Configuration spaces via factorization homology

Gabriel C. Drummond-Cole

CGP

Nov. 27, 2014

Configuration spaces

For M a manifold of dimension d, consider the space of distinct k-tuples in M:

Definition

$$F_k(M) := \{(x_1, \ldots, x_k) \in M^k | x_i \neq x_j \text{ for } i \neq j\}.$$

Configuration spaces

For M a manifold of dimension d, consider the space of distinct k-tuples in M:

Definition

$$F_k(M) := \{(x_1, \ldots, x_k) \in M^k | x_i \neq x_j \text{ for } i \neq j\}.$$

This space has a natural action of the symmetric group S_k.

Configuration spaces

For M a manifold of dimension d, consider the space of distinct k-tuples in M:

Definition

$$F_k(M) := \{(x_1, \ldots, x_k) \in M^k | x_i \neq x_j \text{ for } i \neq j\}.$$

This space has a natural action of the symmetric group S_k. The quotient by this action is the space of k unordered distinct points in M:

Definition

$$B_k(M) := F_k(M)/S_k$$

Examples

Examples

Example

$$B_1(X) = X$$

Examples

Example

$$B_1(X) = X$$

Example

$$B_2(\mathbb{R}^d) \cong S^{d-1}$$

Examples

Example

$$B_1(X) = X$$

Example

$$B_2(\mathbb{R}^d) \cong S^{d-1}$$

Example

$$B_k(\mathbb{R}^2) = K(Br_k, 1)$$

Examples

Example

$$B_1(X) = X$$

Example

$$B_2(\mathbb{R}^d) \cong S^{d-1}$$

Example

$$B_k(\mathbb{R}^2) = K(Br_k, 1)$$

These spaces are useful and interesting.

History: Prehistory

Let's talk about $H_*B_k(M)$ and $H^*B_k(M)$. Isolated special cases have been known for some time.

History: Prehistory

Let's talk about $H_*B_k(M)$ and $H^*B_k(M)$. Isolated special cases have been known for some time.

Theorem (Arnold, 69)

For $M = \mathbb{R}^2$, $H^*F_k(M)$ is the group cohomology of the pure braid group

$$\bigwedge_{1 \leq a < b \leq k} G_{ab} / (G_{ab}G_{bc} + G_{ac}G_{ab} + G_{bc}G_{ac} = 0$$

History: Prehistory

Let's talk about $H_*B_k(M)$ and $H^*B_k(M)$. Isolated special cases have been known for some time.

Theorem (Arnold, 69)

For $M = \mathbb{R}^2$, $H^*F_k(M)$ is the group cohomology of the pure braid group

$$\bigwedge_{1\leq a<b\leq k} G_{ab}/(G_{ab}G_{bc} + G_{ac}G_{ab} + G_{bc}G_{ac} = 0$$

Eventually, results were determined about $M = \mathbb{R}^n$.

History: Prehistory

Let's talk about $H_*B_k(M)$ and $H^*B_k(M)$. Isolated special cases have been known for some time.

Theorem (Arnold, 69)

For $M = \mathbb{R}^2$, $H^*F_k(M)$ is the group cohomology of the pure braid group

$$\bigwedge_{1 \le a < b \le k} G_{ab}/(G_{ab}G_{bc} + G_{ac}G_{ab} + G_{bc}G_{ac} = 0$$

Eventually, results were determined about $M = \mathbb{R}^n$.

Remark

The spaces $B_k(\mathbb{R}^n)$ are homotopy equivalent to the "little n-disks" which had been used by Boardman-Vogt and May to study loop spaces.

History: Prehistory

Let's talk about $H_* B_k(M)$ and $H^* B_k(M)$. Isolated special cases have been known for some time.

Theorem (Arnold, 69)

For $M = \mathbb{R}^2$, $H^* F_k(M)$ is the group cohomology of the pure braid group

$$\bigwedge_{1 \leq a < b \leq k} G_{ab} / (G_{ab} G_{bc} + G_{ac} G_{ab} + G_{bc} G_{ac} = 0$$

Eventually, results were determined about $M = \mathbb{R}^n$.

Remark

The spaces $B_k(\mathbb{R}^n)$ are homotopy equivalent to the "little n-disks" which had been used by Boardman-Vogt and May to study loop spaces. Their homology was determined by F. Cohen (and others?) in the early 70s.

History: Part I: McDuff 1975

The first major work with direct applicability to analyzing these spaces for more general M was McDuff's *Configuration spaces of positive and negative particles*.

History: Part I: McDuff 1975

The first major work with direct applicability to analyzing these spaces for more general M was McDuff's *Configuration spaces of positive and negative particles*.

Insight

There is a map from $B_k(M)$ to $\Gamma\mathrm{Sph}(TM)$, sections of the sphere bundle of the tangent bundle of M

History: Part I: McDuff 1975

The first major work with direct applicability to analyzing these spaces for more general M was McDuff's *Configuration spaces of positive and negative particles*.

Insight

There is a map from $B_k(M)$ to $\Gamma\mathrm{Sph}(TM)$, sections of the sphere bundle of the tangent bundle of M which lands in compactly supported sections of degree k.

History: Part I: McDuff 1975

The first major work with direct applicability to analyzing these spaces for more general M was McDuff's *Configuration spaces of positive and negative particles*.

Insight

There is a map from $B_k(M)$ to $\Gamma\mathrm{Sph}(TM)$, sections of the sphere bundle of the tangent bundle of M which lands in compactly supported sections of degree k.

Theorem

This map induces an isomorphism on H_n for $k \gg n$.

History: Part I: McDuff 1975

The first major work with direct applicability to analyzing these spaces for more general M was McDuff's *Configuration spaces of positive and negative particles*.

Insight

There is a map from $B_k(M)$ to $\Gamma_k^{\mathrm{cs}}\mathrm{Sph}(TM)$, sections of the sphere bundle of the tangent bundle of M which lands in compactly supported sections of degree k.

Theorem

This map induces an isomorphism on H_n for $k \gg n$.

History, Part IIA: Bödigheimer-Cohen-Taylor 1989

In 1989, Bödigheimer, Cohen, and Taylor computed $H_*B_k(M)$.

History, Part IIA: Bödigheimer-Cohen-Taylor 1989

In 1989, Bödigheimer, Cohen, and Taylor computed $H_* B_k(M)$.

Restrictions

This only works when

- The dimension d is odd, or

History, Part IIA: Bödigheimer-Cohen-Taylor 1989

In 1989, Bödigheimer, Cohen, and Taylor computed $H_* B_k(M)$.

Restrictions

This only works when

- The dimension d is odd, or
- The coefficient ring is $\mathbb{F}_2$.

History, Part IIA: Bödigheimer-Cohen-Taylor 1989

In 1989, Bödigheimer, Cohen, and Taylor computed $H_*B_k(M)$.

Restrictions

This only works when

- The dimension d is odd, or
- The coefficient ring is $\mathbb{F}_2$.

