TOPOLOGY IN AUSTRALIA AND SOUTH KOREA, UNIVERSITY OF MELBOURNE, 2017

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1. May 1: Yong-Geun Oh, Lagrangian Floer theory and mirror symmetry on toric manifolds I

Thanks to the organizers for this wonderful event. I hope that this kind of exchange between Australia and Korea continues beyond this year. I want to talk about, in this series of lectures, a brief introduction to homological mirror symmetry in toric manifolds. In the first lecture I'll overview Lagrangian Floer theory. In the second lecture I'll talk about Floer theory on toric manifolds. In the third lecture I'll talk about (homological) mirror symmetry between toric A-models and Landau–Ginzburg B-models.

Okay, so let me say that (M,ω) is a symplectic manifold. So M may not be compact in general. ω is a nondegenerate closed 2-form. There is a special kind of dynamics on symplectic manifolds, called Hamiltonian dynamics. The way this goes is as follows. Let h be a smooth function $M \to \mathbb{R}$. By nondegeneracy we can define a vector field

Definition 1.1. A vector field X is called *Hamiltonian* if it solves the following equation $X \lrcorner \omega = dh$, uniquely solvable by nondegeneracy.

If $H = H(t,x)$ is a time-dependent Hamiltonian, then we can write *Hamilton's equations*

$$\dot{x} = X_{H_t}(x)$$

On the phase space $\mathbb{R}^{2n}$ with symplectic form $\omega_0 = \sum dq_i \wedge dp_i$, the Hamilton equations can be written

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Then one can easily see that $d(X_h \lrcorner \omega) = 0$ by closed-ness of ω, and this is equivalent to $\mathcal{L}_{X_h}\omega = 0$ by Cartan's magic formula, which means that this vector field preserves the symplectic form. So X_h is an (infinitesimal) automorphism of ω.

Definition 1.2. The *symplectic automorphism group* (of *symplectomorphisms*) $\mathrm{Symp}(M,\omega)$ consists of diffeomorphisms ϕ of M so that $\phi^*(\omega) = \omega$. We can define a subset $\mathrm{Ham}(M,\omega)$ of *Hameomorphisms* which are symplectomorphisms ϕ equal to ϕ_H^1, the time one flow for some time dependent Hamiltonian $H(t,x)$, where ϕ_H^t is the flow of $\dot{x} = X_{H_t}(x)$.

When M is compact, ϕ_H^t is alwyas defined. When M is not, we impose some conditions at infinite on H, say, compact support or asymptotic control (linear, say, or quadratic). Then an interesting exercise is that $\mathrm{Ham}(M,\omega)$ is in fact a *subgroup* of $\mathrm{Symp}(M,\omega)$. This is an important object of study for sympletic topologists.

One important problem is to study or develop some intersection theory that is invariant under the action of this group $\mathrm{Ham}(M,\omega)$. It turns out that there is some nice structure on Hamilton's equations which allows us to study this problem in some effective way, having to do with some variational structures of these equations $\dot{x} = X_{H_t}(x)$.

What do I mean by this? Let me start with the classical case, $(\mathbb{R}^{2n},\omega_0)$ where ω_0, defined before, is $d(-\sum p_i dq_i)$. The same property holds, in fact, for any cotangent bundle, with $M = T^*N$. Here you have $\omega_0 = -d\theta$ where $\theta = \sum p_i dq_i$ in canonical coordinates. This is called the "Liouville 1-form."

Then we have a classical action functional on the path space. So this is $\mathcal{A}_H : \mathcal{P}([0,1];T^*N) \to \mathbb{R}$ defined as

$$\mathcal{A}_H(\gamma) = \int \gamma^*\theta - \int_0^1 H(t,\gamma(t))dt$$

where in canonical coordinates this is

$$\int_\gamma \sum p_i dq_i - \int_0^1 H(t,q(t),p(t))dt.$$

Under suitable boundary conditions, or instead by restricting $\mathcal{A}_H$ to a subset of $\mathcal{P}([0,1],T^*N)$, the equation's—the solutions to the boundary value problem

$$\begin{cases} \dot{x} = X_{H_t}(x) \\ x(0) \in ?? \\ x(1) \in ?? \end{cases}$$

are in bijection with critical points of the action functional evaluated somewhere.

To make this statement actually precise we need to specify the boundary conditions. There are two boundary conditions used commonly by physicists and mathematicians these days. The most natural boundary conditions are the following. Look at some relations $\Lambda \subset (M,\omega)\times(M,\omega)$. We call this relation, well, we want it to satisify a "Lagrangian" condition,

Definition 1.3. Let (M,ω) be symplectic. A submanifold $L \subset M$ is called Lagrangian if $\dim L = \frac{1}{2}\dim M$, and $i^*(\omega) = 0$. The natural boundary condition is a Lagrangian submanifold in the product. The Lagrangian boundary condition, is that $x(0)$ and $x(1)$ are in a Lagrangian submanifold Λ. For example,

(1) we can take periodic boundary conditions corresponding to Λ the diagonal.
(2) We can take a two-point boundary condition on T^*N, which says that $q(0)$ and $q(1)$, the projections, are fixed. From the point of view of the Lagrangian boundary conditions, this means that $q(0) = T^*_{q_0}N$ and $q(1) \in T^*_{q_1}N$ so that $\Lambda = T^*_{q_0}N \times T^*_{q_1}N \subset T^*N \times T^*N \cong T^*(N\times N)$.

We get this from a variational principle. The first variation of $\mathcal{A}_H$ $d\mathcal{A}_H(\gamma)(\xi)$ is

$$\int_0^1 \omega(\dot{\gamma}(t),\xi(t))dt - \int_0^1 dH_t(\gamma(t))(\xi(t))dt + \langle\theta(\gamma(1)),\xi(1)\rangle - \langle\theta(\gamma_0),\xi(0).$$

By definition $\theta(x)(\xi)$ is nothing but $p(d\pi(\xi))$. Here $x = qp$. [picture].

Then the boundary terms can be written as

$$\langle\alpha(1),d\pi(\xi(1))\rangle - \langle\alpha(0),d\pi(\xi(0))\rangle$$

for $\gamma = (q, p)$ and $\alpha = p \circ \gamma$.

In the periodic case, these both vanish. In the two-point case, the q is fixed, so the projection this way vanishes.

More generally, the 2-boundary conditions can be amplified to so-called conormal boundary conditions. The fiber here can be replaced by any conormal bundele N^*S for S a submanifold.

One of the fundamental thereoms is the so-called Arnold conjecture, which is that the zero section cannnot be removed from itself by any Hamiltonian diffeomorphism.

Fact 1.1. *(Arnold's conjecture, Hofer–Floer, [unintelligible]–Sikorav,...) For N compact, then $N \cap \phi(N) \neq \varnothing$ for any $\phi \in \mathrm{Ham}(M, \omega)$.*

If $\xi(N) = 0$, then we know there is such a diffeomorphism so this is a very different in the smooth world.

There is the so-called *symplectic creed* of my adviser Alan Weinstein. "Everything is a Lagrangian submanifold." There is more and more evidence for this. One plausible expectation is that a suitabel collection of Lagrangian submanifolds sould recover a large part of the sypmlectic topology of (M, ω).

How is this possible? The best way to motivate this involves using some physical language. The best way to see this is to "study some long-range interactions of Lagrangian branes." What has been known for a long time is that symplectic topology cannot be seen locally. There is the (closed) Darboux theorem and the (open) Darboux–Weinstein theorem. The former says that ω can be written pulled back, $\omega = dq_i \wedge dp_i$ for some chart. The latter says that for any L in $(M, \omega$ there exists a symplectic diffeomorphism Φ so that we can pull back the canonical symplectic form and get $\omega = \Phi^* \omega_0$.

Now we should try to exploit these variational structures in some way. Physics may again be a good way of motivating this kind of study. So we want to measure this long-range interaction between L_0 and L_1 by regarding that as some kind of morphism between L_0 and L_1 in some category.

This is by now called the *Fukaya category*. The important aspect of this is to understand the intersection properties of these two. If you have L_0 and L_1, we want to understand the intersection properties.

Let's first consider the case of transversal intersection $L_0 \pitchfork L_1$ with M compact and L_i compact Lagrangian submanifolds. We consider $CF(L_0, L_1)$ as the category of R-modules over L_0 and L_1 over a ring R, suitable in some way, that I will present later. These are modules with some natural basis. We want to construct a "complex," meaning we want to construct an R-linear map $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$ hopefully satisfying $\partial^2 = 0$. Very often this fails and I'll explain how to handle this in this case, but we'll construct ∂ by studying first order quasi-linear elliptic equations with Lagrangian boundary conditions:

$$
\begin{aligned}
\frac{\partial U}{\partial \tau} + J \frac{\partial U}{\partial t} &= 0 \\
U(\tau, 0) &\in L_0 \\
U(\tau, 1) &\in L_1 \\
U : \mathbb{R} \times [0,1] \to M; & \\
U(-\infty) = q U(\infty) & \qquad = p.
\end{aligned}
$$

. Similarly, we can study $u : \mathbb{R} \times S^1 \to M$, and study (perturbed) Cauchy– equations

$$\frac{\partial U}{\partial \tau} + J\left(\frac{\partial U}{\partial_t} - X_H(U)\right) = 0$$

but this domain is not compact, and so we need to control behavior at infinity by imposing some energy conditions. The relevant energy is given by

$$E_J(U) = \frac{1}{2} \iint \left|\frac{\partial U}{\partial \tau}\right|_J^2 + \left|\frac{\partial U}{\partial t} X_H(U)\right|_J^2 dt d\tau$$

So if we impose finite energy , $E_{(J,H)}(U) < \infty$, together with transversality conditions, then as τ goes to ∞ we go to a closed loop $z(t)$ which satisfies Hamilton's equation $\dot{z} = X_H(z)$. For the open string case this is a *Hamiltonian trajectory* connecting the first Lagrangian to the second one at time 1. Given a Hamiltonian, how do you know if you have a periodic orbit? This turns out to be an efficient way to prove the existence of such orbits, but all you have to do is to show the existence of a solution to this equation.

So when $H = 0$, say, the equation, the only solutions, are constant, say, on cotangent bundles. In general, it will be a problem of so-called Gromov–Witten theory on general (M, ω). Assume you understand Gromov–Witten theory very well, then this gives you a lot of information about periodic orbits.

Now I have to explain what this J is. So J is an almost complex structure $J : TM \to TM$, with $J^2 = -\mathrm{id}$, compatible with ω, so that $g_J := \omega(\bullet, J\bullet)$ is a positive definite symmetric bilinear form, i.e., defines a Riemannian metric. Maybe a better, more flexible structure is, this is

(1) it is positive, so that $\omega(v, Jv) \geq 0$ and equality holds if and only if $v = 0$,
(2) along with Hermitian properties $\omega(J\bullet, J\bullet) = \omega(\bullet, \bullet)$.

And a Fact due to Gromov is that the set $\mathcal{J}$ of such J forms a contractible infinite dimensional manifold.

Later we'll look at this in more detail, but for one minute let's go back to the open string,

$$\begin{aligned} \frac{\partial U}{\partial \tau} + J\left(\frac{\partial U}{\partial t} - X_H(U)\right) &= 0 \\ U(\tau, 0) &\in L_0 \\ U(\tau, 1) &\in L_1 \end{aligned}$$

then $\mathcal{M}(J, , H : Z_-, Z_+)$ is the set of finite energy solutions, which look like [picture].

The construction of this moduli space is fundamental. There are two issues, transversality of M and compactness of M, that one has to understand to construct such a space.

