1. Pretalk

I'll have to make some definitions here and again later this afternoon. What I'll talk about now is string diagrams and moduli space. Let me make a diagram. I'll have a space SD which will sit inside a natural completion $\overline{SD}$. I'll have an equivalence relation, and these will sit inside spaces $LD/\sim$ and $\overline{LD}/\sim$. So $LD/\sim$ is equivalent to the moduli space $\mathscr{M}$ and so we'll take $\overline{LD}$ as our definiton of $\overline{M}$. So I'll talk about $SD/\sim\subset\overline{SD}$, then the LD, then I'll talk about the relation to $\mathscr{M}$, and finally I'll show that $\overline{SD}/\sim\to\overline{LD}/\sim$ is a homotopy equivalence.

Fix g, k, ℓ. Let me define a string diagram (a type of graph) by example. Start with k input circles of length 1, with a marked point on each one. I'll attach "chords" of length one among these circles. I want something connected at the end but the endpoints can coinicide, they can coincide with the marked points.

I want to look at this as a fat graph so I want the data of a cyclic order on the half-edges adjacent to each vertex. This information on a graph is enough to specify an orientable surface with boundary that contains this graph as a deformation retract. Let me draw a picture if you haven't seen it before. Say I take this string diagram with one circle and one chord. I'm not going to specify the cyclic order. I think I'm going to draw the one coming from the orientation of the blackboard. You can thicken the vertices, then thicken all of the edges to bands. The cyclic order tells you how to attach the bands to the disks. I want the ribbon surface to have genus g and $k+\ell$ boundary components, k of which are isotopic to the input circles. I'll say the other ones are outputs. So a string diagram is a metric graph with cyclic orders on each vertex, so that the resulting ribbon surface has genus g and the appropriate boundary conditions. Initially this will be a set. We'll make this a topological space. Let me say what the topology should be. When are two string diagrams close? Fix an underlying cyclically ordered graph. The parameters are where the endpoints land on the input circle.

I want to outline a construction that takes in a string diagram and outputs a metric space. I start by forming the associated ribbon surface. The next thing I want to do, for each chord, I want to stick in an extra vertex in the middle, and I have a cyclic order, which is stupid, at that vertex, but at each of these, I can stick in extra edges. I want those to be half infinite, which I'll draw pointing down. I'll take an infinite strip with correctly corresponding width to the picture that we already have. I'm not changing the topological type by adding these pieces. I'll get this infinite surface. On top it's cylindrical, and we end up getting a pair of pants that contains this string diagram.

I'm going to be a little sloppy about my distinguished vertex. Here's my construction that produces this metric space. Is that picture clear enough? I want to say, what if, I 'm allowed to have two chord endpoints coming together. That's the limit of two chord endpoints that are close coming together. I have, if the two points are really close, one leg that is much thinner than the other. In the limit, the leg get's really thin and the leg just becomes an interval. The construction generally produces a surface, but more, a surface with a conformal structure, so a Riemann surface. This is a smooth Riemannian metric away from some funny gluing points. You end up getting nodes and stuff as well, when you've got topology in these chords.

Let me give you another definition. We say that two string diagrams differ by a slide if, I'm going to draw the picture, which is more meaningful. I'll start being sloppy about distinguished points. Say I have three input circles with two chords with endpoints like this. Now if I slide one endpoint of one chord across the other, that's a slide, and I let this generate an equivalence relation, "slide equivalence," notated $\sim$. The construction that took us to a metric space generically a Riemann surface, this is the same surface if and only if their string diagrams are slide equivalent.

Two string diagrams are equivalent if and only if there is an isometry of their metric spaces preserving the decomposition into strips.

Now we have two ways of thinking about the same equivalence. I think it's easier to think about this in terms of slides. You have this thing that is generically a Riemann surface lurking. One think that could motivate is this picture I've drawn. Here for this picture you get a monkey saddle. You get three ups and three downs. If you want to draw pictures you can see that the two chords in this construction get sewn together.

So now for another definition, let $SD(g,k,\ell)$ be the subset of $\overline{SD}(g,k,\ell)$ if the metric space is a surface.

Proposition 1. *$\Gamma \in SD(g,k,\ell)$ if and only if its chord subgraph is a disjoint union of trees.*

I'm more interested in $\overline{SD}(g,k,\ell)/\sim$, which is a connected cell complex of dimension $4g-4+2k+2\ell$. This is true without slide equivalence except it is not connected. $SD(g,k,\ell)/\sim$ is a union of open cells which is dense in $\overline{SD}(g,k,\ell)$. You may not be able to include all the faces.