In order to calculate, the introduced a configuration space with parameters in an auxiliary target space

Definition

$$B_k(M;X) := F_k(M) \times_{S_k} X^k.$$

History, Part IIA: Bödigheimer-Cohen-Taylor 1989

In 1989, Bödigheimer, Cohen, and Taylor computed $H_* B_k(M)$.

Restrictions

This only works when

- The dimension d is odd, or
- The coefficient ring is $\mathbb{F}_2$.

In order to calculate, the introduced a configuration space with parameters in an auxiliary target space

Definition

$$B_k(M; X) := F_k(M) \times_{S_k} X^k.$$

Typically X has a basepoint and one identifies these spaces by allowing points decorated by the basepoint to disappear. Then if $X = S^0$,

$$B(M; S^0) = \coprod B_k(M).$$

History, Part IIB: Bödigheimer-Cohen-Taylor 1989

Their calculation gave a *somewhat* explicit recipe to calculate the ranks of homology groups.

Results

For d odd and arbitrary even n,

$$H_i B_k(M) \cong Gr_k \bigotimes_{\alpha \in H_*(M)} H_{nk+i} \Omega^{d-|\alpha|} S^{d+n}$$

History, Part IIB: Bödigheimer-Cohen-Taylor 1989

Their calculation gave a *somewhat* explicit recipe to calculate the ranks of homology groups.

Results

For d odd and arbitrary even n,

$$H_i B_k(M) \cong Gr_k \bigotimes_{\alpha \in H_*(M)} H_{nk+i} \Omega^{d-|\alpha|} S^{d+n}$$

In order to give this formula, they use the the Thom space of direct sums of the bundle $B_k(M, \mathbb{R}) \to B_k(M)$

History, Part IIB: Bödigheimer-Cohen-Taylor 1989

Their calculation gave a *somewhat* explicit recipe to calculate the ranks of homology groups.

Results

For d odd and arbitrary even n,

$$H_i B_k(M) \cong Gr_k \bigotimes_{\alpha \in H_*(M)} H_{nk+i} \Omega^{d-|\alpha|} S^{d+n}$$

In order to give this formula, they use the the Thom space of direct sums of the bundle $B_k(M, \mathbb{R}) \to B_k(M)$

Drawbacks

History, Part IIB: Bödigheimer-Cohen-Taylor 1989

Their calculation gave a *somewhat* explicit recipe to calculate the ranks of homology groups.

Results

For d odd and arbitrary even n,

$$H_i B_k(M) \cong Gr_k \bigotimes_{\alpha \in H_*(M)} H_{nk+i} \Omega^{d-|\alpha|} S^{d+n}$$

In order to give this formula, they use the the Thom space of direct sums of the bundle $B_k(M, \mathbb{R}) \to B_k(M)$

Drawbacks

History, Part IIB: Bödigheimer-Cohen-Taylor 1989

Their calculation gave a *somewhat* explicit recipe to calculate the ranks of homology groups.

Results

For d odd and arbitrary even n,

$$H_i B_k(M) \cong Gr_k \bigotimes_{\alpha \in H_*(M)} H_{nk+i} \Omega^{d-|\alpha|} S^{d+n}$$

In order to give this formula, they use the the Thom space of direct sums of the bundle $B_k(M, \mathbb{R}) \to B_k(M)$

Drawbacks

- they need some condition on the Thom space, so M must be orientable, and

History, Part IIB: Bödigheimer-Cohen-Taylor 1989

Their calculation gave a *somewhat* explicit recipe to calculate the ranks of homology groups.

Results

For d odd and arbitrary even n,

$$H_i B_k(M) \cong Gr_k \bigotimes_{\alpha \in H_*(M)} H_{nk+i} \Omega^{d-|\alpha|} S^{d+n}$$

In order to give this formula, they use the the Thom space of direct sums of the bundle $B_k(M, \mathbb{R}) \to B_k(M)$

Drawbacks

- they need some condition on the Thom space, so M must be orientable, and
- in even dimensions, they have to twist their space by a sign representation of S_k, so the results only work for $\mathbb{F}_2$.

History, Part IIC: Bödigheimer-Cohen-Taylor 1989

This isn't as nice as one might hope.

History, Part IIC: Bödigheimer-Cohen-Taylor 1989

This isn't as nice as one might hope. They only calculated the rank of a few homology groups explicitly:

History, Part IIC: Bödigheimer-Cohen-Taylor 1989

This isn't as nice as one might hope. They only calculated the rank of a few homology groups explicitly:

Results

- the groups $H_{\leq 4}B_3(M_g, \mathbb{F}_2)$ and

History, Part IIC: Bödigheimer-Cohen-Taylor 1989

This isn't as nice as one might hope. They only calculated the rank of a few homology groups explicitly:

Results

- the groups $H_{\leq 4}B_3(M_g, \mathbb{F}_2)$ and
- the groups $H_{\leq 9}B_{\leq 10}(S^2, \mathbb{F}_2)$.

History, Part IIC: Bödigheimer-Cohen-Taylor 1989

This isn't as nice as one might hope. They only calculated the rank of a few homology groups explicitly:

Results

- the groups $H_{\leq 4}B_3(M_g, \mathbb{F}_2)$ and
- the groups $H_{\leq 9}B_{\leq 10}(S^2, \mathbb{F}_2)$.

They further showed:

Theorem (H^{odd} determined by β)

the homology groups for odd d were entirely determined by the ranks of the homology groups of M and the dimension of M.

History, Part IIC: Bödigheimer-Cohen-Taylor 1989

This isn't as nice as one might hope. They only calculated the rank of a few homology groups explicitly:

Results

- the groups $H_{\leq 4}B_3(M_g, \mathbb{F}_2)$ and
- the groups $H_{\leq 9}B_{\leq 10}(S^2, \mathbb{F}_2)$.

They further showed:

Theorem (H^{odd} determined by β)

the homology groups for odd d were entirely determined by the ranks of the homology groups of M and the dimension of M.

Theorem (Stability)

For $k \gg n$ the homology group $H_nB_k(M)$ (d odd or $\mathbb{F}_2$ coefficients) is independent of k.

History, Part IIIA: Bödigheimer-Cohen 1988

They were soon able to get the following:

Results

Explicit closed form numerical formulas for the ranks of $H_n(B_k(M), \mathbb{Q})$ for M a once punctured surface of genus g.

These results using the same methods and an application of the Serre spectral sequence.

History, Part IIIA: Bödigheimer-Cohen 1988

They were soon able to get the following:

Results

Explicit closed form numerical formulas for the ranks of $H_n(B_k(M), \mathbb{Q})$ for M a once punctured surface of genus g.

These results using the same methods and an application of the Serre spectral sequence.

Method

They give a chain complex for calculating the configurations with values in an even sphere:

$$H^* B(M; S^{2n}) \cong (Sym(V), d)$$

for explicit V and d, by fitting these spaces into quasifibrations involving $B(D^2, S^{2n})$.

