Domineering

Gabriel C. Drummond-Cole

CGP

May 7, 2014

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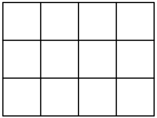

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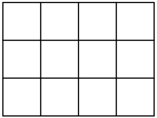

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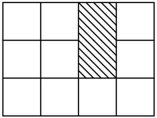

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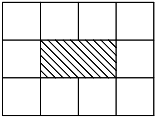

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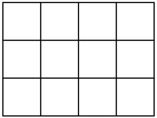

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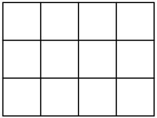

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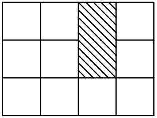

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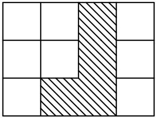

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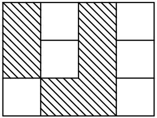

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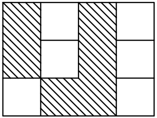

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Questions about domineering

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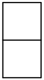

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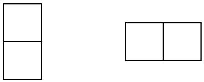

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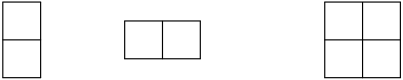

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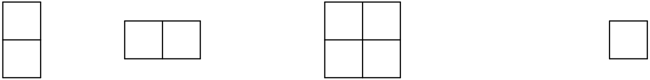

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Answer (Berlekamp-Conway-Guy, older results)

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What is known about rectangular boards?

Rectangular boards

Computer tools can analyze small board sizes.

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	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
2	V	N	N	H	V	N	N	H	V	N	N	H	P	N	N	H	H	N	N	H	H
3	V	N	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
4	V	V	V	N	V	N	V	H	V	H	V	H	P	H	H	H	H	H	H	H	H
5	V	H	V	H	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6	V	N	V	N	V	N	V	H	V	N	N	H	V	H	N						
7	V	N	V	H	V	H	N	H	H	H	H	H	H								
8	V	V	V	V	V	V	V	N	V	H	V										
9	V	H	V	H	V	H	V	H	N	H											
10	V	N	V	V	V	N	V	V	V	N											
11	V	N	V	H	V	N	V	H													
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13	V	P	V	P	V	H	V														
14	V	N	V	V	V	V															
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2	V	N	N	H	V	N	N	H	V	N	N	H	P	N	N	H	H	N	N	H	H
3	V	N	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
4	V	V	V	N	V	N	V	H	V	H	V	H	P	H	H	H	H	H	H	H	H
5	V	H	V	H	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6	V	N	V	N	V	N	V	H	V	N	N	H	V	H	N						
7	V	N	V	H	V	H	N	H	H	H	H	H	H								
8	V	V	V	V	V	V	V	N	V	H	V										
9	V	H	V	H	V	H	V	H	N	H											
10	V	N	V	V	V	N	V	V	V	N											
11	V	N	V	H	V	N	V	H													
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Computer science research on Domineering has focused on improving the speed of analysis of computer programs to extend results further into the array. The most recent results and program in this line are those of Bullock.

Augmenting computer tools

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Proposition (Folklore)

There is a symmetry in outcome classes that comes from interchanging the $m \times n$ board and $n \times m$ boards. This symmetry interchanges the outcome classes V and H and fixes the outcome classes N and P.

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Corollary

A square board must have outcome class either N or P.

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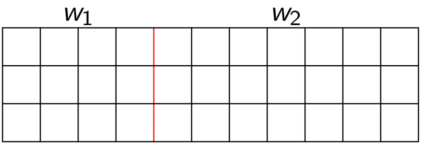

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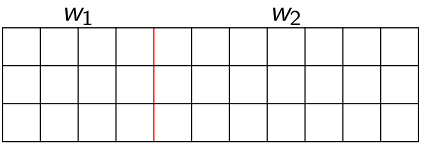
H	*H or P*	*H*
N	*H or P*	*H or N*



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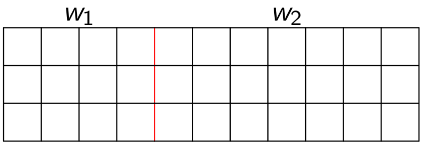