That's part 1. I hope that you got something out of that, because here for part two, replacing SD with LD is a very similar story, just a little more complicated. A string diagram of type (g,k,ℓ) with levels is a diagram with the same condition, we'll keep doing that, but those are my level one chords. I'll attach the next chords as longer. Level two, I'll add more chords, and level three, I'll add more chords. I still have a cyclic order so that the final thing has genus g, k inputs, and ℓ outputs. I keep track of a parameter between each level S_1 between 1 and 2 and think of the length of a chord at level n as having length $1+2\sum S_i$. If the spacing parameter is small, I want to think that they're at the same level. If the spacing parameter gets to zero, we collapse those two levels.

Fixing a combinatorial type is an underlying graph and levelization and also the spacing parameters.

[I don't know how to connect this with anything but Feynman diagrams. Can you talk about motivation?]

I want to take this more elaborate thing and generate a surface. I'll draw a picture rather than give you explicit instructions. Before our strips did one thing for length $\frac{1}{2}$. Now they do something for length S_1 after that, then S_2, and so on. The space is called $\overline{LD}(g,k,\ell)$ which is the space of string diagrams with levels of this type. Again, consider LD as the subset where the construction produces a metric space which is a surface.

We had an equivalence relation that we can state two ways, with slide equivalence. We have this construction that produces generically a surface. Before it was easy to see when two diagrams produce the same metric space. Two string diagrams with levels differ by a slide if (same picture). A chord here can only slide over a chord if it's "below" (higher level number). You can do them at the same level, or the sliding chord must have been attached after the one that is being slid across. Let this generate an equivalence relation.

Proposition 2. *Two string diagrams with levels are equivalent if and only if there is an isometry of the associated metric spaces preserving the decomposition.*

We're interested in $\overline{LD}/\sim$.

Proposition 3. $\overline{SD}(g,k,\ell)/\sim\subset\overline{LD}(g,k,\ell)$ *is a deformation retract.*

You can bring all the spacing parameters to zero. Let me just do a quick example that might say this.

$\overline{LD}(0,3,1)/sim$ is a three torus bundle. I'll describe the base. This has three circles with two chords. The string diagrams with one level sits inside as the spine of the base right here [picture].

Where might this definition come from? It comes from Riemann surfaces. This is a Riemann surface with a convenient metric that gives us the cylindrical shape. Let $\mathscr{M}_{g,k,\ell}$ be the moduli space of Riemann surfaces (which are highly decorated). This is a surface of genus g with $k+\ell$ marked points or punctures, k of which are inputs and ℓ outputs. The inputs are marked by tangent directions. The outputs are decorated by weights. The sum of the weights should be equal to the number of inputs, k.

This information is enough to specify a harmonic function from the Riemann surface to the real numbers. This is a generalized Dirichlet principle. This gives us a harmonic function that takes certain values on the inputs and outputs. This lets us build a metric on the surface. The example is, you could have one input and one output. At the output there's only one weight, this is one, let's say genus one. The harmonic function makes this look like a height function, that's the point. Then I can use that harmonic function to build a string diagram with levels up to slide equivalence. I want to get a graph. This is generically Morse but not always Morse. The critical points, I'll look at the critical graph, gradient flow lines going to and from the critical points.

This is a bit more than what we need. Make a cut above the first critical point. I added half-infinite edges. I don't need the infinite edges, those come automatically. The spacing parameters is differences in critical values. That's where this comes from.

I showed you why you get a homotopy equivalence, but not in general without the bars. If the chords have topology, not generally. We'll only get string diagrams with levels up to equivalence. How can I get a representative that has levels. One choice that I could make is to have one chord attached here, and then a second chord attached, if I perturb just a little bit, I'll get something that looks like this: [picture]. Remember that we have a deformation retraction from shrinking spacing parameters. So here, if you shrink the spacing parameter, you'd get something where they were close, if we went with this one where they're right on top of each other. Your chord subgraph is not a union of trees. The metric space is not genus one, so you shrink this genus to a point. Okay, thank you.

2. Seminar

Thank you for giving me the opportunity. I think that maybe, let me change the title: string topology. I'll talk about string topology and compactified moduli spaces. The connection to moduli space was stressed in the pretalk. The goal is to solve what is known as the master equation $\partial X = X * X$ where X is in the direct sum of homomorphism complexes $\bigoplus Hom(P^{\otimes k}, P^{\otimes \ell})$, where P is a chain complex computing the S^1 equivariant homology of the loop space of a manifold relative to constant loops $H^{S^1}(LM, M)$ where M is a closed, oriented, complete Riemannian manifold and LM is its free loop space. This is what we want to do.

Why might we want to do this. Homotopical algebra tells us that such a solution passes to homology, but on homology the boundary is zero, so $X \star X = 0$, so this would be a packaging of k to ℓ operations, so an algebraic structure on the homology, operations and quadratic relations.

[This is an embarrassing question to be asking. What is equivariant homology?]

$H^{S^1}(LM)$ is the usual homology of the homotopy quotient, $H_*(LM \times ES^1/S^1)$.