2. TriThang Tran: Quantum shuffle algebras and the homology of Hurwitz spaces

Thanks for the invitation to speak and for coming to hear me talk. Today I'm talking about quantum shuffle algebras and the homology of Hurwitz spaces. Can everyone see if I write this size? So this is joint work with Jordan Ellenberg and Craig Westerland. There are a few bits and pieces that go into this project so this talk will possibly be in bits and pieces as well. The first bit I want to talk about is what a quantum shuffle algebra is. The second part is what Hurwitz spaces are, and the third part I want to talk about is the relationship between the two. And lastly, if there's time, I'll put the motivation in the end, talk about the reason we cared about this in the first place, something called Malle's conjecture for function fields. I'm probably going to spend a lot of time in the first two.

2.1. **quantum shuffle algebras.** In this talk, $\mathbf{k}$ is going to be a field. To talk about quantum shuffle algebras, let me start with braided vector spaces

Definition 2.1. A *braided vector space* (V, σ) will be

(1) A vector space V (finite dimensional, for me), and
(2) A braiding $\sigma : V \otimes V \to V \otimes V$

such that

$$(\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma)$$

as a map $V \otimes V \otimes V \to V \otimes V \otimes V$, sometimes called the *braid relation* (probably why this is called a braided vector space)

Some examples, I could define $\sigma(x \otimes y) = y \otimes x$, this is a boring one. I could also choose $\sigma(x \otimes y) = -y \otimes x$.

So given a braided vector space I can get an action of the braid group on n strands, Br_n, on $V^{\otimes n}$. The braid group is

$$\langle \sigma_1, \dots, \sigma_{n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1 \rangle$$

and what's the action of σ_i on $[v_1 | \cdots | v_n]$? It does nothing for the first $i-1$ terms and then applies σ:

$$[v_1 | \cdots | v_{i-1} | \sigma(v_i, v_{i+1}) | v_{i+2} | \cdots | v_n]$$

and because σ satisfies the braid relation this gives me an action of the braid group.

I wanted to talk about quantum shuffle algebras, so given (V, σ) I can form something that I'll call a quantum shuffle algebra, which I do as follows. Define $A(V, \sigma) = A(V) = A$ as, well, as a vector space

$$A = \bigoplus_{n \geq 0} V^{\otimes n}$$

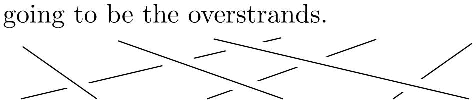
and so this is a vector space, this is the same underlying vector space as the tensor algebra where I concatenate words. I use a different product. We'll use the braid action in this product.

I'll say what it does on $V^{\otimes n} \otimes V^{\otimes m} \to V^{\otimes n+m}$, let me call this $\star$, and

$$[v_1 | \cdots | v_n] \star [w_1 | \cdots | w_m] = \sum_{\tau \in \mathrm{Sh}(n,m) \subset S_{n+m}} \tilde{\tau} [v_1 | \cdots | v_n | w_1 | \cdots | w_m]$$

this sum over (n,m)-shuffles, lifted to the braid group. Here $\mathrm{Sh}(n,m)$ is the set of (n,m)-shuffles, and I can't just take any lift to the braid gruop, and so I should say $\tilde{\tau}$ is the *Matsumoto lift* of S_{n+m} to Br_{n+m}.

So a shuffle is a permutation that keeps the relative ordering of the first n things the same and the last m things the same, but can interleave these two sets [picture]. So I get an element of the braid group by deciding that crossings from left to right are going to be the overstrands.



So Ross checked in 1998 that *star* is associative. This is a bit bashy, but I don't know. We can check this. So we end up with a product on A. Was it Vigleik that mentioned signs? If I take $\sigma(x \otimes y) = -y \otimes x$ then I get

$$[a|b] \star [c] = [a|b|c] - [a|c|b] + [c|a|b]$$

I do have to be a little bit careful. I shouldn't just take any diagram that represents S_{n+m}, I want the identity to go to the identity braid, and this is presentation dependent. I should write my shuffles in this simple form.

The (A, v) we will have will come from *racks.* A *rack* is a set R with a binary operation $\triangleleft$. It satisfies conditions

(1) $a \triangleleft (b \triangleleft c) = (a \triangleleft b) \triangleleft (a \triangleleft c)$ (this is the so-called "shelf"- distributivity condition)
(2) $a \triangleleft a = a$

I struggled thinking saying $\triangleleft$ out loud, we'll write $a \triangleleft b$ as b^a. This is going to make sense right now because I'll give you an example right now. The conditions are instead

(1) $(c^b)^a = (c^a)^{(b^a)}$
(2) $a^a = a$.

The prototypical example is $R = G$ a group, $a^b = b^{-1}ab$, and you can check these conditions easily:

$$(c^b)^a = a^{-1}(b^{-1}cb)a^{-1} = c^{ba} = c^{ab^a}$$

and the second condition is easy. Another example is, I can let R be a union of conjugacy classes of G, let $[a]^{[b]} = [b^{-1}][a][b]$, and given such a rack, let V be the braided vector space generated by R, and let the braiding σ be $V \otimes V \to V \otimes V$ is

$$[a|b] \mapsto [b|a^b].$$

I'll let you check that if you use condition (1) it'll tell you that σ satisfies the braid relation.

These are the quantum shuffle algebras that arise from this rack, relevant for Hurwitz spaces.

2.2. **Hurwitz spaces.** Now let me talk about Hurwitz spaces. Let G be a finite group and c a union of conjugacy classes. For example you could take S_n and the transpositions. So this is $\mathrm{Hur}^c_{G,n}$, and I'll input a positive integer n, and here $\mathrm{Hur}^c_{G,n}$ is a branched G-cover of the complex plane with n branched points. The c means that the monodromy around each branch point lies in c. I want this set up to isomorphism of branched covers. This is supposed to be a set whose points are branched covers of $\mathbb{C}$ (not necessarily connected at this stage). The claim is that this is a space. Maybe instead of giving you a good definition of the topology, let me tell you how I think about the topology. There's a map from $\mathrm{Hur}^c_{G,n} \to \mathrm{Conf}_n(\mathbb{C})$, the spaces of n points in the plane. What's the map? I take the branched cover to its branch locus. If it's an n-branched cover then there are n branch points. So the configuration space is, this, if, I can think of this as sitting inside $\mathbb{C}^n/S_n$, and this map $\mathrm{Hur}^c_{G,n} \to \mathrm{Conf}_n(\mathcal{C})$ is a covering space whose fibers are discrete, so a covering space.

So this space is pretty interesting. Let me give you a sample theorem for why people have cared about this space.

Theorem 2.1. *(Ellenberg–Venkatesh–Westerland '16)*
Let G be a finite group and c a conjugacy class of G such that

(1) $\langle c \rangle = G$*, that is, c generates G,*
(2) *G is* non-splitting*, which I don't want to write down what that is right now, a strong condition on G.*

Then, short-handing, there are constants A, B, and D, so that $H_p(\mathrm{Hur}^c_{G,n}; \mathbb{Q}) \cong H_p(\mathrm{Hur}^c_{G,n+D}, \mathbb{Q})$ for $n \geq Ap + B$.

This is sometimes called a homological stability result. This occurs in many other places in homology, such as in configuration spaces.

Not only did they prove that this space satisfies homological stability, but they also used this to prove a version of the Cohen–Lenstra heuristics for function fields. These are sort of, one of these statements that tell you about the number of groups whose, I'm going to screw up, so, I was going to remember this sentence but I blanked out. It's about counting, if I was given a number field and I was going to count—no, I've blanked out. You can ask Craig, he's right there. This was a number theory problem about number fields that was reworded to be about function fields, and this result helped them prove the function field version.

The point is that understanding the Hurwitz spaces has some cool number theoretic computations you might be interested in. I've told you about Hurwitz spaces and quantum shuffle algebras.

2.3. **Relation between Hurwitz spaces and quantum shuffle algebras.**

Theorem 2.2. *(Ellenberg–T.–Westerland) For any (G, c) and G finite, if I take the rack associated to c, letting $V = \mathbf{k}c$ and σ I gave before be a braided vector space, then the quantum shuffle algebra associated to this vector space, I have $A(V_\epsilon)$, take this quantum shuffle algebra, then $\mathrm{Ext}^{n-q,n}_{A(V_\epsilon)}(\mathbf{k}, \mathbf{k}) \cong H_q(\mathrm{Hur}^c_{G,n}, \mathbf{k})$, where $(V_\epsilon, \sigma_\epsilon) = (V, -\sigma)$.*

The idea is that here's a space whose homology I'm interested in, the Hurwitz space, and here's one way to compute its homology. This would be pretty useless if I couldn't compute these Ext groups, but I can, so it's not so useless.

[Can one describe the product on the right hand side?]

This lifts the product that comes from adding two configurations sort of "operadically," putting two configurations in their own little boxes.

Where does this come from? I can stratify $\mathrm{Hur}^c_{G,n}$ based on where the branch points are. Let's make n a number, say 5, so I can write something down. Let λ be an ordered partition of 5, say $(2, 1, 2)$. So for a partition I'll get a stratum of this space, where, here's the complex plane, and I'll look at branched covers, where I have two points that share one real coordinate, one on its own, and another two sharing another real coordinate. It has to be in this order. So each λ, if I think about this strata, the vertical lines can move about and this is what the cells look like, well, this is what I'd get for C_n, these are the [unintelligible]cells, but I'll get additional information, something telling me the monodromy around each of the branch points. I should really get a monodromy element in c for each of the points. So specifying that stuff I get a cell in $\mathrm{Hur}^c_{G,n}$. The highest dimensional cells correspond to $\lambda = (1, 1, \ldots, 1)$. If I'm thinking about the boundary maps, there are three ways that these can combine, and if I had more than one point, you'd get some sort of shuffling effect.

So roughly speaking, this cell structure which is not a cell structure on $\mathrm{Hur}_{G,n}$, which is really a cell structure on its one-point compactification, then you get something like the bar complex on the shuffle algebra, which is roughly why this is the same.

In the last two minutes I'll say a word about why this is useful.

2.4. **Malle's conjecture for function fields.** Let $X(n)$ be the number of finite field extensions $L/\mathbb{F}_q(t)$ with Galois group G and discriminant less than n. I want the number of such field extensions.

Conjecture 2.1. There exist constants $c_1(G)$, $c_2(G,\epsilon)$, and $a(G)$, such that
$$c_1(G)n^{a(G)} \le X(n) \le c_2(G(n^{a(G)+\epsilon})).$$

and so the second inequality is roughly what Ellenberg–T.–Westerland prove using this count of Hurwitz spaces.

3. Seonjeong Park: Cohomological rigidity of manifolds arisen from right-angled 3-dimensional polytopes

[I do not take notes during slide talks]

4. May 2: Mike Hopkins: Reflection positivity and invertible topological phases

[I do not take notes during slide talks]

5. Calin Lazaroiu: Differential models for open-closed Landau-Ginzburg theories

[I do not take notes during slide talks]

6. Daniel Murfet: A-infinity minimal models and matrix factorisations

I'm going to explain a map from the sort of algebraic subset of what Calin was talking about in the last lecture, from pairs, well, let me just say polynomial functions $W : \mathbb{C}^n \to \mathbb{C}$ with isolated singularities, to A_∞ algebras. This will go via a triangulated or dg category of matrix factorizations and this should really be a 2-functor, with bimodules and so on. The ultimate motivation was understanding higher categorical content related to some work I did with bicategories related to Landau–Ginzburg. So W will be my polynomial and $\mathcal{A}_W$ the algebra (a finite dimensional vector space with higher operations). So A_∞ algebras extract, often, a finite thing which is easier to understand than a big space that you start with.

