Configuration spaces via factorization homology

Gabriel C. Drummond-Cole

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Nov. 27, 2014

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Configurations

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Configuration spaces

For M a manifold of dimension d, consider the space of distinct k-tuples in M:

Definition

$$F_k(M) := \{(x_1, \ldots, x_k) \in M^k | x_i \neq x_j \text{ for } i \neq j\}.$$

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This space has a natural action of the symmetric group S_k .

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This space has a natural action of the symmetric group S_k . The quotient by this action is the space of k unordered distinct points in M:

Definition

$$B_k(M) := F_k(M)/S_k$$

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These spaces are useful and interesting.

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Let's talk about $H_*B_k(M)$ and $H^*B_k(M)$. Isolated special cases have been known for some time.

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Theorem (Arnold, 69)

For $M = \mathbb{R}^2$, $H^*F_k(M)$ is the group cohomology of the pure braid group

$$\bigwedge_{1 \leq a < b \leq k} G_{ab} / (G_{ab}G_{bc} + G_{ac}G_{ab} + G_{bc}G_{ac} = 0$$

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Remark

The spaces $B_k(\mathbb{R}^n)$ are homotopy equivalent to the "little *n*-disks" which had been used by Boardman-Vogt and May to study loop spaces.

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Remark

The spaces $B_k(\mathbb{R}^n)$ are homotopy equivalent to the "little *n*-disks" which had been used by Boardman-Vogt and May to study loop spaces. Their homology was determined by F. Cohen (and others?) in the early 70s.

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Insight

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History, Part IIA: Bödigheimer-Cohen-Taylor 1989 In 1989, Bödigheimer, Cohen, and Taylor computed $H_*B_k(M)$.

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Typically X has a basepoint and one identifies these spaces by allowing points decorated by the basepoint to disappear. Then if $X = S^0$,

$$B(M;S^0)=\coprod B_k(M).$$

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Their calculation gave a *somewhat* explicit recipe to calculate the ranks of homology groups.

Results

For d odd and arbitrary even n,

$$H_iB_k(M)\cong Gr_k\bigotimes_{\alpha\in H_*(M)}H_{nk+i}\Omega^{d-|\alpha|}S^{d+n}$$

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- in even dimensions, they have to twist their space by a sign representation of S_k , so the results only work for \mathbb{F}_2 .

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They further showed:

Theorem (H^{odd} determined by β)

the homology groups for odd d were entirely determined by the ranks of the homology groups of M and the dimension of M.

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Theorem (Stability)

For $k \gg n$ the homology group $H_n B_k(M)$ (d odd or \mathbb{F}_2 coefficients) is independent of k.

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They were soon able to get the following:

Results

Explicit closed form numerical formulas for the ranks of $H_n(B_k(M), \mathbb{Q})$ for M a once punctured surface of genus g.

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Method

They give a chain complex for calculating the configurations with values in an even sphere:

$$H^*B(M; S^{2n}) \cong (Sym(V), d)$$

for explicit V and d, by fitting these spaces into quasifibrations involving $B(D^2, S^{2n})$.

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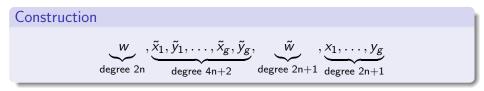
$$H^*B(M; S^{2n}) \cong (Sym(V), d)$$

V is spanned by the following basis:

Construction $\overset{W}{\longrightarrow}, \overset{\tilde{x}_1, \tilde{y}_1, \ldots, \tilde{x}_g, \tilde{y}_g}{\longrightarrow}, \overset{\tilde{W}}{\longrightarrow}, \overset{\tilde{x}_1, \ldots, y_g}{\longrightarrow}$ degree 2n degree 4n+2 degree 2n+1 degree 2n+1

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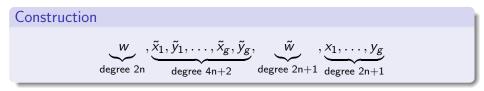


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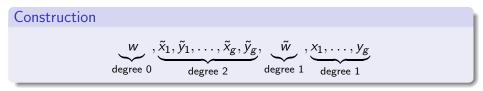
Note that for n = 0 this gives:

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Image: A mathematical states and a mathem

$$H^*B(M; S^0) \cong (Sym(V), d)$$

V is spanned by the following basis:



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History, Part IIIC: Bödigheimer-Cohen 1988 We have $U^* \mathbf{II} \in (M) \simeq (C \oplus (U) = 0$

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Feelings about this

This is so annoying! Wouldn't it be better to just be able to pick out the pieces from (Sym(V), d) directly?

History, Part IVA: Félix-Thomas 2000

Félix-Thomas extended these methods to more general even dimensional manifolds.