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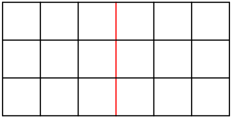

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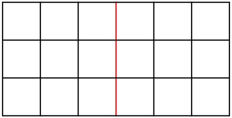

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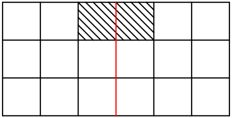

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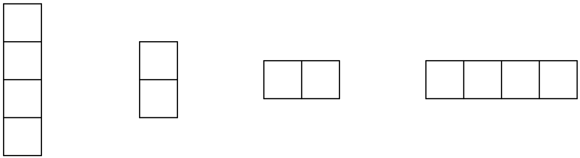

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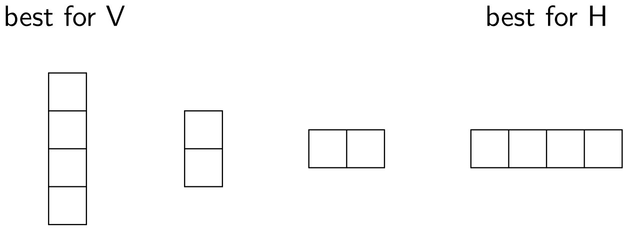

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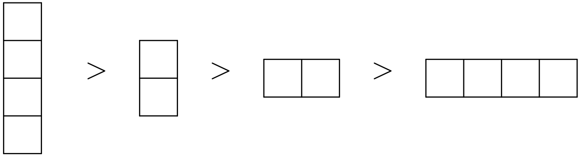

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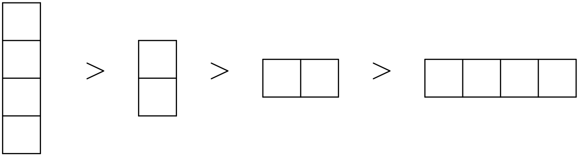

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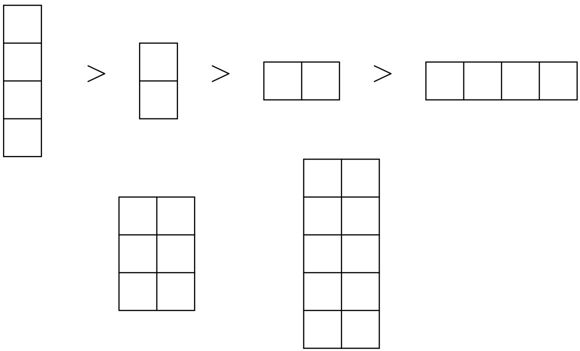

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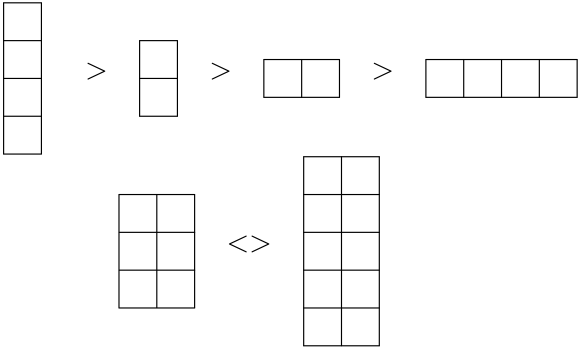

Combinatorial game theory perspective

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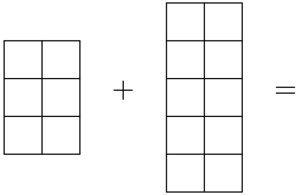

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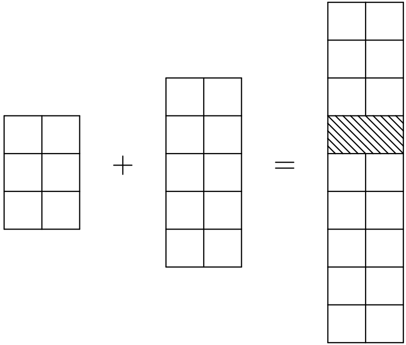

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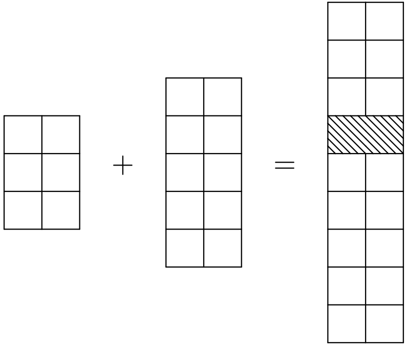

This operation arises naturally in gameplay.