History, Part IIIB: Bödigheimer-Cohen 1988

$$H^*B(M; S^{2n}) \cong (Sym(V), d)$$

History, Part IIIB: Bödigheimer-Cohen 1988

$$H^* B(M; S^{2n}) \cong (Sym(V), d)$$

V is spanned by the following basis:

Construction

$$\underbrace{w}_{\text{degree 2n}}, \underbrace{\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_g, \tilde{y}_g}_{\text{degree 4n+2}}, \underbrace{\tilde{w}}_{\text{degree 2n+1}}, \underbrace{x_1, \dots, y_g}_{\text{degree 2n+1}}$$

History, Part IIIB: Bödigheimer-Cohen 1988

$$H^* B(M; S^{2n}) \cong (Sym(V), d)$$

V is spanned by the following basis:

Construction

$$\underbrace{w}_{\text{degree 2n}}, \underbrace{\tilde{x}_1, \tilde{y}_1, \ldots, \tilde{x}_g, \tilde{y}_g}_{\text{degree 4n+2}}, \underbrace{\tilde{w}}_{\text{degree 2n+1}}, \underbrace{x_1, \ldots, y_g}_{\text{degree 2n+1}}$$

and the differential extends:

$$d\tilde{w} = 2 \sum x_i y_i$$

History, Part IIIB: Bödigheimer-Cohen 1988

$$H^* B(M; S^{2n}) \cong (Sym(V), d)$$

V is spanned by the following basis:

Construction

$$\underbrace{w}_{\text{degree 2n}}, \underbrace{\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_g, \tilde{y}_g}_{\text{degree 4n+2}}, \underbrace{\tilde{w}}_{\text{degree 2n+1}}, \underbrace{x_1, \dots, y_g}_{\text{degree 2n+1}}$$

and the differential extends:

$$d\tilde{w} = 2 \sum x_i y_i$$

Note that for $n = 0$ this gives:

$$H^* \coprod_k B_k(M)$$

History, Part IIIB: Bödigheimer-Cohen 1988

$$H^*B(M;S^0) \cong (Sym(V), d)$$

V is spanned by the following basis:

Construction

$$\underbrace{w}_{\text{degree } 0}, \underbrace{\tilde{x}_1, \tilde{y}_1, \ldots, \tilde{x}_g, \tilde{y}_g}_{\text{degree } 2}, \underbrace{\tilde{w}}_{\text{degree } 1}, \underbrace{x_1, \ldots, y_g}_{\text{degree } 1}$$

and the differential extends:

$$d\tilde{w} = 2\sum x_i y_i$$

Note that for $n = 0$ this gives:

$$H^* \coprod_k B_k(M)$$

History, Part IIIC: Bödigheimer-Cohen 1988

We have

$$H^* \coprod_k B_k(M) \cong (Sym(V), d)$$

History, Part IIIC: Bödigheimer-Cohen 1988

We have

$$H^* \coprod_k B_k(M) \cong (Sym(V), d)$$

Problem

- This doesn't work for closed manifolds because the Serre spectral sequence they use relies heavily on the puncture, and

History, Part IIIC: Bödigheimer-Cohen 1988

We have

$$H^* \coprod_k B_k(M) \cong (Sym(V), d)$$

Problem

- This doesn't work for closed manifolds because the Serre spectral sequence they use relies heavily on the puncture, and
- these results need to use higher spheres to find the right piece within that space.

History, Part IIIC: Bödigheimer-Cohen 1988

We have

$$H^* \coprod_k B_k(M) \cong (Sym(V), d)$$

Problem

- This doesn't work for closed manifolds because the Serre spectral sequence they use relies heavily on the puncture, and
- these results need to use higher spheres to find the right piece within that space.

To calculate $H^* B_k(M)$ it was necessary to take configurations with values in a sphere of dimension equal to the order of the Thom space of d times the the bundle $B_k(M, \mathbb{R}) \to B_k(M)$ and desuspend by k times that order.

History, Part IIIC: Bödigheimer-Cohen 1988

We have

$$H^* \coprod_k B_k(M) \cong (Sym(V), d)$$

Problem

- This doesn't work for closed manifolds because the Serre spectral sequence they use relies heavily on the puncture, and
- these results need to use higher spheres to find the right piece within that space.

To calculate $H^* B_k(M)$ it was necessary to take configurations with values in a sphere of dimension equal to the order of the Thom space of d times the the bundle $B_k(M, \mathbb{R}) \to B_k(M)$ and desuspend by k times that order.

Feelings about this

This is so annoying! Wouldn't it be better to just be able to pick out the pieces from $(Sym(V), d)$ directly?

History, Part IVA: Félix-Thomas 2000

Félix-Thomas extended these methods to more general even dimensional manifolds.

History, Part IVA: Félix-Thomas 2000

Félix-Thomas extended these methods to more general even dimensional manifolds.

Theorem

For M^d an even dimensional nilpotent orientable closed manifold and q positive and even, in rational coefficients we have:

History, Part IVA: Félix-Thomas 2000

Félix-Thomas extended these methods to more general even dimensional manifolds.

Theorem

For M^d an even dimensional nilpotent orientable closed manifold and q positive and even, in rational coefficients we have:

$$\bigoplus H^* B_k(M)[qk] \cong \tilde{H}^* \left(\bigwedge(\underbrace{H_* M[q+d]}_{\text{weight } 1} \oplus \underbrace{H_*[M][2q+2d-1]}_{\text{weight } 2}), d \right)$$

where d goes from the weight 2 factor to the weight 1 factor by the coproduct in $H_* M$.

History, Part IVB: Félix-Thomas 2000

Theorem

For even $q > 0$,

$$\bigoplus H^* B_k(M)[qk] \cong \tilde{H}^* \left(\bigwedge (H_* M[q+d] \oplus H_*[M][2q+2d-1]), d \right)$$

We'd like to pick out the cohomology of $B_k(M)$ for a single value of k.

History, Part IVB: Félix-Thomas 2000

Theorem

For even $q > 0$,

$$\bigoplus H^* B_k(M)[qk] \cong \tilde{H}^* \left(\bigwedge (H_* M[q+d] \oplus H_*[M][2q+2d-1]), d \right)$$

We'd like to pick out the cohomology of $B_k(M)$ for a single value of k.

Theorem

For $q > kd$, there is an isomorphism

$$H^* B_k M[qk] \cong H^* Wt_k \left(\bigwedge (H_* M[q+d] \oplus H_*[M][2q+2d-1]), d \right)$$

History, Part IVB: Félix-Thomas 2000

Theorem

For even $q > 0$,

$$\bigoplus H^* B_k(M)[qk] \cong \tilde{H}^* \left(\bigwedge (H_* M[q+d] \oplus H_*[M][2q+2d-1]), d \right)$$

We'd like to pick out the cohomology of $B_k(M)$ for a single value of k.

Theorem

For $q > kd$, *there is an isomorphism*

$$H^* B_k M[qk] \cong H^* Wt_k \left(\bigwedge (H_* M[q+d] \oplus H_*[M][2q+2d-1]), d \right)$$

Feelings

This is so annoying again! Wouldn't it be easier if q could be zero?