[I ask this question because homotopy theorists often mean something else.]

The method is, I want to bulid a pseudomanifold with appropriate boundary, and map chains on that space in here to the hom complex. We get an anomalous term in some cases, which can be eliminated in some examples and likely in general.

That's why we want to solve the master equation. Let me write a quick outline on this side board. I'll start by talking about string diagrams, SD, $\overline{SD}$, $\overline{LD}$, the string topology construction for $\overline{SD}$, then a quotienting $\sim$ and finally the beginnings of constructing X.

Definition 1. *A string diagram of type g, k, ℓ, is a connected graph, starting with k disjoint circles (input circles) with total length 1 and a marked point on each circle. I'll add chords of length one whose endpoints land on the input circles. I want to give a cyclic order to the half edges on each vertex. This specifies an orientable surface with boundary carrying this graph as a deformation retract.*

The ribbon surface that I obtain should have genus g and $k+\ell$ boundary components, k of which are isotopic to the k input circles. The ℓ will be output circles. Fixing the topological type g, k, and ℓ, we can look at the space of such diagrams. So $\overline{SD}(g,k,l)$ is the space of string diagrams of that type. There's a natural subspace $SD(g,k,\ell)$ which is the subspace where the subgraph looking at the chords is a disjoint union of trees.

The simplest example is $\overline{SD}(0,2,1)$. I want to attach chords so that I get the right Euler characteristic. I can only do that with one chord. What are the parameters? It's only where the endpoints are on the circles. This is the two-torus T^2.

An example I'll want to come back to is $\overline{SD}(0,3,1)$. I have three circles and two chords. I should point out that the input circles are labeled so I can tell them apart. So this space is the total space of a bundle, where the fiber will keep track of the distinguished vertex. The fiber will be the three torus. The base space is three disjoint intervals. Each component corresponds to which input is in the middle.

These spaces $\overline{SD}(g,k,\ell)$ is a cell complex of dimension $4g-4+2k+2\ell$. If you've seen this before, think that every cell is labeled by a combinatorial type. The parameters are where the chord endpoints land. So the decomposition for $\overline{SD}(0,2,1)$ is the standard decomposition for the torus.

Then $SD(g,k,\ell)$ is a union of open cells in $\overline{SD}$, dense. You can't include all the faces without violating the conditions. It's actually true that $\overline{SD}(g,k,\ell) = SD(g,k,\ell)$ only if $(g,k,\ell) = (0,k,1)$.

I want to modify this. I want to take this definition, add one more piece of information, and change the name to a string diagram with levels. With levels I'll start off the same way. I say, these are the chords I attached at level one. I'll attach chords at level two, and say that they are longer. Here's a picture. I'll keep track of levels not by length but by assigning a partial order on the chords and add a spacing parameter in $(0,1]$ between levels. I've left off 0 because if the spacing parameter is 0, I should imagine that I get the same level. We have a corresponding space of these. This is the space $\overline{LD}$, the space of string diagrams with levels of type (g,k,ℓ).

One thing to notice is that a string diagram *is* a string diagram with levels. Note that $\overline{SD}(g,k,\ell) \subset \overline{LD}(g,k,\ell)$. The case $(0,3,1)$ is the first interesting example when we have levels. The base space is diagrams that have two levels. You cross each of your base space components with an interval, and it looks like this: [picture]

Proposition 4. *$\overline{LD}(g,k,\ell)$ is a cell complex of dimension $6g-6+3k+3\ell-1$, and the inclusion $\overline{SD}(g,k,\ell)$ is a deformation retract.*

The retraction we can see very easily by contracting the spacing parameters to 0.

2.1. **string topology.** Now let me talk about string topology. That picture is going to come back later. I should point out that the statements here are about equivariant homology, and there's a non-equivariant story that I'll mix a little bit, hopefully without too much confusion.

A string topology operation would usually go $H^{\otimes k} \to H^{\otimes \ell}$ where H is either $H_*(LM)$ or $H_*^{S^1}(LM, M)$. Where might such an operation come from? Say I had a string diagram, and because I'm conflating the equivariant and non-equivariant pictures, I'll be sloppy about the marked point. If I had a map from the barbell graph to M, if I restrict to the input circles, I get a map from two circles to M, restricting to the outputs I get one circle. If we could extend the two circles to a map that included a chord, then we could reverse the arrow. So string topology reverses that first arrow on H. Once you do that, the next step is easy, restricting to outputs.

One way you could do this is, look at chains of loops. Look at two of them. If they intersect transversally, then along the intersection locus I have one loop by concatenation. It's not true that any two chains intersect transversally. We can always pick transversally intersecting representative cycles so we can do this with homology.