Combinatorial game theory perspective

What structure do game positions have?

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After imposing an equivalence relation, game positions take values in a partially ordered Abelian group $\mathcal{G}$. We'll denote the $\mathcal{G}$ value of G by $|G|$.

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Outcome classes can be read off of group values in the following sense:

positive	negative	zero	incomparable with zero
V	H	P	N

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1	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
2	V	N	N	H	V	N	N	H	V	N	N	H	P	N	N	H	H	N	N	H	H
3	V	N	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
4	V	V	V	N	V	N	V	H	V	H	V	H	P	H	H	H	H	H	H	H	H
5	V	H	V	H	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6	V	N	V	N	V	N	V	H	V	N	N	H	V	H	N						
7	V	N	V	H	V	H	N	H	H	H	H	H	H								
8	V	V	V	V	V	V	V	N	V	H	V										
9	V	H	V	H	V	H	V	H	N	H											
10	V	N	V	V	V	N	V	V	V	N											
11	V	N	V	H	V	N	V	H													
12	V	V	V	V	V	V	V														
13	V	P	V	P	V	H	V														
14	V	N	V	V	V	V															
15	V	N	V	V	V	N															
16	V	V	V	V	V																
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3															H	H	H	H	H	H	H
4										H	V	H	P	H	H	H	H	H	H	H	H
5							H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6							V	H	V	N	N	H	V	H	N						
7					V	H	N	H	H	H	H	H	H								
8					V	V	V	N	V	H	V										
9					V	H	V	H	N	H											
10				V	V	N	V	V	V	N											
11				H	V	N	V	H													
12				V	V	V	V														
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14				V	V	V															
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4										H	V	H	P	H	H	H	H	H	H	H	H
5							H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6							V	H	V	N	N	H	V	H	N						
7					V	H	N	H	H	H	H	H	H								
8					V	V	V	N	V	H	V										
9					V	H	V	H	N	H											
10				V	V	N	V	V	V	N											
11				H	V	N	V	H													
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However, knowing $\mathcal{G}$ values for small boards can still tell us certain things.

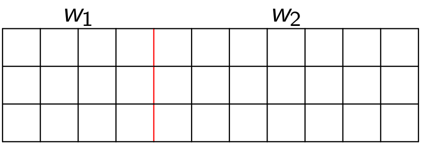
What can $\mathcal{G}$ values tell us?

Recall:

Proposition (Lachmann, Moore, Rapaport, 2000)

For a fixed height,

if width w_1 is:	*and width w_2 is:*	*then width $w_1 + w_2$ is:*
H	*H or P*	*H*
N	*H or P*	*H or N*
P	*P*	*H or P*



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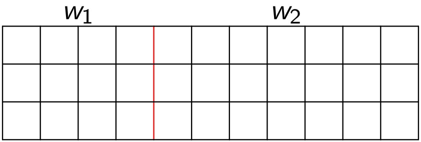

This proposition follows a the calculation in $\mathcal{G}$. For some fixed height let G_m denote the game of width m. Then:

$$|G_m| + |G_n| \geq |G_{m+n}|.$$

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Therefore the 11×14 board has outcome class H.

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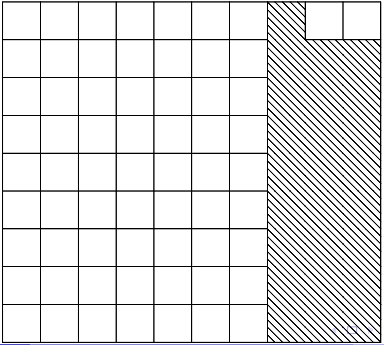

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So the 9×13 board has outcome class H.

