

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Don't worry about the relations because in this talk I'm mainly talking about L_∞ algebras where $\ell_i = 0$ for all i .

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The map f is an isomorphism if and only if f_1 is an isomorphism $V \rightarrow W$.

Reformulation of cumulants

Construction

Consider a classical probability space A as an L_∞ algebra with trivial (zero) L_∞ structure. Do the same for \mathbb{C} .

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Let M_A be the L_∞ map $A \rightarrow A$ where $S^n A \rightarrow A$ is multiplication using the product in A ,

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But that means that M_A is an isomorphism not between (SA, d) and itself, but rather between $(SA, M_A^{-1}dM_A)$ and (SA, d) .

Homotopy random variables

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Example

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Corollary

Joint moments and cumulants only depend on the *homotopy class* of the expectation map and the collection of random variables.

Generalizing the theory

One would like to generalize this theory to deal with different kinds of probability space.

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We hope to define moments, cumulants, and collections of homotopy random variables as before. But...

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Problem (major)

We used a non-canonical identification of the symmetric coalgebra with the symmetric algebra. That is, the map M_A , repeated multiplication, from the commutative *coalgebra* on A to itself. We would know what to do if the domain was the commutative *algebra* on A .

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Problem (?)

Not every kind of probability space should be unital. Furthermore, the meaning of “vanishing joint cumulants” is not clearly independent of \mathcal{P} .

Thank you for listening!