So the property this will have is that the triangulated category of perfect modules over $\mathcal{A}_W$ equivalent to $hmf(W)$, the homotopy category of matrix factorizations of W, or more equivalently, the bounded derived category of coherent sheaves on the critical locus of W, but the tweak is that we mod out by perfect guys $\mathrm{Perf}(W^{-1}(0))$. Orlov called this the triangulated category of singularities.

Singularity is important here because if you're working nonsingularly this quotient is zero.

As I said I'll give a construction of $\mathcal{A}_W$ but what you want to know is the equivalence so that you can consider the bounded derived category via this algebraic structure.

So my motivation was to understand topological string theory better, Calin's work and others, Calin described some way to get a 2d open-closed tft, and there's a richer story with string theory. Just as a tft is a functor, these things are minimal cyclic (Calabi–Yau) strictly unital A_∞ categories. This connection appears first in Herbst–Lazaroiu, then Lerche, Costello, et cetera. In particular I wanted some examples of A_∞ algebras to run through the machine.

The second thing was I wanted to try and understand unfoldings in terms of A_∞ categories. I can ask, if I have a family of isolated singularities, can I get a sheaf of A_∞-algebras on the base space of the unfolding. This should be related to Frobenius manifolds and other interesting things.

I'll start by defining A_∞ algebras and then the category of matrix factorizations, and then I'll explain the assignment which will go by the minimal model theorem. This assignment will go through a certain dg algebra naturally associated to the category of matrix factorizations, and then if I have time I'll talk about modules.

Every talk you prepare there are two pages at the end that every time you give the talk you don't get to give.

Let me say what an A_∞ algebra is.

6.1. **A_∞-algebras.** I don't want to work over a field necessarily. My base commutative ring might be the coordinate ring related to the unfolding. So $\mathbf{k}$ is a commutative $\mathbb{Q}$-algebra.

Definition 6.1. An A_∞-algebra is a $\mathbb{Z}$ or $\mathbb{Z}_2$-graded finitely generated projective $\mathbf{k}$-module $A = \bigoplus_{n\in\mathbb{Z}} A_n$ with linear maps $m_n : A^{\otimes n} \to A$ of degree $2-n$, for n at least 1. So m_1 goes from A to A and is of degree 1, m_2 goes from $A^{\otimes 2}$ to A of degree 0, m_3 is degree -1 and gives a sort of homotopy.

The equation is that for all $n \geq 1$, we have

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0.$$

The $n = 1$ case will say that m_1 is a differential, the $n = 2$ will say that m_1 is a derivation with respect to m_2, so this looks like a dg algebra except it's not strictly associative, m_3 is a homotopy, and so on.

I won't talk about history (as this is a topology conference, you may know this better than me), I want to give some explicit small examples. If $m_n = 0$ for $n \geq 3$ then (A, m_1, m_2) is a dg algebra. It's interesting to talk about modules over an ordinary algebra even if you don't have higher structure. I should say secretly I want everything to be homologically unital.

I didn't say it, but if I have an A_∞-algebra, then taking the homology gives me an associative algebra and I want a unit for that algebra.

Definition 6.2. A is *minimal* if $m_1 = 0$.

Nature gives you ones with m_1, and you're interested in the invariant homology, the passage in between the thing you're given which is infinite dimensional and the other end where everything is finite dimensional, that's where the interesting mathematics comes.

Let me give an example of a minimal A_∞ algebra. I said earlier $\mathbb{Z}$ or $\mathbb{Z}_2$-graded, I'll in fact be $\mathbb{Z}_2$-graded.

For $d > 2$, let $|\epsilon| = 1$, and $A^{(d)} = \mathbf{k}[\epsilon]/\epsilon^2 = \mathbf{k} \oplus \mathbf{k}\epsilon$. So $m_n = 0$ unless $n \in \{2, d\}$. So m_2 is the usual product, and m_d is zero on the basis vectors except $m_d(\epsilon, \ldots, \epsilon) = (-1)^{d-1}1$.

This is an A_∞-algebra, and this will be the $\mathcal{A}_W$ for $W = x^d + y^2 + z^2$.

Let me say briefly what A_∞-modules are. So given an A_∞ algebra, an A_∞-module is a $\mathbb{Z}$ or $\mathbb{Z}_2$-graded finitely generated projective $\mathbf{k}$-module M with operations $m_n^M : A^{\otimes n-1} \otimes M \to M$ of degree $2-n$ for all $n \geq 1$, satisfying the same identities interpreted appropriately. The first operator, if $t \neq 0$, then the first operator acts on some tensor not including an M. If $t = 0$, then you use the operator involving M. So things are like a module but the two things that are normally equal are only homotopic.

So now $\mathrm{per}(A)$ is the triangulated subcategory of $H^0(\mathrm{Mod}_\infty A)$ generated by A. If you had a ring, this would give you finitely generated complexes of projective modules. You cook up a dg category, its homology is a triangulated category, and then you take the subcategory generated by that one object (with summands as well as direct sums, mapping cones, and suspensions). This is the category of *perfect modules*.

This is like the usual triangulated structure.

Where would I naturally run across an A_∞-algebra? There are a variety of ways, but I come to them via algebraic geometry rather than algebraic topology. The way they appear to me is as follows. I assert that we start life caring about certain triangulated categories, and the one we love best is the bounded derived category of coherent sheaves on a Noetherian scheme, or second to that the homotopy category of matrix factorizations, and joking aside, these encode ext and tors that we care about properly. So we care about triangulated categories. There are constructions that require us to go beyond triangulated categories. If you have a dg category whose homology is your given triangulated category, you call this an enhancement, and these are often more complicated. We want to enhance the category to understand it, and in many cases there's a generator, that's in the sense I described earlier, a particular coherent sheaf or complex, where I can get every guy from this guy by these operations. The existence of a generator is pretty common. If we have a generator, we get an algebra, the endomorphisms of the generator. The perfect dg modules over these endomorphisms, under hypotheses, should give us back the triangulated categories. This sounds like a good place to stop, except that these endomorphisms are infinite dimensional. What we want to get down to is something honestly finite dimensional, and if we take the cohomology of that complex and add higher operations, we get an A_∞ algebra. That's the minimal model theorem, that there is always a way of doing that so that this guy is quasi-isomorphic to the original dg algebra, so then its perfect modules, the upshot, call it A, the A_∞-algebra, it has the same triangulated category we started with as its perfect modules.

Under some hypotheses, so, we end up with a finite dimensional vector space and this encodes everything there is to know about the original triangulated category.

Definition 6.3. Say that $W \in \mathbf{k}[x_1, \ldots, x_n]$ is a *potential* if, for $f_i = \partial_{x_i} W$,

(1) For $\mathbf{k}$ Noetherian, $f_1, \ldots, f_n$ are quasi-regular, so that a maximal ideal containing all f_i's, localizing at them gives me something regular.
(2) the critical locus $k[\underline{x}]/(f_1, \ldots, f_n)$ is finitely generated projective
(3) the Koszul complex $K(f_1, \ldots, f_n)$ is exact outside degree 0.

Some examples.

(1) $\mathbf{k} = \mathbb{C}$ and the critical points of W are isolated.

(2) $\mathbf{k} = \mathbb{C}[t]$ and $W = x^2 + y^3 - 3t^2y + 2t^3$ is a potential. You can see that $\mathbf{k}[x,y,t]/(\partial_x W, \partial_y W) = \mathbb{C}[t] \oplus \mathbb{C}[t]y$.

Let me define the category of matrix factorizations. Everything I'll say for the rest of the talk will have W a potential.

Definition 6.4. the dg category $\mathrm{mf}(W)$ of *matrix factorizations* of W has objects (X, d_X) with X a finitely generated projective $\mathbf{k}[\underline{x}]$-module, $\mathbb{Z}_2$-graded, and d_X as an odd operator $d_X^2 = W 1_X$. You can think $d_X = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$ with $uv = vu = W \cdot 1_X$

The morphisms are $\mathrm{Hom}_{k[\underline{x}](X,Y)}$ with $d_{Hom}(\alpha) = d_Y \alpha - (-1)^{|\alpha|} \alpha d_X$.

Then $\mathrm{hmf}(W) = H^0(\mathrm{mf}(W))$.

Now I want to find a generator. From any, I put up an equivalence earlier which is maybe a good place to start,

$$\mathrm{hmf}(W) \cong \mathbb{D}^b(\mathrm{coh}(W^{-1}(0)))/\operatorname{Perf}(W^{-1}(0))$$

this is a theorem of Buchereitz (unpublished) and later Orlov. So in there on the right is $\mathbf{k}(P)$ for $P \in \operatorname{Sing} W^{-1}(0)$. This corresponds to something called $\mathbf{k}(P)^{\mathrm{stab}}$ on the left, and I'll tell you what it is.

Given a singular point P as $W = \sum_{i=1}^n (x_i - P_i) W_P^i$ with $W_P^i \in \mathfrak{m}_P^2$. There are some details that aren't quite right unless I work completely locally at each point in the critical locus.

I'll use the decomposition, which isn't unique, to write down a factorization, and then I get a matrix factorization

$$\mathbf{k}(P)^{\mathrm{stab}} \coloneqq \left(\mathbf{k}[\underline{x}] \otimes_{\mathbf{k}} \bigwedge (\mathbf{k}\varphi_1 \otimes \cdots \otimes \mathbf{k}\varphi_n), \sum_{i=1}^n (x_i - P_i)\varphi_i^* + \sum W_P^i \varphi_i \right)$$

So if I square the operator, I get $(x_i - P_i) W_P^i$ which is W.

Let me stick to the case of a single singular point at P for simplicity; otherwise I'd need to do more. Then the theorem is that $\mathbf{k}(P)^{\mathrm{stab}}$ generates $\mathrm{hmf}(W)$. There are proofs of this in Orlov, in Keller–Murfet–Van den Bergh, in Dyckerhoff, and so we're free to go on to the next step. Now we can try to take its endomorphisms and take the homology.

There's a big distinction between polynomials and power series that I can evade by replacing $\mathrm{hmf}(W)$ with the Karoubi completion $\mathrm{hmf}(W)^{\omega}$.

So now I want to give you a sketch of what the answer looks like, and the main content is what the homotopy retract looks like.

Definition 6.5. $\mathcal{A}_W$ is the minimal model of $\operatorname{End}(\mathbf{k}(P)^{\mathrm{stab}})$, meaning that $\mathcal{A}_W$ is an A_∞-algebra homotopy equivalent over $\mathbf{k}$ to the endomorphisms of $\mathbf{k}(P)_W^{\mathrm{stab}}$.

Of course this doesn't really pick out a representative and in fact I have a particular construction in mind.

I won't give you a closed formula for the higher products for every W, but what I'll tell you is the right homotopy retract to use in order to generate answers and then some of the answers.

So write $\iota : \mathbf{k}[\underline{x}] \to \mathbf{k}[\underline{x}]/(f_1, \ldots, f_n)$ with $f_i = \partial_{x_i} W$. I'll also need $S \coloneqq \bigwedge(\mathbf{k}\mathcal{O}_1 \oplus \cdots \oplus \mathbf{k}\mathcal{O}_n)$ with $|\mathcal{O}_i| = 1$. Let X be in $\mathrm{hmf}(W)$.

Theorem 6.1. *(Dyckerhoff–M. '09, M. '15) There is a strict homotopy retract of $\mathbb{Z}_2$-graded complexes over $\mathbf{k}$*

$$S \otimes_{\mathbf{k}} End(X) \leftrightarrows \iota^* \operatorname{End}(X)$$

where p maps to the right, i to the left, and H is the self-map. Becausef of [unintelligible], we have that the right hand side is a finitely generated projective module.