Current status of known outcome classes

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
2	V	N	N	H	V	N	N	H	V	N	N	H	P	N	N	H	H	N	N	H	H
3	V	N	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
4	V	V	V	N	V	N	V	H	V	H	V	H	P	H	H	H	H	H	H	H	H
5	V	H	V	H	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6	V	N	V	N	V	N	V	H	V	N	N	H	V	H	N	H		NH	NH	H	
7	V	N	V	H	V	H	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H
8	V	V	V	V	V	V	V	N	V	H	V		V			H		NH		H	
9	V	H	V	H	V	H	V	H	N	H	NH	H	NH	H	NH	H	NH	H	NH	H	NH
10	V	N	V	V	V	N	V	V	V	N	NV		V							H	
11	V	N	V	H	V	N	V	H	NV	NH	NP	H	-V	NH	NH	H	NH	NH	NH	H	NH
12	V	V	V	V	V	V	V		V		V	NP	V								
13	V	P	V	P	V	H	V	H	NV	H	-H	H	NP	H	-V	H	-V	H	NH	H	NH
14	V	N	V	V	V	V	V		V		NV		V	NP							
15	V	N	V	V	V	N	V		NV		NV		-H		NP						
16	V	V	V	V	V	V	V	V	V		V		V			NP					
17	V	V	V	V	V		V		NV		NV		-H				NP				
18	V	N	V	V	V	NV	V	NV	V		NV		V					NP			
19	V	N	V	V	V	NV	V		NV		NV		NV						NP		
20	V	V	V	V	V	V	V	V	V	V	V		V							NP	
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1	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
2	V	N	N	H	V	N	N	H	V	N	N	H	P	N	N	H	H	N	N	H	H
3	V	N	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
4	V	V	V	N	V	N	V	H	V	H	V	H	P	H	H	H	H	H	H	H	H
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6	V	N	V	N	V	N	V	H	V	N	N	H	V	H	N	H		NH	NH	H	
7	V	N	V	H	V	H	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H
8	V	V	V	V	V	V	V	N	V	H	V		V			H		NH		H	
9	V	H	V	H	V	H	V	H	N	H	NH	H	NH	H	NH	H	NH	H	NH	H	NH
10	V	N	V	V	V	N	V	V	V	N	NV		V							H	
11	V	N	V	H	V	N	V	H	NV	NH	NP	H	-V	NH	NH	H	NH	NH	NH	H	NH
12	V	V	V	V	V	V	V		V		V	NP	V								
13	V	P	V	P	V	H	V	H	NV	H	-H	H	NP	H	-V	H	-V	H	NH	H	NH
14	V	N	V	V	V	V	V		V		NV		V	NP							
15	V	N	V	V	V	N	V		NV		NV		-H		NP						
16	V	V	V	V	V	V	V	V	V		V		V			NP					
17	V	V	V	V	V		V		NV		NV		-H				NP				
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19	V	N	V	V	V	NV	V		NV		NV		NV						NP		
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Using these methods we can obtain a few new outcome classes.

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	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
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5	V	H	V	H	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6	V	N	V	N	V	N	V	H	V	N	N	H	V	H	N	H		NH	N	H	
7	V	N	V	H	V	H	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H
8	V	V	V	V	V	V	V	N	V	H	V		V			H		NH		H	
9	V	H	V	H	V	H	V	H	N	H	NH	H	H	H	H	H	H	H	H	H	H
10	V	N	V	V	V	N	V	V	V	N	NV		V		NV					H	
11	V	N	V	H	V	N	V	H	NV	NH	NP	H	-V	H	NH	H	NH	H	NH	H	NH
12	V	V	V	V	V	V	V		V		V	NP	V								
13	V	P	V	P	V	H	V	H	V	H	-H	H	NP	H	-V	H	-V	H	NH	H	NH
14	V	N	V	V	V	V	V		V		V		V	NP	NV						
15	V	N	V	V	V	N	V		V	NH	NV		-H	NH	NP			NH			
16	V	V	V	V	V	V	V	V	V		V		V			NP					
17	V	V	V	V	V		V		V		NV		-H				NP				
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4	V	V	V	N	V	N	V	H	V	H	V	H	P	H	H	H	H	H	H	H	H
5	V	H	V	H	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6	V	N	V	N	V	N	V	H	V	N	N	H	V	H	N	H		NH	N	H	
7	V	N	V	H	V	H	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H
8	V	V	V	V	V	V	V	N	V	H	V		V			H		NH		H	
9	V	H	V	H	V	H	V	H	N	H	NH	H	H	H	H	H	H	H	H	H	H
10	V	N	V	V	V	N	V	V	V	N	NV		V		NV					H	
11	V	N	V	H	V	N	V	H	NV	NH	NP	H	-V	H	NH	H	NH	H	NH	H	NH
12	V	V	V	V	V	V	V		V		V	NP	V								
13	V	P	V	P	V	H	V	H	V	H	-H	H	NP	H	-V	H	-V	H	NH	H	NH
14	V	N	V	V	V	V	V		V		V		V	NP	NV						
15	V	N	V	V	V	N	V		V	NH	NV		-H	NH	NP			NH			
16	V	V	V	V	V	V	V	V	V		V		V			NP					
17	V	V	V	V	V		V		V		NV		-H				NP				
18	V	N	V	V	V	NV	V	NV	V		V		V		NV			NP			
19	V	N	V	V	V	N	V		V		NV		NV						NP		
20	V	V	V	V	V	V	V	V	V	V	V		V							NP	
21	V	V	V	V	V		V		V		NV		NV								NP