Chas Sullivan did this with $\overline{SD}(g,k,\ell)$ for $(0,k,1)$, $(0,1,k)$, $(0,2,2)$, and $(1,1,1)$. Cohen-Godin did this for $SD(g,k,\ell)$ and Godin did this for the moduli space $\mathscr{M}$. We have a chain complex P computing $H_*^{S^1}(LM,M)$. This P is for pair. I won't go too much into it. We also have a cellular map from $C_*(\overline{SD}(g,k,\ell)) \to \bigoplus(P^{\otimes k}, P^{\otimes \ell})$. This is the string topology map, and involves picking a representative for the Thom class of the diagonal.

Remember that $\overline{SD}(0,2,1)$, it's the torus with the standard decomposition. So if I apply the string topology operation to the top chain of that torus, I'll call that B for string bracket. B goes $P \otimes P \to P$. It's a two to one operation like we were trying to describe.

2.2. **equivalence relation.** I said I wanted to define an equivalence relation. Two such differ by a slide if, well, [picture]. We can slide a chord with a higher numbered level slide over a chord with a lower numbered level. I'm interested in quotienting my picture from before by my equivalence relation. So $\overline{LD}(0,3,1)$ is a torus bundle over the space we get via those identifications. Similarly, we have $\overline{SD}(0,3,1)$ sitting inside of this.

Proposition 5. *$\overline{LD}(g,k,\ell)/\sim$ is a pseudomanifold with boundary which either has an output of chords or one of the spacing parameters is* 1. *The dimension, again, is* $6g-6+3k+3\ell-1$. *I don't lose this deformation retract property by taking slide equivalence, so $\overline{SD}/\sim$ is a deformation retract of $\overline{LD}/\sim$, and there is an injective map $\mathscr{M} \to \overline{LD}(g,k,\ell)/\sim$ whose image is open and dense.*

We have this relationship to moduli space. I want to be explicit about one thing. If I define $\overline{\mathscr{M}}(g,k,\ell)$ to be $\overline{LD}(g,k,\ell)/\sim$, then $\overline{SD}(g,k,\ell)/\sim \subset \overline{\mathscr{M}}(g,k,\ell)$ as a deformation retract. Without the bars, this is not true.

I want, now, to use $\overline{LD}(g,k,\ell)/\sim$ to build X. What I'll do is start doing this in an example. I'll make an assumption that isn't okay. I can talk about two chains being slide equivalent. If $C \sim C'$ is in $C_*(\overline{SD}(g,k,\ell))$, I have operations applied to these guys, and the statement is that $ST(C) - ST(C') = \delta b$ for some b in the homomorphism complex $\bigoplus Hom(P^{\otimes k}, P^{\otimes \ell})$. Imagine b were 0. Then the string topology construction agrees on slide equivalent chains. Then $ST(C) = ST(C')$, so the string topology construction passes to cellular chains on the quotient.

I have the string topology construction defined on this core of the deformation retraction and I want to extend it to the whole thing. I want to extend this to the nice space that I circled up there, which is our compactification of moduli space. I don't care about all chains, just about the fundamental chain of $\overline{LD}(g,k,\ell)/\sim$. I have a 3-torus bundle over the picture I drew before, and I mainly care about the base. The string bracket is a two to one operation. you can compose this with itself. Recall that B goes $P \otimes P \to P$, so I can consider the three ways of composing B with itself. Call this $B \star_i B$. I want to say that there exist chains c_i in $C_*(\overline{SD}(0,3,1)/\sim)$ and chains B_i in $Hom(P^{\otimes 3}, P)$, so that $ST(c_i) - B \star_i B = \delta B_i$. Let me pick out the three c_i. I can do this in the picture here: [picture].

What that says is that if I look at the fundamental chain of the pseudomanifold $[\overline{LD}(0,3,1)]$, if I take the string topology operation and then the boundary, that's the sum of the compositions $\sum B \star_i B$. This is a Jacobi type relation.

This falls into a more general setting. I'm almost done. That picture is the upshot. That space has boundary that is a sum of compositions. In general what we would get, if we took the string topology construction applied to the fundamental chain of $\overline{LD}(g,k,\ell)/\sim$, and called that $X(g,k,\ell)$, then $\delta X(g,k,\ell) = \sum X(g',k',\ell') \star X(g'',k'',\ell'')$. If I let X be the formal sum, then $\delta X = X \star X$, which would be a solution to the master equation, but we were bad because the string topology operations don't agree for slide equivalent chains. I can take two slide equivalent cells, and stick in an extra cell sending it to the thing that's a boundary. Now I've got more boundary. We don't quite take the quotient. You actually get an extra term $A(g,k,\ell)$. It's bad but it's not so bad. You can fill in and extend the string topology operation over A in this case. $A(g,k,\ell)$ is zero for $(0,k,1),(0,1,k),(0,2,2)$, and $(0,1,1)$. In the latter cases you do it by hand. Maybe that's a good place to stop.