I'm going to work from two assumptions, that we like cohomology theories, and that these are better if we have some idea of what the cocycles are.

Let me take as an example complex K-theory. Take the monoid $Vect_{\mathbb{C}}(X)$, the equivalence classes of $\mathbb{C}$-vector bundle over X. Then $K^0(X)$ is the Grothiendieck completion here, but if you take a step back [unintelligible]cocycles.

One other thing I want to say, if you have a vector bundle E with connection, this gives a 1-dimensional TFT. If you have a point in X, this gives E_x, and a 1-manifold I gives a map $E_x \to E_y$.

What is a higher dimensional analog? This is a big problem that has been worked on for decades. People would like this where this had surface transport. People can make bundle objects that satisfy surface transport, but usually Meyer-Vietoris fails. This is a picture of what it would be nice to do with the results I'm talking about today.

We'll take $\mathbb{C}$ and replace it with R a connective ring spectrum. First, I'll describe the analog for the vector bundles. We'll replace E a bundle over X with $(E \downarrow X)$, an R -bundle over X. This will be a family of spectra, each one of which will be a module over R. I'll develop this precisely but fairly quickly. Before I go on, are there any questions? Is this cool? I'm assuming that people are comfortable with a spectrum. What would it mean for a spectrum to be parameterized by a space X? For us a parameterized space is a space Y with a map to X. An ex-space is a space Y over X with a section s which we think of as giving base points in each fiber. A parameterized spectrum E over X is ex-spaces $E(n) \to X$ with $\Sigma_X E(n) \to E(n+1)$. This is a fiberwise suspension. You could write this domain as $S^1 \wedge_X E(n)$. I want to live in a world with smash products. You should do this using diagram spectra. I won't spell this out precisely.

Let's note that over each point of x, this is a map from $\Sigma E_x(n) \to E_x(n+1)$, so E_x is an ordinary spectrum. To be an R-bundle, we want each of these to be further, a module over R, $R \wedge_X E \to E$. The intuition is that you smash each fiber with R. Over $x \in X$, this looks like $R \wedge E_x \to E_x$. As a complex vector bundle varies smoothly over the base space, this is a collection of modules over R, with maybe twisting where they're glued together. Lend me some slack that these are geometric objects. Let's see if we can get a cohomology theory, like K-theory.

Okay, one thing before we go on, an equivalence of vector bundles is a map of total spaces that is a fiberwise isomorphism. Similarly, $E \to E'$ over X is a weak equivalence if $\pi_* E_x \to pi_* E'_x$ is an isomorphism for each x. This is the notion of equivalence we'll use.

Now let's start on the other end. For K-theory we took the group completion of this monoid. At the beginning, I didn't say, for complex K-theory, $K^0(X)$ is $[X, \Omega^\infty ku]$ where $\Omega^\infty ku = \mathbb{Z} \times BU$, which is the group completion $gr(\amalg_{n \geq 0} BU(n))$ which is the group completion of the classifying space for complex vector spaces, which is $\Omega^\infty K_{alg}(\mathbb{C})$, as a topological ring.

We'll replace $\mathbb{C}$ with a ring spectrum R. So we'll look at the algebraic K-theory of R, $K(R)$. What is that? I'll say one way to describe it by $\Omega^\infty K(R) = K_0(R) \times BGL_\infty R^+$. $GL_\infty R$ is an A_∞-space, the colimit of $GL_n R$. This can be delooped, and you can make its plus construction in the sense of Quillen, which kills the commutator subgroup of π_1. Once you do that you can deloop it into a spectrum, and that's the algebraic K-theory spectrum. These should be classifying spaces for the R-bundles I have.

Let me state the main result connecting these threads.

Theorem 1. *As a spectrum, we get a cohomology theory.* $K(R)^0(X) \cong gr[\textit{virtual free } R\textit{-bundles over } X]$. *I haven't said what virtual and free mean. This is the group completion under* $\vee$. *Let me say what virtual and free mean to make this totally precise.*

An R-bundle $E \downarrow X$ is free if $E_x \cong R^{\vee n}$ for some n. Think of this as being like $E_x \cong \mathbb{C}^{\oplus n}$. One problem is that the plus construction does something to the homotopy type. The plus construction doesn't do anything to U because U is Abelian. We'll have to pass to a cover of X because of the plus. A virtual R bundle over X is a an R-bundle $E \downarrow \tilde{X}$ with a fibration $p : \tilde{X} \to X$ where $\tilde{H}(\text{fiber}(p)) = 0$.

I want to say something about what goes into proving it. There's a version of this theorem due to [unintelligible], Bass, and Dundes. In the special case where R is connective complex K-theory, they did this. You can see that they aren't the same at every rank, that ku bundles and 2-bundles are the same stably.

The main thing to do is understand the role of $BGL_\infty R$. WE'll fix a fiber type M (this is an R-module). There is an A_∞-space $End_R(M)$ which you can think of as $Hom_R(M, M)$. The A_∞ structure comes from composition. We'll look at the grouplike A_∞-space where we look at $Aut_R(M)$. These are the units of the ring spectrum of R-module maps from M to itself, $GL_1 F_R(M, M)$. This is something that comes with a delooping $B\ Aut_R(M)$. This is the classifying space for R-bundles with fiber M. So $[R\text{-bundles with fiber } M \text{ over } X] \cong [X, B\ Aut_R M]$.