So I assume $pi = 1$ and $ip = \partial H + H\partial$; here H comes from choosing $\nabla : \mathbf{k}[\underline{x}] \to \mathbf{k}[\underline{x}] \otimes_{\mathbf{k}[\underline{f}]} \Omega^1_{\mathbf{k}[\underline{f}]/\mathbf{k}}$

The minimal model theorem, given such a data, this is a homotopy equivalence, and I have something finitely generated over $\mathbf{k}$ on the right, and I'll get higher products on the right and the resulting one on the right will be quasi-isomorphic to the original dg algebra. So I get something $\{m_n\}_{n\geq 1}$ on $\iota^*(\operatorname{End}(X))$.

So I want to start with $\mathbf{k}(P)^{\text{stab}}$, and get down, but I have at the top $S \otimes \operatorname{End}(X)$, and you can find within this $\mathbf{k} \cdot 1 \otimes \operatorname{End}(X)$ and then the corresponding piece downstairs will be an A_∞ model.

I'm out of time. Let me just draw, I'll tell you the output for this case. I should mention, in case I forget, Dyckerhoff started this in his thesis, there are many versions in mirror symmetry. Efimov does this for one particular W, and the underlying $\mathbb{Z}_2$-graded free module is some exterior algebra.

So taking $X = \mathbf{k}(P)^{\text{stab}}$ with $P = \underline{0}$, then $\mathcal{A}_W$ is $\bigwedge(\mathbf{k}\varphi_1 \oplus \cdots \oplus \mathbf{k}\varphi_n)$. Take an auxiliary space $\mathcal{H} = \mathcal{A}_W \otimes \bigwedge(\mathbf{k}\theta_1 \oplus \cdots \oplus \mathbf{k}\theta_n) \oplus \mathbf{k}[\underline{x}]$.

The interactions in the Feynman calculus, $W = \sum_i x_i W^i$, the operators, each W^i is $\sum_\gamma W^i(\gamma) x^\gamma$ where $W^i \in \mathbf{k}[\underline{x}]$ and γ_i are numbers. So you get for an operator on $\mathcal{H}$,

$$\frac{-1}{|\gamma|} W^i(\gamma)\theta_j \partial_{x_j}(x^\gamma)\varphi_i^*$$

(picture) and then for this picture you get

$$\theta_i \partial_{x_i}$$

and for this one

$$\varphi_i^* \otimes \theta_i^*$$

and then the numbers you get is by summing over all ways of drawing these vertices in your trees and using these $W^i(\gamma)$. For generic W the calculations are too involved to do by hand. I'll stop here.

7. Yong-Geun Oh, Lagrangian Floer theory and mirror symmetry on toric manifolds II

So I'd like to continue, the statement I made at the end of the last lecture; the basic statement was that the Fukaya category, which I'll denote $\operatorname{Fuk}(M, \omega)$, is a filtered curved A_∞-category. I want to try to explain what I mean by this in this lecture. Last time I said that the objects of this category are some Lagrangian submanifolds, but we need to restrict our discussion to a certain class of Lagrangian submanifold and then decorate them with some additional data. So the resulting objects will be (L, b), where b is called a *bounding cochain* and this is going to be a solution to a certain A_∞ Maurer–Cartan equation. The morphisms we'll define are going to be $\mathrm{CF}_\Lambda((L_0,b_0),(L_1,b_1))$ along with η_{k_1,k_2} and $\xi_{k_1,x_2\geq 0}$, some bimodule operations. The products

$$m_k : \mathrm{CF}_\Lambda((L_0,b_0),(L_1,b_1)) \otimes \cdots \otimes \mathrm{CF}_\Lambda((L_{k-1},b_{k-1}),(L_k,b_k)) \\ \to \mathrm{CF}_\Lambda((L_0,b_0),(L_k,b_k)).$$

So maybe the instructive thing is to look at disks with $k+1$ marked points [pictures] along with maps into M which take the marked points to intersection points of L_i and L_{i+1} and which take the boundary between two marked points into the appropriate Lagrangian. This map of the disk will satisfy the Cauchy–Riemann equation $\bar\partial_{J,j)}W = 0$ where j is a complex structure on the disk and J a compatible almost complex structure on (M,ω).

To describe how m_k is defined, I need to describe two pieces of topological data. Let $\mathcal{L}$ be the chain $(L_0, L_1, \ldots, L_p)$, the chain of Lagrangians, and $\vec{p} = (p_{01}, \ldots, p_{k,k+1})$ (where $k+1 \equiv 0$) the intersection points.

Definition 7.1. Let $\pi_2(\mathcal{L},\vec{p})$ be the set of homotopy classes of smooth maps with the given boundary conditions and asymptotic conditions at each marked point. I'll denote by B an element in $\pi_2(\mathcal{L},\vec{p})$.

There are two characteristic numbers associated to this picture. One is the symplectic area $[\omega] : \pi_2(\mathcal{L},\vec{p}) \to \mathbb{R}$ which is $[\omega](B) = \int_W \omega$ where $[W] = B$, and the Maslov index, some kind of winding number $\mu_{(\mathcal{L},\vec{p})} : \pi_2(\mathcal{L},\vec{p})$, the polygonal Maslov index, which I won't explain.

Then I define a moduli space

$$\widetilde{\mathcal{M}}_{k+1}(\mathcal{L},\vec{p} : B) = \\ \Big\{((W,j), z_0, \ldots, z_k) | \\ \bar\partial_J W = 0, \\ [W] = B, \\ W(\overline{z_{i-1}z_i} \subset L_{i-1}, \\ w(z_{i-1}) = p_{i-1,i}, \\ [\omega](B) = E_{(J,j)}(W) < \infty\Big\} \\ /PSL(2,\mathbb{R})$$

Then the "virtual dimension" of $\widetilde{\mathcal{M}}_{k+1}(\mathcal{L},\vec{p} : B) = n + \mu_{(\mathcal{L},\vec{p})}(B) + k - 2$.

We'll compactify $\widetilde{\mathcal{M}}_{k+1}$ by including all "bubble configurations," which gives the so-called *Gromov compactification*.

Then

$$m_k = \sum_{B\in\pi_2(\mathcal{L},\vec{b}):\mu_{\mathcal{L}}(B)=0} m_{k,B} T^{\omega(B)}.$$

This homotopy group is not unique, there's a countable subset. The area changes as the homotopy class changes. So this formal parameter T encodes the area. I'm not going to concern the grading, but if you look at all those Lagrangian submanifolds, oriented, and this has a natural $\mathbb{Z}_2$-grading. If every L is oriented, then this $\mathrm{CF}(L_{i-1}, L_i)$ has a degree 0 and a degree 1 part, but the trouble is that, I already talked about the transverse intersection case, but we cannot talk about identity or units. To incorporate the unit, we must consider the case when the Lagrangians coincide.

So when $L \pitchfork L'$ you look at Λ-modules over $L_0 \cap L_1$; otherwise you can look at $\Omega(L, \Lambda)$ when $L = L'$.

So the best way to control this discussion is to restrict to a finite collection of Lagrangian submanifolds. Let's take the special case, where $\mathcal{L}$ is a single Lagrangian submanifold L. Then I want to discuss the A_∞ operations for this case. So we'll define a *filtered* and *curved* A_∞-algebra structure on L.

Let's specialize all these m_k operations when this chain is just a single Lagrangian. Then you replace this with a differential form on L. So we can define

$$m_k : \Omega(L) \otimes \cdots \otimes \Omega(L) \to \Omega(L)$$

where m_k is again decomposed over homotopy classes $\beta \in \pi_2(M, L)$ of $m_{k,\beta} T^{\omega(\beta)}$ which is defined by considering the moduli space as a correspondence. There is a natural evaluation map $\mathrm{ev}_i : \overline{\mathcal{M}}_{k+1}(\mathcal{L}, \vec{p}, B) \to L$ by mapping $(w, \vec{z}) \mapsto w(z_i)$, which gives the following correspondence

$$\begin{tikzcd} & \overline{\mathcal{M}}_{k+1}(\mathcal{L}, \vec{p} : B) \arrow[dl, "\mathrm{ev}_\dagger = (\mathrm{ev}_1, \ldots, \mathrm{ev}_k)"'] \arrow[dr, "\mathrm{ev}_0"] & \\ L \times \cdots \times L & & L \end{tikzcd}$$

So how is this defined? So

$$m_{k,\beta}(\alpha_1, \ldots, \alpha_k) \coloneqq (\mathrm{ev}_0)_!(ev_1^* \alpha_1 \wedge \cdots \wedge \mathrm{ev}_k^* \alpha_k).$$

and to have the pushforward you need some discussion on virtual transversality matters.

In this talk I'm not going to get into this.

So what is m_0? This is parrticularly important in this enhancement of Lagrangian submanifolds, so what is ev_0, when $k = 0$, this is $m_0(1)$, so this is nothing but, this uses the evaluation of a disk with one marked point. You just think of the pushforward of the function 1, here, this is $(\mathrm{ev}_0)_!(1_{\overline{\mathcal{M}}})$, and this is nothing but the chain you get from the boundary values of the moduli space of holomorphic disks. So this chain $[\mathrm{ev}_0 : \mathcal{M}_1(\beta)] = m_{0,\beta}$.

Especially when the Maslov index of β is 2, then in this case the dimension $\dim \mathcal{M}_1(\beta) = n = \dim L$. So this kind of disk with Maslov index 2 will play an important role.

The A_∞ relations are then nothing but a consequence of the degeneration of, if you look at the configuration space of k points in D^2, or $k+1$ points, $z_0, \ldots, z_k$, then the compactification of this one is well-understood. Each boundary component is obtained by a degeneration into two irreducible components. If I try to compactify my moduli space of maps, then a similar decomposition holds. If this is the only kind of degeneration for the space of maps, then the A_∞ relations hold without $k = 0$. Unfortunately there are other components for the moduli space of maps. If you include that case, you have to consider the curved A_∞ relations, which always hold.

Our, one of the main interests, is that whether m_1 satisfies $m_1^2 = 0$. But that does not always hold. Let me write down the first two curved A_∞ relations, which go like this. The first one is $m_1(m_0(1)) = 0$. The second one geos $m_2(m_0(1), x) + (-1)^{|x|'}(m_2(x, m_0(1))) = m_1^2(x)$. So if $m_0(1) \neq 0$ then $m_1^2(x)$ is not generally zero. This will affect the bimodule operation.

So what do we do? Let's first look at examples, how this m_0 appears in practice.

So here is an example, with $M = \mathbb{C}$, and $L_0 = \mathbb{R}$ and $L_1 = S^1$. Then there are two intersection points $p = -1$ and $q = 1$. Then B_+ is the homotopy class of the upper and B_- the lower semidisk. So I want to compute, well, $\mathrm{CF}(L_0, L_1) = \Lambda\{p, q\}$. So $m_1(p)$, in this case, there is one homotopy class, we need to find, here $k = 1$, so there is this $\mathbb{R}$-translation, I have to look at $\mu_{\mathcal{L}}(p, q) = 1$, so that the moduli space is isolated. By the Riemann mapping theorem, this B_+ is the only such disk. So $m_1(p) = T^{\omega(B_+)}q$ and similarly $m_1(q) = T^{\omega(B_-)}p$. So if you compose these, youget $m_1^2(p) = T^{\omega(B_+)+\omega(B_+)}p = T^{2\pi}p$, and similarly for q. So here we see that $m_1^2 = T^{2\pi}\mathbf{1}$. This is a basic example of a matrix factorization. This configuration provides a natural thing. This is not always the case. To achieve this kind of story, you have to restrict to a certain class of Lagrangian submanifolds.