History, Part VA: Totaro 1996

This work gave the first meaningful statements about the cohomology *rings* of the configuration spaces.

History, Part VA: Totaro 1996

This work gave the first meaningful statements about the cohomology *rings* of the configuration spaces. Totaro worked primarily with $F_k(M)$, using the Leray spectral sequence of the inclusion $F_k(M) \subset M^k$.

History, Part VA: Totaro 1996

This work gave the first meaningful statements about the cohomology *rings* of the configuration spaces. Totaro worked primarily with $F_k(M)$, using the Leray spectral sequence of the inclusion $F_k(M) \subset M^k$.

Limitations

He wants the spectral sequence to degenerate so he works over $\mathbb{Q}$ with very nice manifolds: smooth complex projective varieties.

History, Part VA: Totaro 1996

This work gave the first meaningful statements about the cohomology *rings* of the configuration spaces. Totaro worked primarily with $F_k(M)$, using the Leray spectral sequence of the inclusion $F_k(M) \subset M^k$.

Limitations

He wants the spectral sequence to degenerate so he works over $\mathbb{Q}$ with very nice manifolds: smooth complex projective varieties.

Method

The spectral sequence is compatible with *ℓ-adic weight* and because we are working with smooth projective varieties, the ℓ-adic weight is fairly degenerate.

History, Part VA: Totaro 1996

This work gave the first meaningful statements about the cohomology *rings* of the configuration spaces. Totaro worked primarily with $F_k(M)$, using the Leray spectral sequence of the inclusion $F_k(M) \subset M^k$.

Limitations

He wants the spectral sequence to degenerate so he works over $\mathbb{Q}$ with very nice manifolds: smooth complex projective varieties.

Method

The spectral sequence is compatible with *ℓ-adic weight* and because we are working with smooth projective varieties, the ℓ-adic weight is fairly degenerate. This is gives degeneration of the spectral sequence and allows us to recover the ring structure.

History, VB: Totaro 1996

Totaro's results imply results for $B_k(M)$ as well.

History, VB: Totaro 1996

Totaro's results imply results for $B_k(M)$ as well.
Unfortunately, they are not very explicit.

Theorem

$$H^* B_k(M) \cong H_*((H^*(M^k)[G_{ab}]/\sim, d)^{S_k})$$

where $a \leq b \leq k$,

History, VB: Totaro 1996

Totaro's results imply results for $B_k(M)$ as well.
Unfortunately, they are not very explicit.

Theorem

$$H^*B_k(M) \cong H_*((H^*(M^k)[G_{ab}]/\sim, d)^{S_k})$$

where $a \leq b \leq k$*,*

$$\begin{aligned} G_{ab}G_{ac} + G_{bc}G_{ab} + G_{ac}G_{bc} &\sim 0 \\ G_{ab}(\pi_a^* - \pi_b^*)(x) &= 0 \\ dG_{ab} = \pi_{ab}^*\Delta & \end{aligned}$$

History, VB: Totaro 1996

Totaro's results imply results for $B_k(M)$ as well.
Unfortunately, they are not very explicit.

Theorem

$$H^* B_k(M) \cong H_*((H^*(M^k)[G_{ab}]/\sim, d)^{S_k})$$

where $a \leq b \leq k$*,*

$$\begin{aligned} G_{ab}G_{ac} + G_{bc}G_{ab} + G_{ac}G_{bc} &\sim 0 \\ G_{ab}(\pi_a^* - \pi_b^*)(x) &= 0 \\ dG_{ab} = \pi_{ab}^*\Delta \end{aligned}$$

For instance, stability is not at all clear from this picture.

History, Part VIA: Félix Tanré 2006

These authors combined the earlier lines of argument with Totaro's multiplicative results.

History, Part VIA: Félix Tanré 2006

These authors combined the earlier lines of argument with Totaro's multiplicative results.

Theorem

If M is odd dimensional, with coefficients in $\mathbb{Q}$ or $\mathbb{F}_p$ with $p > k$,

$$H^* B_k M \cong \bigwedge^k H^*(M)$$

History, Part VIB: Félix Tanré 2006

There is a more complicated statement for even dimensional manifolds that is only valid up to associated graded.

History, Part VIB: Félix Tanré 2006

There is a more complicated statement for even dimensional manifolds that is only valid up to associated graded. There is a special case that is somewhat easier to state.

History, Part VIB: Félix Tanré 2006

There is a more complicated statement for even dimensional manifolds that is only valid up to associated graded. There is a special case that is somewhat easier to state.

Theorem

For M a smooth projective complex variety, there is an isomorphism of algebras (in rational coefficients)

$$H^* B_k M \cong H^* Wt_k \left(\bigwedge (H^* M \oplus H^* M[N-1]), d \right)$$

History, Part VIB: Félix Tanré 2006

There is a more complicated statement for even dimensional manifolds that is only valid up to associated graded. There is a special case that is somewhat easier to state.

Theorem

For M a smooth projective complex variety, there is an isomorphism of algebras (in rational coefficients)

$$H^* B_k M \cong H^* Wt_k \left(\bigwedge (H^* M \oplus H^* M[N-1]), d \right)$$

with differential induced by the coproduct and a messy formula for the product that nevertheless only depends on the product on $H^* M$.

Example

For $k \geq 4$,

$$H^* B_k(\mathbb{CP}^2) \cong \bigwedge (x, y)/x^3.$$

End of history

I have left out many steps in this history:

End of history

I have left out many steps in this history: Segal (70s)

End of history

I have left out many steps in this history: Segal (70s), Bendersky-Gitler (91)

End of history

I have left out many steps in this history: Segal (70s), Bendersky-Gitler (91), Fulton-MacPherson (94)

End of history

I have left out many steps in this history: Segal (70s), Bendersky-Gitler (91), Fulton-MacPherson (94), Kriz (94)

End of history

I have left out many steps in this history: Segal (70s), Bendersky-Gitler (91), Fulton-MacPherson (94), Kriz (94), Salvatore (01)

End of history

I have left out many steps in this history: Segal (70s), Bendersky-Gitler (91), Fulton-MacPherson (94), Kriz (94), Salvatore (01), Napolitano (03)

End of history

I have left out many steps in this history: Segal (70s), Bendersky-Gitler (91), Fulton-MacPherson (94), Kriz (94), Salvatore (01), Napolitano (03), Church (11), Randal-Williams (11)

End of history

I have left out many steps in this history: Segal (70s), Bendersky-Gitler (91), Fulton-MacPherson (94), Kriz (94), Salvatore (01), Napolitano (03), Church (11), Randal-Williams (11), others?