Now, given an R bundle with fiber $R^{\vee n}$ over X, we get a map $X \to B\ Aut_R(R^\vee n) = BGL_n R \to BGL_\infty R^+$.

Conversely, given a map to the plus construction, realize and take the pullback

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d] & BGL_\infty R \arrow[d] \\ X \arrow[r] & BGL_\infty R^+ \end{tikzcd}$$

This classification result is in some sense the heart of the matter. In the 50s to 70s there was some work about what the most general kinds of bundle one could do classifications of. I'm following Peter May but this goes back to Dold and Hurewicz. If G is a topological monoid ($G = hAut(Y)$), then $[X, BG] \cong [\text{principal } G\text{-fibrations over } X]$. This is usually stated your fiber looks like Y and the automorphisms are the automorphisms of Y. Every A_∞ space can be realized as a topological monoid. It comes down to building the universal G-bundle. This is what it comes down to building. I want to think of $EG = B(*, G, G)$ and $BG = B(*, G, *)$. The bottom guy you have a q-fold copies of G and above you have one extra G in the q space.

Suppose you have a "derived version" $\boxtimes$ of the cartesian product $\times$ so that a monoid over $\boxtimes$ is an A_∞ space. If you look at where the Cartesian product is, if you replace it with $\boxtimes$, you'll get a universal G-bundle, where G is an A_∞ space. This will give a classification theorem for principal G-fibrations where G is an A_∞ space.

Now we're talking about translating normal algebra to A_∞ notions. You can do this but then you're not using topological spaces any more. You have to pass to Quillen equivalent categories where $\boxtimes$ exists. There's symmetric monoidal categories of spectra, each of which has a corresponding category of spaces.

Σ-spectra	orthogonal spectra	EKMM modules
$\mathbb{I}$-spaces	$\mathcal{I}$-spaces	$*$-modules.

There are equivalences among these categories that commute, and all of these forget to spaces and that's also a Quillen equivalence.

The bottom row comes with a symmetric monoidal product. Just as a monoid under $\wedge_S$ is a ring spectrum, a monoid under $\boxtimes$ is an A_∞ space.

I've pushed myself further and further into abstraction. $\mathbb{I}$ is the category of finite sets and injective functions. An $\mathbb{I}$-space is a functor from $\mathbb{I}$ to spaces. This is a contractible category because of the empty set. So this will give [unintelligible]. Given a symmetric spectrum, $\Omega^\infty Y$ is the functor that sends n to $\Omega^n Y_n$. You get the injections using the spectrum structure maps.

We wanted a model of spaces where we can use $\boxtimes$, and this works if we use this category of spaces.

Finally, let's connect this to R-bundles. If we think about principal $Aut_R M$ fibrations over X, out of these things we can build R-bundles over X with fiber M. The classification theorem on the left should give the classification theorem on the right. This has to happen within the world of $\mathcal{I}$-spaces or $\mathbb{I}$-spaces.

There is a fiberwise suspension functor (making the suspension functor fiberwise) which is a left $\Sigma^\infty_+ Aut_R M$-module with fiber $\Sigma^\infty_+ Aut_R M$. Then you can smash with M over this, which gives you a left R-bundle over X with fiber M.

To go back you have to restrict to things that are equivalences on the fiber. This should be maps from M to E that are equivalences on each fiber. This forces you to land in spaces. You're working stably.

Look at maps $M \to E$, fiberwise, restrict to equivalences, these will be principal $Aut_R M$ fibrations.

I want to say something about how this connects to other sorts of geometry. There's been a lot of work on categorified geometry. People on the n-lab, their sort of work. This is developed to answer this question about surface transport.

One notion of categorification, we could replace $\mathbb{C}$ with $(Vect_{\mathbb{C}}, \oplus, \otimes)$. This is a ring category. You can define $K(Vect_{\mathbb{C}})$ and there's a description of the cocycles in terms of 2-vector spaces. Baas-Dundes-Rogers give a description of 0-cocycles for this theory in terms of 2-bundles. This is a Steenrod transition description of a bundle. You describe transition functions, replace complex numbers with complex vector spaces, and then demand that the determinants not be zero in some sense. Then there is a result that $K(Vect_{\mathbb{C}}) \cong K(ku)$, so stably 2-vector bundles over X are the same as ku-bundles over X. This is meaningful only in the virtual sense. I'm looking for a geometric realization of this.

Maybe one last thing. This is work in progress and quite conjectural. Look at maps $F(X, K(R))$, and look at a trace (like in algebraic K-theory to THH). With Andrew Blumberg and [unintelligible]I hope to land in a relative version $THH^R(F(X,R)) \cong F(LX, R)$. What is the effect on π_0? This gives $K(R)^0(X) \to R^0(LX)$ where LX is $Map(S^1, X)$. The Norwegians talk about this a lot, but there has never been a construction that realizes surface transport without already having [unintelligible]built in.

You'd have $[(E \downarrow X)] \mapsto [V_* \to LX]$. Then a map of a surface into X should give a map $V_{\gamma_1} \otimes \cdots \otimes V_{\gamma_p} \to V_{\gamma_1} \otimes \cdots \otimes V_{\gamma_q}$.