So to make our story more rich, we have to allow, to deform the original definition of the m_k operations. So we deform m_k and there are two ways, one is so-called "boundary deformations" and the other "bulk deformations." Here we consider more marked points, the original version has $k + 1$ marked points, but we insert many possible insertions. If we insert more such things on the boundary, those are boundary deformations, but you can do them on the interior and make "bulk deformations." So you look at a holomorphic disk, let me define this. For a given $b \in \Omega(L)$, we can deform m_k^b, which you can think of this as

$$m_k^b(x_1, \dots, x_k) = \sum m(b, \dots, b, x_1, b, \dots, b, x_2, \dots, x_k, b, \dots, b)$$

and then this doesn't make sense unless you make some sense of convergence, and this only converges in the T-adic topology. So Λ is the "universal Novikov ring"

$$\Lambda = \{\sum a_i T^{\lambda_i} : a_i \in R; \lambda_0 \le \lambda_1 \le \cdots \le\to \infty\}$$

and then we have

$$\Lambda_0 = \{\sum a_i T^{\lambda_i} : \lambda_i \ge 0\}$$

and

$$\Lambda_+ = \{\sum a_i T^{\lambda_i} : \lambda_i > 0\}$$

So the fact is that R is an algebraically closed field, then Λ is as well. This is far from Noetherian but enters symplectic topology in a natural way.

The proposition is

Proposition 7.1. *For any b of degree 1, then $\{m_k^b\}$ again defines a curved A_∞ algebra*

Definition 7.2. We say L is *unobstructed* if there is a b such that $m_0^b(1) = 0$.

If you go back to the definition of b, this equation is equivalent to $\sum m(b, \dots, b) = 0$, and for convergence of this, I need the T-adic valuation of b to be positive. This is the A_∞ Maurer–Cartan equation. It's useful for the Calabi–Yau case, but for the Fano case, you need

Definition 7.3. L is *weakly unobstructed* if there is a b such that $\sum m_k(b, \dots, b) = \lambda_b\mathbf{1}$. Then we say that such a Lagrangian submanifold is weakly unobstructed.

The solution is not unique, let me write $\mathcal{M}^{\mathrm{weak}}(L)$ as the set of gauge equivalence classes of solutions of the equation. Then by definition, for each element, a potential function $\mathcal{PO}_L$ goes from $\mathcal{M}^{\mathrm{weak}}(L) \to \Lambda_+$, saying $\mathcal{PO}_L(b) = \lambda_b$, and we call elements b in this solution space a *weak bounding cochain.* So we have L equipped with a potential $\mathcal{PO}_L$, and (L, b) is an object in the Fukaya category.

We can compute the potential functions or their restriction to some subspace of the Maurer–Cartan moduli space for the toric case. Given the moment map we can compute the potential function and in fact in this case we have a natural inclusion, an embedding $\Phi : H^1(L;\Lambda_+) \hookrightarrow \mathcal{M}^{\text{weak}}(L)$ and then physicists' Landau–Ginzburg potentials, after $\mathbb{C}$-reductions, are precisely $W_L := \mathcal{PO}_L \circ \Phi$.

8. May 3: Cheol-Hyun Cho: Globalizing local mirror functors

Thank you very much, I'd like to thank the organizers for the invitation. My talk today is about the connection between Yong-Geun and Daniel Murfet's talks yesterday. It's about homological mirror symmetry. This tries to relate what is called the Fukaya category of a symplectic manifold X with a matrix factorization category of a potential function W. There are several versions of this; there's a version where X is a symplectic space and the mirror is Landau–Ginzburg. Maybe what I want to do today is to give a geometric way to go from one side to the other side and so on.

Before moving to our approach, let me explain the general philosophy behind this kind of correspondence. This mirror picture was conjectured by Kontsevich; then the main question is why and how.

The most convincing, successful philosophy was the SYZ formalism, together with some viewpoints of Auroux. This approach says suppose you have a symplectic manifold X, maybe Kähler. D is its anticanonical divisor. We take the complement $X\backslash D$ and try to find a Lagrangian torus fibration structure for $X\backslash D$, so some base with fibers Lagrangian tori. The easies example is $\mathbb{CP}^1$, and then D is two points, and $\mathbb{CP}^1\backslash D$, this is a circle fibration. Then the mirror takes the dual torus fibration, with the same base but the dual torus, which you can think of as a kind of $U(1)$ holonomy along the tori, but let me skip that part.

You are kind of choosing a flat complex line bundle, and the choices of holonomy form another torus. So we get another circle fibration, drawn this way: [picture] and so the dual space we can think of as $Y = \mathbb{C}^*$, and now the fact that we took out the anticanonical divisor enters the picture. So we'll look at holomorphic disks which contract to the boundary. In our example, each fiber supports two holomorphic disks, and if you parameterize the base of the fibration by $[0,1]$ then the areas are approximately u and $1-u$ for the fiber u. So the potential function is something like $1\cdot T^u h + 1\cdot T^{1-u}\frac{1}{h}$ where h is the holonomy. We can write this as $W = z + \frac{T^1}{z}$.

So this approach, the difficult part is finding the Lagrangian torus fibration of $X\backslash D$. Then also it's difficult to go to the dual torus fibration. We want to send this Lagrangian to matrix factorizations or sheaves and that part is also not easy. The idea of constructing the potential from holomorphic disks, this appeared in the last lecture of Yong-Geun Oh. Together we classified holomorphic disks in toric manifolds [which gives a way to do this kind of thing in this case].

Let me explain a different setting to address "how?" This is Abouzaid following Fukaya. If you have the base B and a torus fibration [picture] without any singular fibers, then suppose we have another Lagrangian, we want to send this to some sheaf on the other side, on $T^\vee$, living on this space, and this is not a correspondence between the space and the Landau–Ginzburg model. Morally speaking the stalk at the point u and holonomy h is a Floer complex of (L_u, b) with L, with generators intersections and differentials which count holomorphic disks. This is difficult for many reasons, involving singularities and so on, but Abouzaid does this.

The approach I want to talk about today is more elementary than this. This is joint work with my former student Hong as well as with Lau. It's best explained by examples. The first example is when the symplectic manifold is $\mathbb{R}^2$ and we choose a Lagrangian, the circle. Then the potential $W_{\mathbb{L}}$ associated to this Lagrangian will come from counting disks. There is an obvious holomorphic disk bounded by the cicrle, so we get $1 \cdot T^{\text{area}}$, and I'd like this to be a function, so I'll put a complex line bundle over $\mathbb{L}$ with flat connection, and holonomy $z \in \mathbb{C}^*$. I'll choose a specific flat connection that does nothing around the circle except at one point where it changes by z (in one direction). Then when we write the potential, we want to write $W_{\mathbb{L}} = 1 \cdot T^{\text{area}} z$.

Nothing fancy so far. I claim that we can send curves, we can think of this z as living in $H^1(L, \mathbb{C}^*)$. Then I'd like to construct some map from $\text{Fuk}(\mathbb{R}^2) \to MF(z = W_{\mathbb{L}})$.

In $\mathbb{R}^2$ we can consider some curve passing through our Lagrangian $\mathbb{L}$, and then we can look at the intersection points and the holomorphic strips between them. Let me call these two points [picture] p and q and we'll have one map from $\langle p \rangle$ to $\langle q \rangle$, which has decoration z because the point with the holonomy is in that section. So this is

$$\langle p \rangle \xrightarrow{z} \langle q \rangle \xrightarrow{1} \langle p \rangle.$$

which is a factorization of z.

An interesting perturbation is this [picture]. So you can think of this example as another matrix factorization of the polynomial z. The approach is to fix a Lagrangian $\mathbb{L}$ and then study the Maurer–Cartan elements of $\mathbb{L}$, and what we obtain is a potential function related to this data, $W_{\mathbb{L}}(b)$, and we automatically get an A_∞-functor from $\text{Fuk}(X) \to \text{MF}(W_{\mathbb{L}}^b)$. This is why we call this a "localized" mirror functor, it uses $\mathbb{L}$. If you avoid $\mathbb{L}$ it says nothing.

So this approach, maybe I can explain a little bit more. This approach is algebraic, so you can somehow start as follows. Let $\mathcal{A}$ be any A_∞-category, an A_∞-algebra with several objects, and then choose an object $\mathbb{L}$ and choose a distinguished set of generators $X_1, \ldots, X_n$ in $\text{Hom}^{\text{odd}}(\mathbb{L}, \mathbb{L})$. I look at a linear combintation $b = x_1 X_1 + \cdots + x_n X_n$

Then we solve the Maurer–Cartan equation (possbly the weak one) $m_0(1) + m_1(b) + m_2(b,b) + \cdots = 0$ or $W_{\mathbb{L}}(b) \cdot \text{mf}\, 1$.

This is a restriction on X. The m_k operations cancel out except for the multiple of the unit. This Maurer–Cartan equation is somehow a unit relation.

So what do I want to say? Let Y be the solution space of the Maurer–Cartan equation; then $W_{\mathbb{L}}$ is a natural function from Y to Λ (or maybe $\mathbb{C}$ by reduction as in yesterday's talk) which sends b to $W_{\mathbb{L}}(b)$. This all depends on $\mathbb{L}$. For example, Y also depends on $\mathbb{L}$. Then we get a functor $\mathbb{F}^{\mathbb{L}}$ from $\mathcal{A}$ to $MF(W_{\mathbb{L}})$ which at the object level takes A to $(\text{Hom}((\mathbb{L}, b), A), m_1^{b,0})$. Let me say what $m_1^{b,0}$ is. [picture]

On the level of morphisms, if you have an element α from A_1 to A_2, then we need a map from $\text{Hom}((\mathbb{L}, b), A_1)$ to $\text{Hom}((\mathbb{L}, b), A_2)$ and so you can just push forward with α.

There are higher maps as well. The curved Yoneda embedding is a map to matrix factorizations. This A_∞ category, the target has no higher operations, but there are higher operations.

Algebraically this is quite simple, just the curved Yoneda embedding. Let me now move to the case of punctured Riemann surfaces. It's a similar example but with a different flavor. So let me look at the three-punctured sphere. This is our symplectic manifold with natural symplectic form. Here one considers what is called the *wrapped* Fukaya category. I will not explain the details but what goes on is that objects are non-compact Lagrangians, and the Hom spaces are infinite dimensional and come from twisting along the puncture, which provides many intersection points. We want to find a similar functor to the matrix factorization category. The Lagrangian we choose is [picture]. This is an *immersed* Lagrangian. We have chosen the object. What is the preferred generator? [pictures]. My Maurer–Cartan element is $xX + yY + zZ$ and I want to look at my equation, choosing a generic point and counting. [picture]. The potential we get is $W_{\mathbb{L}} = 1 \cdot xyz$. If I multiply these two angles, I get contributions here. The Maurer–Cartan equation gives you cancellation like this [pictures].

Now how does the functor work? In a very similar way. [pictures]. It turns out that in this case, the Fukaya category of the three-punctured sphere mapping to matrix factorizations of xyz is an equivalence. Mirror symmetry on this space is the work of Auroux Abouzaid Etingof Katzarkoff Orlov and also Bocklandt (?) and also Heather Lee. Our approach gives a direct way to compare these categories.

Now let's move on to the case with several punctures. For this we do a pair of pants decomposition fro this Riemann surface and then choose the same skeleton-like thing you did on each pair of pants. You get a functor from the wrapped Fukaya category to, you get $W_{\mathbb{L}_1}$ and $W_{\mathbb{L}_2}$ and so on. The answer you get from this picture is something toric Calabi–Yau. You get one chart for each pair of pants glued to make something toric Calabi–Yau. The W extends to the whole thing and so we get a potential $W : Y \to \mathbb{C}$.