Factorization homology: motivation

Definition (Motivation)

A *homology theory* valued in R is a symmetric monoidal functor C from $(Top, \coprod)$ to $(Ch_R, \oplus)$

Factorization homology: motivation

Definition (Motivation)

A *homology theory* valued in R is a symmetric monoidal functor C from $(Top, \coprod)$ to $(Ch_R, \oplus)$which satisfies the gluing axiom for cofibrations $X \leftarrow Y \rightarrow Z$:

Factorization homology: motivation

Definition (Motivation)

A *homology theory* valued in R is a symmetric monoidal functor C from $(Top, \coprod)$ to $(Ch_R, \oplus)$which satisfies the gluing axiom for cofibrations $X \leftarrow Y \rightarrow Z$:

$$C(X) \oplus_{C(Y)} C(Z) \rightarrow C(X \sqcup_Y Z)$$

is a weak equivalence of chain complexes.

Factorization homology: motivation

Definition (Motivation)

A *homology theory* valued in R is a symmetric monoidal functor C from $(Top, \coprod)$ to $(Ch_R, \oplus)$which satisfies the gluing axiom for cofibrations $X \leftarrow Y \rightarrow Z$:

$$C(X) \oplus_{C(Y)} C(Z) \rightarrow C(X \sqcup_Y Z)$$

is a weak equivalence of chain complexes.

Theorem (Motivation)

There is an equivalence between the category of homology theories valued in R and chain complexes in R given by

Factorization homology: motivation

Definition (Motivation)

A *homology theory* valued in R is a symmetric monoidal functor C from $(Top, \coprod)$ to $(Ch_R, \oplus)$which satisfies the gluing axiom for cofibrations $X \leftarrow Y \to Z$:

$$C(X) \oplus_{C(Y)} C(Z) \to C(X \sqcup_Y Z)$$

is a weak equivalence of chain complexes.

Theorem (Motivation)

There is an equivalence between the category of homology theories valued in R and chain complexes in R given by

- $C \mapsto C(\mathrm{pt})$ *and*

Factorization homology: motivation

Definition (Motivation)

A *homology theory* valued in R is a symmetric monoidal functor C from $(Top, \coprod)$ to $(Ch_R, \oplus)$which satisfies the gluing axiom for cofibrations $X \leftarrow Y \rightarrow Z$:

$$C(X) \oplus_{C(Y)} C(Z) \rightarrow C(X \sqcup_Y Z)$$

is a weak equivalence of chain complexes.

Theorem (Motivation)

There is an equivalence between the category of homology theories valued in R and chain complexes in R given by

- $C \mapsto C(\mathrm{pt})$ *and*
- $V \mapsto C_*^{\mathrm{sing}}(-, V))$.

Factorization homology: axiomatics

Definition (Factorization homology)

A dimension d *factorization homology theory* valued in R is a symmetric monoidal functor $\int$ from $(\mathbf{Mfld}_d, \mathrm{emb}, \coprod)$ to $(Ch_R, \otimes)$

Factorization homology: axiomatics

Definition (Factorization homology)

A dimension d *factorization homology theory* valued in R is a symmetric monoidal functor $\int$ from $(\mathbf{Mfld}_d, \mathrm{emb}, \coprod)$ to $(Ch_R, \otimes)$ which satisfies the gluing axiom for decompositions $W = X \cup_{Y\times\mathbb{R}} Z$:

Factorization homology: axiomatics

Definition (Factorization homology)

A dimension d *factorization homology theory* valued in R is a symmetric monoidal functor $\int$ from $(\mathbf{Mfld}_d, \mathrm{emb}, \coprod)$ to $(Ch_R, \otimes)$ which satisfies the gluing axiom for decompositions $W = X \cup_{Y\times\mathbb{R}} Z$:

$$\int X \otimes_{\int(Y\times\mathbb{R})} \int(Z) \to \int(W)$$

is a weak equivalence of chain complexes.

Factorization homology: axiomatics

Definition (Factorization homology)

A dimension d *factorization homology theory* valued in R is a symmetric monoidal functor $\int$ from $(\mathbf{Mfld}_d, \mathrm{emb}, \coprod)$ to $(Ch_R, \otimes)$ which satisfies the gluing axiom for decompositions $W = X \cup_{Y\times\mathbb{R}} Z$:

$$\int X \otimes_{\int(Y\times\mathbb{R})} \int(Z) \to \int(W)$$

is a weak equivalence of chain complexes.

Theorem

There is an equivalence between the category of dimension d factorization homology theories valued in R and d-disk algebras in R given by

Factorization homology: axiomatics

Definition (Factorization homology)

A dimension d *factorization homology theory* valued in R is a symmetric monoidal functor $\int$ from $(\mathbf{Mfld}_d, \mathrm{emb}, \coprod)$ to $(Ch_R, \otimes)$ which satisfies the gluing axiom for decompositions $W = X \cup_{Y \times \mathbb{R}} Z$:

$$\int X \otimes_{\int(Y \times \mathbb{R})} \int(Z) \to \int(W)$$

is a weak equivalence of chain complexes.

Theorem

There is an equivalence between the category of dimension d factorization homology theories valued in R and d-disk algebras in R given by

- $\int \mapsto \int(\mathbb{R}^d)$ *and*

Factorization homology: axiomatics

Definition (Factorization homology)

A dimension d *factorization homology theory* valued in R is a symmetric monoidal functor $\int$ from $(\mathbf{Mfld}_d, \mathrm{emb}, \coprod)$ to $(Ch_R, \otimes)$ which satisfies the gluing axiom for decompositions $W = X \cup_{Y \times \mathbb{R}} Z$:

$$\int X \otimes_{\int (Y \times \mathbb{R})} \int (Z) \to \int (W)$$

is a weak equivalence of chain complexes.

Theorem

There is an equivalence between the category of dimension d factorization homology theories valued in R and d-disk algebras in R given by

- $\int \mapsto \int (\mathbb{R}^d)$ *and*
- $A \mapsto \int_{(-)} A$.

Factorization homology: axiomatics

Theorem

There is an equivalence between the category of dimension d factorization homology theories valued in R and d-disk algebras in R given by

- $\int \mapsto \int(\mathbb{R}^d)$ *and*
- $A \mapsto \int_{(-)} A$.

The presentation I will be using is due to Francis.

Factorization homology: axiomatics

Theorem (Francis)

There is an equivalence between the category of dimension d factorization homology theories valued in R and d-disk algebras in R given by

- $\int \mapsto \int(\mathbb{R}^d)$ *and*
- $A \mapsto \int_{(-)} A$.

The presentation I will be using is due to Francis.

Factorization homology: axiomatics

Theorem (Lurie, Francis)

There is an equivalence between the category of dimension d factorization homology theories valued in R and d-disk algebras in R given by

- $\int \mapsto \int(\mathbb{R}^d)$ *and*
- $A \mapsto \int_{(-)} A$.

The presentation I will be using is due to Francis. There are also many variations with different names like topological chiral homology

Factorization homology: axiomatics

Theorem (Lurie, Morrison-Walker, Francis)

There is an equivalence between the category of dimension d factorization homology theories valued in R and d-disk algebras in R given by

- $\int \mapsto \int(\mathbb{R}^d)$ *and*
- $A \mapsto \int_{(-)} A$.