Using these methods we can obtain a few new outcome classes. We also get the outcome class for all sufficiently wide boards of height 6 (Lachmann et. al. and Bullock had partial results)

Future directions

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
2	V	N	N	H	V	N	N	H	V	N	N	H	P	N	N	H	H	N	N	H	H
3	V	N	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
4	V	V	V	N	V	N	V	H	V	H	V	H	P	H	H	H	H	H	H	H	H
5	V	H	V	H	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6	V	N	V	N	V	N	V	H	V	N	N	H	V	H	N	H		NH	N	H	
7	V	N	V	H	V	H	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H
8	V	V	V	V	V	V	V	N	V	H	V		V			H		NH		H	
9	V	H	V	H	V	H	V	H	N	H	NH	H	H	H	H	H	H	H	H	H	H
10	V	N	V	V	V	N	V	V	V	N	NV		V		NV					H	
11	V	N	V	H	V	N	V	H	NV	NH	NP	H	-V	H	NH	H	NH	H	NH	H	NH
12	V	V	V	V	V	V	V		V		V	NP	V								
13	V	P	V	P	V	H	V	H	V	H	-H	H	NP	H	-V	H	-V	H	NH	H	NH
14	V	N	V	V	V	V	V		V		V		V	NP	NV						
15	V	N	V	V	V	N	V		V	NH	NV		-H	NH	NP			NH			
16	V	V	V	V	V	V	V	V	V		V		V			NP					
17	V	V	V	V	V		V		V		NV		-H				NP				
18	V	N	V	V	V	NV	V	NV	V		V		V		NV			NP			
19	V	N	V	V	V	N	V		V		NV		NV						NP		
20	V	V	V	V	V	V	V	V	V	V	V		V							NP	
21	V	V	V	V	V		V		V		NV		NV								NP

We can look for patterns and make conjectures.

Future directions

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
2	V	N	N	H	V	N	N	H	V	N	N	H	P	N	N	H	H	N	N	H	H
3	V	N	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
4	V	V	V	N	V	N	V	H	V	H	V	H	P	H	H	H	H	H	H	H	H
5	V	H	V	H	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6	V	N	V	N	V	N	V	H	V	N	N	H	V	H	N	H		NH	N	H	
7	V	N	V	H	V	H	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H
8	V	V	V	V	V	V	V	N	V	H	V		V			H		NH		H	
9	V	H	V	H	V	H	V	H	N	H	NH	H	H	H	H	H	H	H	H	H	H
10	V	N	V	V	V	N	V	V	V	N	NV		V		NV					H	
11	V	N	V	H	V	N	V	H	NV	NH	NP	H	-V	H	NH	H	NH	H	NH	H	NH
12	V	V	V	V	V	V	V		V		V	NP	V								
13	V	P	V	P	V	H	V	H	V	H	-H	H	NP	H	-V	H	-V	H	NH	H	NH
14	V	N	V	V	V	V	V		V		V		V	NP	NV						
15	V	N	V	V	V	N	V		V	NH	NV		-H	NH	NP			NH			
16	V	V	V	V	V	V	V	V	V		V		V			NP					
17	V	V	V	V	V		V		V		NV		-H				NP				
18	V	N	V	V	V	NV	V	NV	V		V		V		NV			NP			
19	V	N	V	V	V	N	V		V		NV		NV						NP		
20	V	V	V	V	V	V	V	V	V	V	V		V							NP	
21	V	V	V	V	V		V		V		NV		NV								NP

We can look for patterns and make conjectures. It is expected that for a fixed height, as width increases, eventually every outcome class is H.