Let me explain a bit why toric Calabi–Yaus appear. The strategy is the following. Let's consider the four-punctured sphere. [many pictures].

9. Clarisson Rizzie Canlubo: Non-commutative coverings of classical spaces

Thank you, this is joint work with Ryszard Nest. This is fancy name for finitely generated projective Hopf–Galois extensions, by which I mean extensions by algebroids, and the classical spaces are $C(X)$ for some topological space X. The details here, this is locally compact Hausdorff space, an affine scheme, a manifold, and we should interpret C appropriately as the correct kinds of functions (bounded, smooth, et cetera).

I won't define a Hopf algebroid. I'll follow one of the inequivalent definitions, one of Böhm. These generalize Hopf algebras and groupoids. They can be thought of as Hopf algebras over a base algebra R over $\mathbf{k}$, not necessarily commutative. They are also the analogue of groupoids in non-commutative geometry, just as Hopf algebras are the analogues of groups. You might think these are Hopf algebra objects in some category, but they are not. But the category of R-modules if R is non-commutative, the coproducts and products don't match nicely in bimodules because of non-commutativity.

If you have a groupoid over X, this gives you a group G for every point x in X. In non-commutative geometry, you'd expect that if you have a Hopf algebroid over $C(X)$ that you'd get a Hopf algebra over $\mathbf{k}$; this isn't quite the case; you get *coupled* Hopf algebras. Let me talk to you about that.

This is (H_1, H_2, C); let me say that $H_1 = (H, m, 1, \Delta_1, \epsilon_1, S_1)$ and H_2 has the same underlying algebra but maybe the other parts are different: that is, we have $H_2 = (H, m, 1, \Delta_2, \epsilon_2, S_2)$ and a $\mathbf{k}$-linear bijection $C : H \to H$, the *coupling map*. This satisfies the following coupling conditions

(1)

$$\begin{tikzcd} & H \otimes H \arrow[r, "c\otimes\mathrm{id}"] & H \otimes H \arrow[dr, "m"] & \\ H \arrow[ur, "\Delta_1"] \arrow[rr, "\epsilon_2"] & & \mathbf{k} \arrow[r, "1"] & H \end{tikzcd}$$

$$\begin{tikzcd} H \arrow[rr, "\epsilon_2"] \arrow[dr, "\Delta_2"] & & \mathbf{k} \arrow[r, "1"] & H \\ & H \otimes H \arrow[r, "\mathrm{id}\otimes c"] & H \otimes H \arrow[ur, "m"] & \end{tikzcd}$$

(2)

$$\begin{tikzcd} H \arrow[r, "\Delta_1"] \arrow[d, "\Delta_2"] & H \otimes H \arrow[d, "\Delta_2\otimes\mathrm{id}"] \\ H \otimes H \arrow[r, "\mathrm{id}\otimes\Delta_1"'] & H \otimes H \otimes H \end{tikzcd}$$

and the similar one on the other side.

Examples are

(1) given a Hopf algebra H, take (H, H, S).
(2) given H and $\sigma : H \to \mathbf{k}$, take $H_1 = H$ but for H_2 let Δ_2 take h to $h_{(1)} \otimes \sigma(s(h_{(2)})h_3$ and $\epsilon_2 = \sigma$ and S_2 send $h \mapsto \sigma(h_{(1)})s(h_{(2)})\sigma(h_{(3)})$. I claim these are coupled with the map $h \mapsto \sigma(h_{(1)})s(h_{(2)})$.

An interesting question is when H_1 and H_2 coincide.

(1) If Δ_1 is counital with respect to ϵ_2 then they coincide by the Eckmann–Hilton argument for coalgebras.
(2) If $\mathcal{H}_1$ is the group algebra of a group or its dual then these coincide.

An interesting fact is that any two of H_1, H_2, and C determine the third.

In the work I'm talking about I studied Hopf–Galois extensions by Hopf algebroids, which I can't explain because I'm not talking about what Hopf algebroids are. If you're given $A \subset B$ extensions of $\mathbf{k}$-algebras and H a $\mathbf{k}$-Hopf algebra, and B an H-comodule algebra, i.e., $\rho : B \to B \otimes H$. We say B/A is *H-Galois* if

(1) $B^{\mathrm{co}\,\rho} = A$
(2) $B \otimes_A B \to B \otimes H$ via $a \otimes b \mapsto (a \otimes 1)\rho(b)$

[note: I must have missed something?]

If you're given a coupled Hopf algebra, being Galois with respect to H_1 is equivalent to being Galois with respect to H_2.

Part of $\mathcal{H}$ being a Hopf algebroid over something, over $C(X)$, there is a $\mathbf{k}$-algebra map from $C(X) \to \mathcal{H}$. Even if $C(X)$ is commutative the image need not be central in $\mathcal{H}$. So the first case is if $C(X)$ is central, let me tell you what happens in that case. In this case, $\mathcal{H} = \Gamma(X, E)$ where $E \to X$ is a finite rank vector bundle over X. This is given by Serre–Swan but says more. Not only is $\mathcal{H}$ the global sections of a finite vector bundle, but th fibers are coupled Hopf algebras, and the operations are all pointwise.

The more interesting case, when the image of $C(X)$ is non-central. I mean *possibly* non-central so this includes the other case. What I did in this case only works when $\mathbf{k} = \mathbb{C}$ and X is compact Hausdorff. Before I give the definition, let me talk about (topological) Hopf categories.

Let me define these categories first. I don't know who introduced them but I first read about them in a paper by Caenepeel (sp?)–Joost–Batista.

Given X a set, a *Hopf category* $\mathbb{H}$ is a complex vector-space enriched (small) category with object set X, together with a functors $\Delta : \mathbb{H} \to \mathbb{H} \otimes_X \mathbb{H}$. This codomain has objects X and the hom sets are pointwise tensor products of hom sets:

$$(\mathbb{H} \otimes_X \mathbb{H})_{(x,y)} = \mathbb{H}_{(x,y)} \otimes \mathbb{H}_{(x,y)}.$$

and $\epsilon : \mathbb{H} \to \mathbb{I}^{\times}$, where $\mathbb{I}^{\times}_{(x,y)} = \mathbb{C}$ for all $x, y \in X$ and $S : \mathbb{H} \to \mathbb{H}^{\mathrm{op}}$ such that Δ is coassociative (you can relax this up to a natural isomorphism):

$$\begin{tikzcd}
\mathbb{H} \arrow[r, "\Delta"] \arrow[d, "\Delta"] & \mathbb{H} \otimes_X \mathbb{H} \arrow[d, "\Delta\otimes\mathrm{id}"] \\
\mathbb{H} \otimes_X \mathbb{H} \arrow[r, "\mathrm{id}\otimes\Delta"'] & \mathbb{H} \otimes_X \mathbb{H} \otimes_X \otimes\mathbb{H}
\end{tikzcd}$$

and unit

$$\begin{tikzcd}
\mathbb{H} \arrow[r, equal] \arrow[d, equal] \arrow[dr] & \mathbb{H} \otimes_X \mathbb{I}^{\times} \\
\mathbb{I}^{\times} \otimes_X \mathbb{H} & \mathbb{H} \otimes_X \mathbb{H} \arrow[l, "\epsilon\otimes\mathrm{id}"] \arrow[u, "\mathrm{id}\otimes\epsilon"']
\end{tikzcd}$$

and antipode conditions as well.

Let me give an example; for $X = \{1, \ldots, n\}$, let $\mathbb{H}_{(x,y)} = \mathbb{C}e_{xy}$. This is trivially a Hopf category but I want to say something more interesting. Given such a Hopf category, I can take the direct sum of Hom sets, $B = \bigoplus \mathbb{H}_{(x,y)}$. This is a Hopf algebroid. Let me define

$$e_{pq} \cdot e_{st} = \begin{cases} e_{pt} & q = s \\ 0 & q \neq s \end{cases}$$

which tells you $B \cong M_n(\mathbb{C})$. Then B is a Hopf algebroid over the diagonal matrices, that is, $\mathbb{C}^n$, that is, $C(X)$, the functions on this discrete set. This is a nice example because the general situation is a generalization of this example.

For X a topological space and $\mathcal{O}_X$ the sheaf of continous $\mathbb{C}$-valued functions on X, a Hopf category $\mathbb{H}$ over X is *topological* if there is a sheaf of $\mathcal{O}_X$-bimodules over $X \times X$ such that

(1) for any f and g in $\mathcal{O}_X(\quad)$, $\sigma \in \mathfrak{H}(U)$, $U(X \times X)$,
$$(f \cdot \sigma \cdot g)(x, y) = f(x)\sigma(x, y)g(y)$$
for (x, y) in U.

(2) $\mathbb{H}_{x,y}$ is the *fiber* of $\mathfrak{H}$ is (x, y) in $X \times X$.

(3) the product, unit, coproduct, counit, and antipode of $\mathbb{H}$ should be induced from from

$$\mathfrak{H} \otimes_{\mathcal{O}_X} \mathfrak{H} \xrightarrow{\circ'} \mathfrak{H}$$
$$\mathcal{O}_X \xrightarrow{\eta'} \mathfrak{H}$$
$$\mathfrak{H} \xrightarrow{\Delta'} \mathfrak{H} \otimes \mathcal{O}_{X\times X}\mathfrak{H}$$
$$\mathfrak{H} \xrightarrow{\epsilon'} \mathcal{O}_X$$
$$\mathfrak{H} \xrightarrow{S} \mathfrak{H}^{\mathrm{op}}$$
$$X \times X \xrightarrow{\text{flip}} X \times X$$

Let me go back to the case where $\mathcal{H}$ is a Hopf algebroid over $C(X)$ and the image is non-central. We have a left and right action of $C(X)$ on $\mathcal{H}$, each is finitely generated projective. Then $\mathcal{H} = \Gamma(X, E)$ for some bundle E from Serre-Swan. These two actions commute to $C(X)$ acts on the fibers, but not simply by multiplication in each fiber. This means that $C(X) \to \mathrm{End}(E)$ (via the right action), but since $C(X)$ is commutative this factors thorugh some maximal Abelian subalgebra D_n of $\mathrm{End}(E)$.

Sparing you the details of the computation, [unintelligible], trivializing vector bundles, choosing bases for the fibers, there are several arguments, sparing you all the details let me tell you what you will get. So $\mathcal{D}$ is a sheaf of complex vector spaces over $X \times X$ supported on a closed subset $\mathcal{Z}$ in $X \times X$. Let me tell you roughly how we get $\mathcal{E}$.

Using a theorem in complex analysis, since things in $C(X)$ are simultaneously diagonalizable. For every x you have a vector space decomposed into several subspace, joint eigenspaces for $C(X)$ viewed as a collection of operators. If I denote by E_x the fiber of the vector bundle over x, and then $E_{(x,y)}$ is the eigensubspace of E_x corresponding to some eigenvalue. E_x is finite dimensional so there are only finitely many of these.

This is symmetric, because I can run the argument, I picked out the left action and then used the right action on the bundle I get. The same thing would work in the other direction. $\mathcal{Z}$ is also a union of diagonal subsets of $X \times X$. I mean by a diagonal subset a subset so that π_1, π_2 both map T to X.

So [unintelligible]I can find $\mathbb{H}_{x,y} = E_{(x,y)}$ and I claim that this is a topological Hopf category. With this I have the following.