The presentation I will be using is due to Francis. There are also many variations with different names like topological chiral homology, blob homology

Factorization homology: axiomatics

Theorem (Lurie, Morrison-Walker, Ginot-Tradler-Zeinalian, Francis)

There is an equivalence between the category of dimension d factorization homology theories valued in R and d-disk algebras in R given by

- $\int \mapsto \int(\mathbb{R}^d)$ *and*
- $A \mapsto \int_{(-)} A$.

The presentation I will be using is due to Francis. There are also many variations with different names like topological chiral homology, blob homology, derived higher Hochschild homology

Factorization homology: axiomatics

Theorem (Lurie, Morrison-Walker, Ginot-Tradler-Zeinalian, Francis, ???)

There is an equivalence between the category of dimension d factorization homology theories valued in R and d-disk algebras in R given by

- $\int \mapsto \int(\mathbb{R}^d)$ *and*
- $A \mapsto \int_{(-)} A$.

The presentation I will be using is due to Francis. There are also many variations with different names like topological chiral homology, blob homology, derived higher Hochschild homology, others?

Factorization homology: axiomatics

Theorem (Beilinson-Drinfeld, Lurie, Morrison-Walker, Ginot-Tradler-Zeinalian, Francis, ???)

There is an equivalence between the category of dimension d factorization homology theories valued in R and d-disk algebras in R given by

- $\int \mapsto \int(\mathbb{R}^d)$ *and*
- $A \mapsto \int_{(-)} A$.

The presentation I will be using is due to Francis. There are also many variations with different names like topological chiral homology, blob homology, derived higher Hochschild homology, others? Some inspiration comes from chiral homology

Factorization homology: axiomatics

Theorem (Beilinson-Drinfeld, Costello-Gwilliam, Lurie, Morrison-Walker, Ginot-Tradler-Zeinalian, Francis, ???)

There is an equivalence between the category of dimension d factorization homology theories valued in R and d-disk algebras in R given by

- $\int \mapsto \int(\mathbb{R}^d)$ *and*
- $A \mapsto \int_{(-)} A$.

The presentation I will be using is due to Francis. There are also many variations with different names like topological chiral homology, blob homology, derived higher Hochschild homology, others? Some inspiration comes from chiral homology and its reformulation/popularization as factorization algebras.

Factorization homology: axiomatics

Theorem (Francis, others?)

There is an equivalence between the category of dimension d factorization homology theories valued in R and d-disk algebras in R given by

- $\int \mapsto \int(\mathbb{R}^d)$ *and*
- $A \mapsto \int_{(-)} A$.

The presentation I will be using is due to Francis. There are also many variations with different names like topological chiral homology, blob homology, derived higher Hochschild homology, others? Some inspiration comes from chiral homology and its reformulation/popularization as factorization algebras.

d-disk algebras

Definition (Fancy)

A *d-disk* algebra is a symmetric monoidal functor from the full $(\infty, 1)$-subcategory of $\mathbf{Mfld}_d$ whose objects are disjoint unions of $\mathbb{R}^d$.

d-disk algebras

Definition (Fancy)

A *d-disk* algebra is a symmetric monoidal functor from the full $(\infty, 1)$-subcategory of $\mathbf{Mfld}_d$ whose objects are disjoint unions of $\mathbb{R}^d$.

Definition (Simpler, less elegant)

A *d-disk* algebra in Ch_R is an algebra over the operad made up of chains of the configurations of d-disks embedded in the d-disk

d-disk algebras

Definition (Fancy)

A *d-disk* algebra is a symmetric monoidal functor from the full $(\infty, 1)$-subcategory of $\mathbf{Mfld}_d$ whose objects are disjoint unions of $\mathbb{R}^d$.

Definition (Simpler, less elegant)

A *d-disk* algebra in Ch_R is an algebra over the operad made up of chains of the configurations of d-disks embedded in the d-disk

Warning

This is related to but different from the "little disks" operad because the little disks have trivial $\mathcal{P}(1)$ but $Emb(\mathbb{R}^d, \mathbb{R}^d) \cong O(d)$.

d-disk algebras

Definition (Simpler, less elegant)

A *d-disk* algebra in Ch_R is an algebra over the operad made up of chains of the configurations of d-disks embedded in the d-disk

Warning

This is related to but different from the "little disks" operad because the little disks have trivial $\mathcal{P}(1)$ but $Emb(\mathbb{R}^d, \mathbb{R}^d) \cong O(d)$.

Example

Let $R = \mathbb{Q}$ and specialize to oriented manifolds and orientation preserving embeddings. Then:

d-disk algebras

Definition (Simpler, less elegant)

A *d-disk* algebra in Ch_R is an algebra over the operad made up of chains of the configurations of d-disks embedded in the d-disk

Warning

This is related to but different from the "little disks" operad because the little disks have trivial $\mathcal{P}(1)$ but $Emb(\mathbb{R}^d, \mathbb{R}^d) \cong O(d)$.

Example

Let $R = \mathbb{Q}$ and specialize to oriented manifolds and orientation preserving embeddings. Then:

- a 1-disk algebra is an A_∞ algebra

d-disk algebras

Definition (Simpler, less elegant)

A *d-disk* algebra in Ch_R is an algebra over the operad made up of chains of the configurations of d-disks embedded in the d-disk

Warning

This is related to but different from the "little disks" operad because the little disks have trivial $\mathcal{P}(1)$ but $Emb(\mathbb{R}^d, \mathbb{R}^d) \cong O(d)$.

Example

Let $R = \mathbb{Q}$ and specialize to oriented manifolds and orientation preserving embeddings. Then:

- a 1-disk algebra is an A_∞ algebra
- a 2-disk algebra is a BV_∞ algebra

Construction of factorization homology

Construction (inexplicit)

Factorization homology with coefficients in the d-disk algebra A

$$\int_M A$$

can be constructed as a homotopy colimit over the category of embeddings of disjoint copies of $\mathbb{R}^d$ into M

Construction of factorization homology

Construction (inexplicit)

Factorization homology with coefficients in the d-disk algebra A

$$\int_M A$$

can be constructed as a homotopy colimit over the category of embeddings of disjoint copies of $\mathbb{R}^d$ into M

$$\operatorname*{colim}_{\coprod^n \mathbb{R}^d \to M} A^{\otimes n}$$

Construction of factorization homology

Construction (inexplicit)

Factorization homology with coefficients in the d-disk algebra A

$$\int_M A$$

can be constructed as a homotopy colimit over the category of embeddings of disjoint copies of $\mathbb{R}^d$ into M

$$\operatorname*{colim}_{\coprod^n \mathbb{R}^d \to M} A^{\otimes n}$$

We will basically only need formal properties of this definition.