Future directions

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
2	V	N	N	H	V	N	N	H	V	N	N	H	P	N	N	H	H	N	N	H	H
3	V	N	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
4	V	V	V	N	V	N	V	H	V	H	V	H	P	H	H	H	H	H	H	H	H
5	V	H	V	H	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6	V	N	V	N	V	N	V	H	V	N	N	H	V	H	N	H		NH	N	H	
7	V	N	V	H	V	H	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H
8	V	V	V	V	V	V	V	N	V	H	V		V			H		NH		H	
9	V	H	V	H	V	H	V	H	N	H	NH	H	H	H	H	H	H	H	H	H	H
10	V	N	V	V	V	N	V	V	V	N	NV		V		NV					H	
11	V	N	V	H	V	N	V	H	NV	NH	NP	H	-V	H	NH	H	NH	H	NH	H	NH
12	V	V	V	V	V	V	V		V		V	NP	V								
13	V	P	V	P	V	H	V	H	V	H	-H	H	NP	H	-V	H	-V	H	NH	H	NH
14	V	N	V	V	V	V	V		V		V		V	NP	NV						
15	V	N	V	V	V	N	V		V	NH	NV		-H	NH	NP			NH			
16	V	V	V	V	V	V	V	V	V		V		V			NP					
17	V	V	V	V	V		V		V		NV		-H				NP				
18	V	N	V	V	V	NV	V	NV	V		V		V		NV			NP			
19	V	N	V	V	V	N	V		V		NV		NV						NP		
20	V	V	V	V	V	V	V	V	V	V	V		V							NP	
21	V	V	V	V	V		V		V		NV		NV								NP

We can look for patterns and make conjectures. It is expected that for a fixed height, as width increases, eventually every outcome class is *H*. Most conjectures are of this general type.

Future directions

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
2	V	N	N	H	V	N	N	H	V	N	N	H	P	N	N	H	H	N	N	H	H
3	V	N	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
4	V	V	V	N	V	N	V	H	V	H	V	H	P	H	H	H	H	H	H	H	H
5	V	H	V	H	P	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6	V	N	V	N	V	N	V	H	V	N	N	H	V	H	N	H		NH	N	H	
7	V	N	V	H	V	H	N	H	H	H	H	H	H	H	H	H	H	H	H	H	H
8	V	V	V	V	V	V	V	N	V	H	V		V			H		NH		H	
9	V	H	V	H	V	H	V	H	N	H	NH	H	H	H	H	H	H	H	H	H	H
10	V	N	V	V	V	N	V	V	V	N	NV		V		NV					H	
11	V	N	V	H	V	N	V	H	NV	NH	NP	H	-V	H	NH	H	NH	H	NH	H	NH
12	V	V	V	V	V	V	V		V		V	NP	V								
13	V	P	V	P	V	H	V	H	V	H	-H	H	NP	H	-V	H	-V	H	NH	H	NH
14	V	N	V	V	V	V	V		V		V		V	NP	NV						
15	V	N	V	V	V	N	V		V	NH	NV		-H	NH	NP			NH			
16	V	V	V	V	V	V	V	V	V		V		V			NP					
17	V	V	V	V	V		V		V		NV		-H				NP				
18	V	N	V	V	V	NV	V	NV	V		V		V		NV			NP			
19	V	N	V	V	V	N	V		V		NV		NV						NP		
20	V	V	V	V	V	V	V	V	V	V	V		V							NP	
21	V	V	V	V	V		V		V		NV		NV								NP

Theorem (D., 2013)

There is at most one prime such that the set of boards of height p contains infinitely many games with outcome class V.

Thank you for your attention!