Theorem 9.1. *Hopf algebroids over $C(X)$ are in bijection with topological Hopf categories over X.*

This is a bijection, far from being functorial. I have no way of proving that this is functorial. More than this, though, the other half is

Theorem 9.2. *Galois extensions of Hopf algebroids are in bijection with Galois extensions of Hopf categories over X.*

10. Zsuzsanna Dancso: Categorified and quantized lattices of integer cuts and flows / Does homological algebra have untappend potential in lattice theory?

Once in a while I'm tempted to talk about the things that are on my mind instead of the things I can give a nice polished talk about, I'm a slow learner; this usually turns out poorly but here goes. This is joint non-results with Tony Licata. I'll start with utter nonsense. Then I'll talk about an example I sort of understand, and hopefully I'll have time to give you a list of mysteries.

10.1. **utter nonsense.** For the rest of this talk, let me declare what I mean by a lattice, which is a finitely generated free Abelian group with a symmetric non-degenerate bilinear form (this is not nonsense, this is the truth). Say you take some algebra A, with some properties, Artinian, maybe Koszul for some of this to work, let me not be too precise, and if you take the category of finitely generated modules over A, then you can consider the Grothendieck group of this category, $K_0(\text{A- mod})$, the free Abelian group generated by equivalence classes of modules module $[Y]=[X]+[Z]$ for a short exact sequence $0\to X\to Y\to Z\to 0$.

On this group there is a pairing called the Euler pairing, which is just

$$\langle [X],[Y]\rangle = \sum(-1)^i \dim \operatorname{Ext}^i_A(X,Y)$$

and "often" this makes K_0 into a lattice. I won't go into often, this is not even well-defined, everything should have finite projective resolutions, this is not always symmetric, so this is often a lattice. You might wonder what notions and statements in lattice theory are secretly governed by this homological algebra behind the scenes. Here are some questions, not very well-formed.

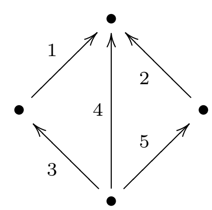
- Given a lattice can you find an algebra to give that as the Grothendieck group?
- Can we use this construction to learn more about lattices
- What notions in homological algebra correspond to interesting notions in lattice theory?
- Often there's more structure on the homological algebra level. Algebras are often graded in a natural way. Then you can try to change your algebras, give them, say, a q-deformation. What is the extra structure on the lattice side?

All we have is one construction for a concrete example. I think that's a very nice example, which has a lot of interesting features. We wonder if this construction is a concrete example for a much more general theory of categorified or homological lattices.

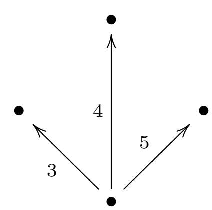
10.2. **Lattices of integer flows and cuts associated to a graph.** Before we find corresponding homological algebra I need to tell you about combinatorics and graph theory. Let G be a finite graph, possibly with multiple edges and loops. By $E(G)$ I'll denote the edge set, number these. We choose an orientation, meaning we choose a direction for every edge, making the graph into a 1-dimensional CW complex. If you have a 1-dimensional CW complex, then you get the first homology, which sits inside the free Abelian group generated by the edges. This is a free Abelian group. To make this a lattice we need an inner product. We build that by giving this thing a Euclidean structure, making the edges an orthonormal basis. You then have an an inner product on edges. You can restrict that to H_1 and you get what we call the lattice of integer flows. This is the first homology of the graph with the inner product induced by making the edges orthonormal. You take cycles in the graph, taking each edge with sign according to whether the cycle agrees or disagrees with the orientation on the edges.

The orthogonal complement of the flows is called integer cuts; they have a combinatorial description but I'll cut that.

One more bit of combinatorics I need to tell you is a nice way to choose bases for these lattices. If you have a graph



and I choose a spanning tree

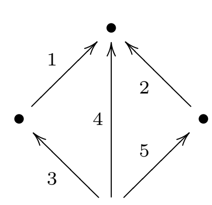


then any edge outside of the spanning tree gives me a "fundamental cycle" made up of that edge and edges from the spanning tree. So in this graph $e_1 \rightsquigarrow e_1 - e_4 + e_3$ and $e_2 \rightsquigarrow e_2 - e_4 + e_5$. For flows you get a basis from $e_i \notin T$ for T the spanning tree; for cuts you get a basis from $e_i \in T$.

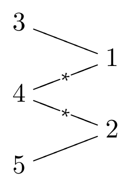
So you have the span of fundamental cycles $\mathcal{F}(G)$, you have the fundamental cuts $\mathcal{C}(G)$, and they both sit inside $\mathbb{Z}^{E(G)}$, and to this picture we'll construct an algebra $A_{G,T}$ so that inside $K_0(A\text{-}\mod)$ we'll have the span of some indecomposable projectives and the span of some complementary simples. Then inside you get the Euler pairing, and then the inner products of these two pieces match up with the fundamental cycles and the fundamental cuts. This is an example of lattice gluing (the determinant is one) and the fundamental cycles and cuts are orthogonal complements in this unimodular lattice. So the example is some kind of "homological lattice gluing" (which is a thing that doesn't exist yet but one day Tony and I hope to find it).

Next I'd like to show you the construction, which is very simple.

Okay so what is the construction? Let me draw the graph one more time?



To this graph G we'll associate a bipartite graph, which encodes all the combinatorial information necessary to remember these two lattices, this is $B_{\Gamma,T}$ so depends on the spanning tree as well. So the vertices of the bipartite graph correspond to edges of the original graph, with tree edges on the left. You connect non-tree edges to the edges in the tree in their fundamental cycle.



where the $*$ keeps track of the sign of e_4 in each of the fundamental cycles. Then $A_{G,T}$ is the path algebra of $B_{G,T}$, where paths concatenate to zero if their ends don't match up. Moreover, starting on the left and ending on the left should give zero. This algebra is graded by path length. This is also $\mathbb{Z}/2$-graded by signs. I promised modules. So $A_{G,T}$- mod is finitely generated graded modules. Before passing to the Grothendieck group let me say

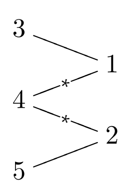
Proposition 10.1. *Both the simple modules and the indecomposable projectives are in one to one correspondence with the edges of G.*

The simple L_i for edge i is the field $\mathbb{C}$ with the action as follows. The idempotent e_i (the constant path) acts a the identity and everything else acts as zero.

The identity of the algebra is the sum of the idempotents, which are pairwise orthogonal. You can use that to split the algebra itself into a direct sum. $A_{G,T} \cong \bigoplus A_{G,T}e_i$, the paths that end at i. and those are indecomposable projectives. Those are all the simples, all the indecomposable projectives, and one more thing it's useful to state is that the homomorphisms between two indecomposable projectives are the paths from i to j. How does that act? By composition with that path. That's a fairly simple exercise.

So here comes the only interesting thing I'm going to say in this talk. This, if you look at it, counts the number of common edges between the fundamental cycle of edge i and edge j with signs, if edges i and j are not in the tree. This is exactly $\langle C_i, C_j\rangle$ in the lattice of flows.

Let me explain that a little bit.



Look at P_1 and P_2. A path from P_1 to P_2 uniquely corresponds to a point on the left, which is in the fundamental cycle of each of these. You count this with stars which tells you about the contribution of that edge to the fundamental cycle.

You can phrase this a bit more algebraically and it gives you our initial statement. The P_i and P_j are graded, and $\mathrm{Hom}(P_i, P_j)$ is a graded vector space by $\mathbb{Z}$ and $\mathbb{Z}/2$. If you take the graded dimension, this is a polynomial in $\mathbb{Z}[q, q^{-1}, t]/t^2 = 1$. The q is for $\mathbb{Z}$ and t for $\mathbb{Z}/2$. If you take the graded dimension $\operatorname{qdim}\mathrm{Hom}(P_i, P_j)|_{q=1,t=-1}$

you get $\langle C_i, C_j \rangle$ in $\mathcal{F}(G)$, and this is a graded version of the Euler form,

$$\sum(-1)^i \operatorname{qdim} \operatorname{Ext}^i(P_i, P_j).$$

These modules are projective so you can just write Hom, you don't need Ext. So that gives us the main statement with a little bit more precision.

I suppose a talk should involve a theorem so I'll call the correspondence I described a theorem. If you take indecomposable projectives for edges not in the tree, and then take the graded Euler form on K_0 of A-mod with this evaluation at $q = 1$ and $t = -1$ you get exactly the lattice I described.

There's some extra aesthetic that I don't have too much time to talk about. It's true that A is a Koszul algebra, and there is a notion of duality, and the duality has to do with swapping the bipartition, which, if this was a planar graph, corresponds to planar duality; in the non-planar case it corresponds to matroid duality.

10.3. **Two minutes of mysteries.** What if we don't evaluate at $q = 1$, then you get that K_0 is a "q-lattice" which is a free $\mathbb{Z}[q, q^{-1}]$-module, a non-degenerate semi-linear form.

There's a classical theorem (from 2010) saying that [unintelligible]is a complete invariant of graphs, which says that q-flow lattices are isomorphic then you get an isomorphism of graph and spanning tree.

The second question is what this has to do with topology. The lattices have a lot to do with topology. There is a theorem of Josh Green saying that the classical flow lattice is a complete isomorphism invariant of graphs. The Tate graph construction, this lets you draw an alternating knot diagram. If you take 2-isomorphism classes of graphs you get mutation classes in knot theory. Proving this combinatorial theorem about graphs let him prove that Heegaard Floer was a complete mutation invariant of alternating knots which is a strong thing in knot theory.

The third question, what happens when you change the spanning tree. The bipartite graph and the spanning tree change dramatically. If you pass from considering modules to complexes, that doesn't really help, but if you consider matrix factorizations, then there is a partial canonicality statement that you can prove. It's not true that the category of matrix factorizations are equivalent.

There ought also to be a connection to zig-zag algebras, that would be nice to explore, and another very interesting subject, the last mystery, is the graphical Riemann–Roch theorem which is a graphical analogue of the Riemann–Roch theorem and it has a statement in terms of cut and flow lattices. I think we can formulate a categorical version of that statement. There's a lot to do and very little that's done and that's all that I wanted to say.

11. Mike Hopkins: Analogies between higher categories and chromatic homotopy theory

So this is going to be kind of a weird talk, but I want to use that as a little bit of a way to motivate and explain something about chromatic homotopy theory.

Let me remind you of something that came up in the first talk. We were looking for this hierarchy of categories C_n. So C_0 was $\mathbb{C}$ and C_1 is vector spaces, and C_n sholud be $C_{n+1}(1,1)$. There was this hypothesis or ansatz that the invertible objects in C_n had something to do with $I\mathbb{Z}(1)$, that $\operatorname{Pic} C_n$ is like $S^{n+1}I\mathbb{Z}(1)$. This $I\mathbb{Z}(1)$ has something to do with homotopy groups of spheres and is a formal object.

The analogy I want to talk about is somehow along these lines and is unreasonably effective. So a space X gives rise to an ∞-category where the objects are the points of X, the 1-morphisms are paths, the 2-morphisms are paths between paths, et cetera. A k-morphism is some sort of map of a k-dimensional complex to X. The spaces are the categories where all the paths are reversible, the morphisms invertible. These things are trying to be spaces but some paths you can only go along in one direction and you can't go back, like that game chutes and ladders.