Factorization homology in the free d-disk algebra

Let X be a space. Consider $C_*(X)$ as a chain complex with trivial $O(d)$ action. Then:

Theorem (Ayala-Francis)

$$\int_M \mathrm{Free}(C_*X) \cong \bigoplus_k C_*(B_k(M;X))$$

Factorization homology in the free d-disk algebra

Let X be a space. Consider $C_*(X)$ as a chain complex with trivial $O(d)$ action. Then:

Theorem (Ayala-Francis)

$$\int_M \mathrm{Free}(C_* X) \cong \bigoplus_k C_*(B_k(M; X))$$

Corollary

With $X = \mathrm{pt}$ we get:

$$\int_M \mathrm{Free}(\mathbb{F}) \cong \bigoplus_k C_*(B_k(M)).$$

Factorization homology in the free d-disk algebra

$$\int_M \mathrm{Free}(C_* X) \cong \bigoplus_k C_*(B_k(M; X))$$

Sketch of proof.

Factorization homology in the free d-disk algebra

$$\int_M \mathrm{Free}(C_* X) \cong \bigoplus_k C_*(B_k(M; X))$$

Sketch of proof.

$$\mathrm{Free}(V) \cong C_*(Emb(\coprod^k \mathbb{R}^d, \mathbb{R}^d)) \otimes_{S_k \ltimes O(d)^k} V^{\otimes k}$$

Factorization homology in the free d-disk algebra

$$\int_M \mathrm{Free}(C_* X) \cong \bigoplus_k C_*(B_k(M; X))$$

Sketch of proof.

$$\mathrm{Free}(V) \cong C_*(Emb(\coprod^k \mathbb{R}^d, \mathbb{R}^d)) \otimes_{S_k \ltimes O(d)^k} V^{\otimes k}$$

$$\xrightarrow{O(d) \text{ acts trivially on } V} \bigoplus_k C_*(Emb(k, \mathbb{R}^d)) \otimes_{S_k} V^{\otimes k}$$

Factorization homology in the free d-disk algebra

$$\int_M \mathrm{Free}(C_*X) \cong \bigoplus_k C_*(B_k(M;X))$$

Sketch of proof.

$$\mathrm{Free}(V) \cong C_*(Emb(\coprod^k \mathbb{R}^d, \mathbb{R}^d)) \otimes_{S_k \ltimes O(d)^k} V^{\otimes k}$$

$$\xrightarrow{O(d) \text{ acts trivially on } V} \bigoplus_k C_*(Emb(k, \mathbb{R}^d)) \otimes_{S_k} V^{\otimes k}$$

So factorization homology gives

$$\operatorname*{colim}_{\coprod^n \mathbb{R}^d \to M} \bigotimes^n \left(\bigoplus_{k_i} C_*(Emb(k_i, \mathbb{R}^d)) \otimes_{S_{k_i}} V^{\otimes k_i} \right)$$

Factorization homology for the free disk algebra

Theorem

$$\int_M \mathrm{Free}(C_* X) \cong \bigoplus_k C_*(B_k(M;X))$$

Proof.

$$\operatorname*{colim}_{\coprod^n \mathbb{R}^d \to M} \bigotimes^n \left(\bigoplus_{k_i} C_*(Emb(k_i, \mathbb{R}^d)) \otimes_{S_{k_i}} V^{\otimes k_i} \right)$$

Factorization homology for the free disk algebra

Theorem

$$\int_M \mathrm{Free}(C_* X) \cong \bigoplus_k C_*(B_k(M;X))$$

Proof.

$$\operatorname*{colim}_{\coprod^n \mathbb{R}^d \to M} \bigotimes^n \left(\bigoplus_{k_i} C_*(Emb(k_i, \mathbb{R}^d)) \otimes_{S_{k_i}} V^{\otimes k_i} \right)$$

$$\cong \operatorname*{colim}_{\coprod^n \mathbb{R}^d \to M} \bigoplus_K C_*(Emb(K, \coprod^n \mathbb{R}^d)) \otimes_{S_K} V^{\otimes K}$$

Factorization homology for the free disk algebra

Theorem

$$\int_M \mathrm{Free}(C_*X) \cong \bigoplus_k C_*(B_k(M;X))$$

Proof.

$$\operatorname*{colim}_{\coprod^n \mathbb{R}^d \to M} \bigotimes^n \left(\bigoplus_{k_i} C_*(Emb(k_i, \mathbb{R}^d)) \otimes_{S_{k_i}} V^{\otimes k_i} \right)$$

$$\cong \operatorname*{colim}_{\coprod^n \mathbb{R}^d \to M} \bigoplus_K C_*(Emb(K, \coprod^n \mathbb{R}^d)) \otimes_{S_K} V^{\otimes K}$$

$$\cong \bigoplus_K \left(\operatorname*{colim}_{\coprod^n \mathbb{R}^d \to M} Emb(K, \coprod^n \mathbb{R}^d) \right) \otimes_{S_K} V^{\otimes K}$$

Factorization homology for the free disk algebra

Theorem

$$\int_M \mathrm{Free}(C_* X) \cong \bigoplus_k C_*(B_k(M; X))$$

Proof.

$$\bigoplus_K \left(\operatorname*{colim}_{\coprod^n \mathbb{R}^d \to M} Emb(K, \coprod^n \mathbb{R}^d)) \right) \otimes_{S_K} V^{\otimes K}$$

Factorization homology for the free disk algebra

Theorem

$$\int_M \mathrm{Free}(C_* X) \cong \bigoplus_k C_*(B_k(M; X))$$

Proof.

$$\bigoplus_K \left(\operatorname*{colim}_{\coprod^n \mathbb{R}^d \to M} Emb(K, \coprod^n \mathbb{R}^d) \right) \otimes_{S_K} V^{\otimes K}$$

$$\cong \bigoplus_K (C_*(Emb(K, M))) \otimes_{S_K} V^{\otimes K}$$

□

Lie and enveloping algebras

Let $\mathfrak{g}$ be a Lie algebra. There is a universal enveloping d-algebra $U_d(\mathfrak{g})$

Lie and enveloping algebras

Let $\mathfrak{g}$ be a Lie algebra. There is a universal enveloping d-algebra $U_d(\mathfrak{g})$

Theorem (Francis-Gaitsgory, Knudsen)

Factorization homology for (compact orientable) M with coefficients in $U_d(\mathfrak{g})$ is the same as Chevalley-Eilenberg chains of $\mathfrak{g}$-valued cochains of M:

$$\int_M U_d(\mathfrak{g}) \cong CE_*(C_*(M)[-d] \otimes \mathfrak{g})$$

Lie and enveloping algebras

Let $\mathfrak{g}$ be a Lie algebra. There is a universal enveloping d-algebra $U_d(\mathfrak{g})$

Theorem (Francis-Gaitsgory, Knudsen)

Factorization homology for (compact orientable) M with coefficients in $U_d(\mathfrak{g})$ is the same as Chevalley-Eilenberg chains of $\mathfrak{g}$-valued cochains of M:

$$\int_M U_d(\mathfrak{g}) \cong CE_*(C_*(M)[-d] \otimes \mathfrak{g})$$

In characteristic zero, this is

$$CE(\mathfrak{g}) := (Sym(\mathfrak{g}[1]), \partial_{CE})$$

Lie and enveloping algebras

Let $\mathfrak{g}$ be a Lie algebra. There is a universal enveloping d-algebra $U_d(\mathfrak{g})$