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$$\bigoplus H^*B_k(M)[qk] \cong \tilde{H}^*\left(\bigwedge(\underbrace{H_*M[q+d]}_{weight \ 1} \oplus \underbrace{H_*[M][2q+2d-1]}_{weight \ 2}), d\right)$$

where d goes from the weight 2 factor to the weight 1 factor by the coproduct in H_*M .

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History, Part IVB: Félix-Thomas 2000

Theorem

For even q > 0,

$$igoplus H^*B_k(\mathcal{M})[qk]\cong ilde{H}^*\left(igwedge(H_*\mathcal{M}[q+d]\oplus H_*[\mathcal{M}][2q+2d-1]),d
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Feelings

This is so annoying again! Wouldn't it be easier if q could be zero?

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$$H^*B_k(M) \cong H_*((H^*(M^k)[G_{ab}]/\sim, d)^{S_k})$$

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For instance, stability is not at all clear from this picture.

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Theorem

If M is odd dimensional, with coefficients in \mathbb{Q} or \mathbb{F}_p with p > k,

$$H^*B_kM\cong \bigwedge^k H^*(M)$$

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For M a smooth projective complex variety, there is an isomorphism of algebras (in rational coefficients)

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with differential induced by the coproduct and a messy formula for the product that nevertheless only depends on the product on H^*M .

Example

For $k \geq 4$,

$$H^*B_k(\mathbb{CP}^2)\cong \bigwedge(x,y)/x^3.$$

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End of history

I have left out many steps in this history:

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Definition (Motivation)
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A *d*-disk algebra is a symmetric monoidal functor from the full $(\infty, 1)$ -subcategory of **Mfld**_d whose objects are disjoint unions of \mathbb{R}^d .

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Construction of factorization homology

Construction (inexplicit)

Factorization homology with coefficients in the d-disk algebra A

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We will basically only need formal properties of this definition.

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Theorem (Ayala-Francis)
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Theorem (Ayala-Francis)

$$\int_{M} \operatorname{Free}(C_*X) \cong \bigoplus_{k} C_*(B_k(M;X))$$

Corollary

With X = pt we get:

$$\int_{M} \operatorname{Free}(\mathbb{F}) \cong \bigoplus_{k} C_{*}(B_{k}(M)).$$

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So factorization homology gives

$$\operatorname{colim}_{\coprod^{n} \mathbb{R}^{d} \to M} \bigotimes^{n} \left(\bigoplus_{k_{i}} C_{*}(Emb(k_{i}, \mathbb{R}^{d})) \otimes_{S_{k_{i}}} V^{\otimes k_{i}} \right)$$

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Proof.

$$\bigoplus_{\mathcal{K}} \left(\operatorname{colim}_{\prod^n \mathbb{R}^d \to M} \operatorname{Emb}(\mathcal{K}, \prod^n \mathbb{R}^d)) \right) \otimes_{S_{\mathcal{K}}} V^{\otimes \mathcal{K}}$$

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Let \mathfrak{g} be a Lie algebra. There is a universal enveloping *d*-algebra $U_d(\mathfrak{g})$

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Let \mathfrak{g} be a Lie algebra. There is a universal enveloping *d*-algebra $U_d(\mathfrak{g})$

Theorem (Francis-Gaitsgory, Knudsen)

Factorization homology for (compact orientable) M with coefficients in $U_d(\mathfrak{g})$ is the same as Chevalley-Eilenberg chains of \mathfrak{g} -valued cochains of M:

$$\int_{M} U_{d}(\mathfrak{g}) \cong CE_{*}(C_{*}(M)[-d] \otimes \mathfrak{g})$$

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Factorization homology for (compact orientable) M with coefficients in $U_d(\mathfrak{g})$ is the same as Chevalley-Eilenberg chains of \mathfrak{g} -valued cochains of M:

$$\int_{M} U_{d}(\mathfrak{g}) \cong CE_{*}(C_{*}(M)[-d] \otimes \mathfrak{g})$$

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This fact comes from a characterization of \int_M in terms of Koszul duality of $Disk_n$ -algebras and cochains on M due to Beilinson-Drinfeld.

Gabriel C. Drummond-Cole (CGP)

Configurations

 $\bigoplus_k C_*(B_k(M)) \cong$

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If d is even dimensional, $Lie(\mathbb{F}[d-1])$ is two dimensional, concentrated in degrees d-1 and 2d-2. Taking homology, we get:

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$$\bigoplus H_*B_k(M) \cong H_*(Sym(H_*(M) \oplus H_*(M)[d-1]), \partial_{CE})$$

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Next

Let's do some explicit calculations!