Theorem (Francis-Gaitsgory, Knudsen)

Factorization homology for (compact orientable) M with coefficients in $U_d(\mathfrak{g})$ is the same as Chevalley-Eilenberg chains of $\mathfrak{g}$-valued cochains of M:

$$\int_M U_d(\mathfrak{g}) \cong CE_*(C_*(M)[-d] \otimes \mathfrak{g})$$

In characteristic zero, this is

$$CE(\mathfrak{g}) := (Sym(\mathfrak{g}[1]), \partial_{CE})$$

The differential comes from the Lie bracket:

$$\partial_{CE}(a[1]b[1]) = [a,b][1]$$

Lie and enveloping algebras

Let $\mathfrak{g}$ be a Lie algebra. There is a universal enveloping d-algebra $U_d(\mathfrak{g})$

Theorem (Francis-Gaitsgory, Knudsen)

Factorization homology for (compact orientable) M with coefficients in $U_d(\mathfrak{g})$ is the same as Chevalley-Eilenberg chains of $\mathfrak{g}$-valued cochains of M:

$$\int_M U_d(\mathfrak{g}) \cong CE_*(C_*(M)[-d] \otimes \mathfrak{g})$$

In characteristic zero, this is

$$CE(\mathfrak{g}) := (Sym(\mathfrak{g}[1]), \partial_{CE})$$

The differential comes from the Lie bracket:

$$\partial_{CE}(a[1]b[1]) = [a, b][1]$$

This fact comes from a characterization of $\int_M$ in terms of Koszul duality of $Disk_n$-algebras and cochains on M due to Beilinson-Drinfeld.

Putting it all together

$$\bigoplus_k C_*(B_k(M)) \cong$$

Putting it all together

$$\bigoplus_k C_*(B_k(M)) \cong$$

$$\int_M \mathrm{Free}(\mathbb{F}) \cong$$

Putting it all together

$$\bigoplus_k C_*(B_k(M)) \cong$$

$$\int_M \mathrm{Free}(\mathbb{F}) \cong$$

$$\int_M U_d(Lie(\mathbb{F}[d-1])) \cong$$

Putting it all together

$$\bigoplus_k C_*(B_k(M)) \cong$$

$$\int_M \mathrm{Free}(\mathbb{F}) \cong$$

$$\int_M U_d(Lie(\mathbb{F}[d-1])) \cong$$

$$CE_*\left(C_*(M)[-d] \otimes Lie(\mathbb{F}[d-1])\right)$$

Putting it all together

$$\bigoplus_k C_*(B_k(M)) \cong$$
$$\int_M \mathrm{Free}(\mathbb{F}) \cong$$
$$\int_M U_d(Lie(\mathbb{F}[d-1])) \cong$$
$$CE_*\left(C_*(M)[-d] \otimes Lie(\mathbb{F}[d-1])\right)$$

If d is even dimensional, $Lie(\mathbb{F}[d-1])$ is two dimensional, concentrated in degrees $d-1$ and $2d-2$.

Putting it all together

$$\bigoplus_k C_*(B_k(M)) \cong$$
$$\int_M \mathrm{Free}(\mathbb{F}) \cong$$
$$\int_M U_d(Lie(\mathbb{F}[d-1])) \cong$$
$$CE_*\left(C_*(M)[-d] \otimes Lie(\mathbb{F}[d-1])\right)$$

If d is even dimensional, $Lie(\mathbb{F}[d-1])$ is two dimensional, concentrated in degrees $d-1$ and $2d-2$. Taking homology, we get:

Summary

$$\bigoplus H_* B_k(M) \cong H_*(Sym(H_*(M) \oplus H_*(M)[d-1]), \partial_{CE})$$

Conclusion

Summary

$$\bigoplus H_* B_k(M) \cong H_*(Sym(H_*(M) \oplus H_*(M)[d-1]), \partial_{CE})$$

This is compatible with the weight on both sides so finally we get

Conclusion

Summary

$$\bigoplus H_* B_k(M) \cong H_*(Sym(H_*(M) \oplus H_*(M)[d-1]), \partial_{CE})$$

This is compatible with the weight on both sides so finally we get

Theorem (Francis, Knudsen)

Rationally,

$$H_* B_k(M) \cong H_* Wt_k(Sym(H_*(M) \oplus H_*(M)[d-1]), \partial_{CE})$$

Conclusion

Summary

$$\bigoplus H_* B_k(M) \cong H_*(Sym(H_*(M) \oplus H_*(M)[d-1]), \partial_{CE})$$

This is compatible with the weight on both sides so finally we get

Theorem (Francis, Knudsen)

Rationally,

$$H_* B_k(M) \cong H_* Wt_k(Sym(H_*(M) \oplus H_*(M)[d-1]), \partial_{CE})$$

As written this is valid for compact orientable even dimensional manifolds without boundary.

Conclusion

Summary

$$\bigoplus H_* B_k(M) \cong H_*(Sym(H_*(M) \oplus H_*(M)[d-1]), \partial_{CE})$$

This is compatible with the weight on both sides so finally we get

Theorem (Francis, Knudsen)

Rationally,

$$H_* B_k(M) \cong H_* Wt_k(Sym(H_*(M) \oplus H_*(M)[d-1]), \partial_{CE})$$

As written this is valid for compact orientable even dimensional manifolds without boundary. But Knudsen's setup works coherently for any dimension whether or not the manifold is compact, orientable, or has boundary.

Conclusion

Summary

$$\bigoplus H_* B_k(M) \cong H_*(Sym(H_*(M) \oplus H_*(M)[d-1]), \partial_{CE})$$

This is compatible with the weight on both sides so finally we get

Theorem (Francis, Knudsen)

Rationally,

$$H_* B_k(M) \cong H_* Wt_k(Sym(H_*(M) \oplus H_*(M)[d-1]), \partial_{CE})$$

As written this is valid for compact orientable even dimensional manifolds without boundary. But Knudsen's setup works coherently for any dimension whether or not the manifold is compact, orientable, or has boundary. The only requirement is finite total homological dimension.

Conclusion

Summary

$$\bigoplus H_* B_k(M) \cong H_*(Sym(H_*(M) \oplus H_*(M)[d-1]), \partial_{CE})$$

This is compatible with the weight on both sides so finally we get

Theorem (Francis, Knudsen)

Rationally,

$$H_* B_k(M) \cong H_* Wt_k(Sym(H_*(M) \oplus H_*(M)[d-1]), \partial_{CE})$$

As written this is valid for compact orientable even dimensional manifolds without boundary. But Knudsen's setup works coherently for any dimension whether or not the manifold is compact, orientable, or has boundary. The only requirement is finite total homological dimension.

Next

Let's do some explicit calculations!