If you have a category, if $\mathcal{C}$ is a category, say an ∞, n-category, or I'll just say a category, there are two ways you could make a space. I'll make up notation. You could have ${}^{\kappa}\mathcal{C} \subset \mathcal{C}$, the largest subcategory where all morphisms are invertible. That's the closest space that sits inside $\mathcal{C}$. There's also something in the other direction C^+ where you invert all the morphism that accepts a functor from $\mathcal{C}$. For example if $\mathcal{C}$ is $\mathrm{Vect}_{\mathbb{C}}$, then ${}^{\kappa}\mathcal{C}$ is $\amalg_{n\geq 0} BGL_n\mathbb{C}$ and C^+ is just the trivial category because the map to the 0 vector space becomes invertible.

But these weren't arbitrary categories, they were symmetric monoidal. So what does a symmetric monoidal structure correspond to in terms of spaces, one where all morphisms are invertible. That corresponds to a space with an infinitely homotopy commutative monoid structure. When I was in the back of the room the computer was no problem to see over but can you see this far down on the board? Are you typing this right now?

There are really two symmetric monoidal structures right here, if you took Whitney sum instead of tensor product, this would correspond to $\amalg BGL_n\mathbb{C}$ as an infinitely commutative monoid with block matrices. This is not commutative, going from $BGL_n\mathbb{C} \times BGL_m\mathbb{C}$ to $BGL_{n+m}\mathbb{C}$. These are not commutative, but they're homotopic, the two inclusions, and eventually infinitely homotopic.

Such a space is trying to be an infinite loop space. In fact it has enough structure to have a classifying space, and if you take loops on that classifying space, it is an infinity loop space. You could form ΩBX, some people call that X^+, and that group completes X under the monoidal structure here, it's analagous to the other $+$, it's completing the sum, not the multiplication. If $X = \amalg BGL_n\mathbb{C}$, then X^+ is $\mathbb{Z} \times BGL\mathbb{C} = \Omega^\infty K$, the classifying space for K-theory.

Every one of these steps loses information and there are choices about how I made these steps that are a mystery. We want to have these categories C_n and do something to get a spectrum, but there's still some mystery. I can take the invertible morphisms or invert all morphisms. I could have taken invertible objects under sum or invert all objects under sum. I chose opposite decisions. On the morphisms I chose inverting and for $\oplus$ I chose invertibles and I got K-theory. But why do these ones?

So anyway, K-theory is the first stage of chromatic homotopy theory and there's a remarkable connection to higher homotopy theory and that's what I want to talk about to prepare for my next talk. Let me press on.

Note that $C_0 = \mathbb{C}$ has a sum and a product. That's more structure. Then C_1, some kind of super vector spaces, has a direct sum and a tensor product, and I'm also supposed to think of this as being some kind of ring. One hopes that all the C_ns are some kind of ring. On plus, it's the right thing to do is to invert everything, and for times, you want to restrict to invertible things or just live it alone. There's a temptation to invert the algebra because of the relationship with field theory, it's not a good thing to do, I'm sorry, I can't talk about that, I forgot what happens that's bad in Vect.

Anyway the thing I want to talk about is that the space model we get here is an E_∞ ring spectrum, you have an infinitely coherent addition, an infinitely coherent multiplication, and it has the property that homotopy classes of maps into E is a cohomology theory. K-theory is the prototype of all of this.

So I want to get to one thing that we can already say is kind of interesting. I talked a lot in the last lecture about reality checks. The one reality check we want is the Picard group of C_n, the invertible objects under tensor product, the groupoid of objects invertible under $\otimes$, this is supposed to turn out to be, so, so, forgive me, I don't know why, I'm hesitant to write this, it's supposed to be the zeroth space of the spctrum $\Omega^\infty(S^{n+1}I\mathbb{Z}(1))$, this thing, it doesn't really matter what this space is, even a professional homotopy theorist will have a hard time knowing when he or she has run into this. There's a nice criterion for when you run into this space. I'll say an easy theorem in topology and then reinterpret it in this language.

Theorem 11.1. *The following are equivalent:*

(1) *E (a spectrum) is equivalent to $I\mathbb{Z}(1)$ (the S^{n+1} is the analogue of shift in triangulated categories)*
(2) *E is coconnected: $\pi_n E = 0$ for $n \gg 0$, and $\mathrm{Map}(H\mathbb{Z}, E) = H\mathbb{Z}$, where $H\mathbb{Z}$ is the Eilenberg–MacLane spectrum.*

This gives you something that you can state without using homotopy theory. The Picard space, this has an infinitely homotopy coherent multiplication. It's got a group law, not strictly commutative, but infinitely homotopy commutative. The Eilenberg–MacLane spectrum is actually commutative. You might think of the mapping space as picking out the actual center of E. But there's already a name for things like that, those are called *strict Picard categories.*

If $\mathcal{C}$ is a symmetric monoidal ∞, n-category, using $\otimes$ to denote the symmetric monoidal structure. There's probably an $\oplus$ although I don't need to state anything. $\mathrm{Pic}\,\mathcal{C}$ will be the $X \in \mathcal{C}$ such that there exists Y with $X \otimes Y \sim \mathbf{1}$. There are also the strictly invertible elements, which are

$$\mathrm{Strict}\,\mathrm{Pic}\,\mathcal{C} = \mathrm{Map}(H\mathbb{Z}, \mathrm{Pic}\,\mathcal{C}).$$

You can have an A_∞-algebra, and you can replace your chain complex with something quasi-isomorphic with differential zero. You can also take the ∞ out of the A_∞ if you'll make the chain complex bigger. These are both descriptions of A_∞-algebras. We have the notion of just an algebra, taking the infinity out of both sides. So Pic has infinitely homotopy commutative product and in the strict case it's actually commutative.

For an example, take sVect, the category of $\mathbb{Z}/2$-graded vector spaces. What is $\mathrm{Pic}\,\mathcal{C}$? If an object is invertible, it has to have dimension 1, so it has two objects, which are $\mathbb{C}$ in dimension 0 and in dimension 1. But the $\mathcal{C}_1$ is not in the strictly commutative part, because the flip map $\mathbb{C}_1 \otimes \mathbb{C}_1 \to \mathbb{C}_1 \otimes \mathbb{C}_1$ is not the identity, this has a sign. So because of this, in the strict Picard category you only have one one dimensional vector space.

So the way that you can recognize that you have this crazy Anderson dual of the sphere is that the strict Picard category looks like an Eilenberg MacLane space. There's some theorem, I don't want to quite call it a theorem, so what do you do when you don't want to call it a theorem? You call it a theorem and don't write the paper.

Theorem 11.2. *saying*
$$\operatorname{Pic}\mathcal{C} \sim \Omega^\infty(S^{n+1}I\mathbb{Z}(1))$$
is the same as saying
$$\operatorname{Strict}\operatorname{Pic}\mathcal{C} = K(\mathbb{Z}, n+1).$$

I haven't gotten to anything having to do with chromatic homotopy theory but we'll get there.

So we're looking for categories where the strict units have this particularly nice form. I'm trying to think about how to do this next segue, I don't know a great way to motivate it, the category C_1 was supposed to correspond to K-theory. Then C_n is supposed to correspond to some other multiplicative cohomology theory, some other E_∞ ring spectrum. In chromatic homotopy theory there is a candidate for this, which is Morava E_n-theory, some higher version of K-theory. The nature of this arrow is nonexistent. There are many calculations you can do, this has been a philosophy going back to the mid-1980s, but I want to sort of evolve this analogy for you and state a theorem that I think is very surprising about Morava E-theory that makes this relationship look unreasonably precise. But we'll get to that in 20 minutes.

So what's up with this Morava E-theory? We're kind of hoping to see an Eilenberg–MacLane space in the strict units. There's supposed to be a map from $K(\mathbb{Z}, 2)$ to K-theory, and so this is $BU(1) = \mathbb{CP}^\infty$ and this map to K theory is the tautological line bundle. Then the fact that pulling back L along multiplication on $\mathbb{CP}^\infty$, the map classifying tensor product of line bundles, this bundle pulls back (of course) to $L \otimes L$.

I want to replace 2 with $n+1$. You can just set out to look for it and that turns out not to work very well.

I don't know a good way to motivate this other than saying that asking for the Eilenberg–MacLane space to sit in the strict units isn't enough structure.

What are *all* maps from $\mathbb{CP}^\infty$ to K? This is $\mathbb{Z}[[x]]$ where $x = 1 - L$ for the tautological line bundle L. What about maps
$$[\mathbb{CP}^\infty \times \mathbb{CP}^\infty, K]?$$
that's $\mathbb{Z}[[x, y]]$ where $x = 1 - L_1$ and $y = 1 - L_2$ and x goes to $x + y - xy$ pulling back from L to $L_1 \otimes L_2$. This is a special case of a formal group law, a formal power series, $F(x, y) = x + y + \cdots + a_{ij}x^i y^j + \cdots$

If you write $x +_F y = F(x, y)$, then the things this satisfies is that it's unital (with zero), associative, and commutative. Some examples are $F(x, y) = x + y$ and $F(x, y) = x + y - xy$.

So are these isomorphic? If you expect the answer is no, then you might want an invariant. There's a good way to discuss formal group laws, I want to pick this angle we're at to look at this from. Pick a prime p and I want to take the algebraic closure of the p-adic number and complete it, and this is $\mathbb{C}_p$, and D_p is the set of all x with p-adic valuation greater than 0. You can think of D_p, something like the unit disk, and a formal group law is the same thing as a Lie group structure on D_p with 0 as the unit. By Lie group, I mean an analytic function for the product, these will have to be in some extension of the p-adic numbers. This lets you think of this as a physical group sitting there. If we have one of these Fs then we can look at the kernel, at the elements of order p. When $F(x,y) = x+y$, the elements of order p are $\{0\}$. When $F(x,y)1-(1-x)(1-y)$, then that's cyclic of order p, it consists of 1 minus pth roots of unity. I can see that these aren't isomorphic then.

For F a formal group you can look at the rank of the kernel of multiplication by p, which gives you "height," this is the n in Morava E-theory.

How can you tell what the height of a formal group is? The standard notation is $|p|_F(x) = \overbrace{x +_F \cdots +_F x}_{p}$ which is some formal power series $|p|_F(x) = px + \cdots + a_i x^i + \cdots$ We want the zeros of that analytic function. Above the number i, you make a dot for the p-adic valuation of a_i. It started with px, and then you look at the convex hull of this set of points. Let me do an example. Let's not look at that series, let's look at $(x-p^\alpha)^n$. That has n roots all with valuation α. If I do this badly I'll get

$$x^n + p^\alpha x^{n-1} + p^{2\alpha}x^{n-2} + \cdots + p^{n\alpha}$$

and I can tell that no one is paying attention any more because this is wrong. But it's almost right, I just need to multiply by something, and all that's going to do is raise the p-adic valuation. So what I find is that the valuation of all the coefficients lies above the line between $(0,n\alpha)$ and $(n,0)$. So this line which has slope $\frac{-1}{\alpha}$ lies in the convex hull and everything else is above.

So what can happen if you have a formal group of height n? How can I get a formal group of height n? Let's just do height 2. The analytic function multiplication by p, this starts with px, the height is 2, so there is a total of p^2 points killed by p. One of them is zero with infinite p-adic valuation and there are p^2-1 other ones. It has to end at p^2, this polygon. The Newton polygon might look like this [picture] but there might be another point killed by p with lower valuation, in which case so does every multiple of it. The break would have to happen at p. If one has the valuation then $p-1$ of them will. The general thing, you can only change slope at powers of p. This is supposed to motivate that there are $n-1$ parameters that can move around and specify this Lie group structure on the disk.

That's supposed to motivate, it's not a proof, but it's supposed to motivate the existence of a universal height n formal group law, which lives over (something like)

$$\mathbb{Z}_p[[u_1, \dots u_{n-1}]]$$

where the $n-1$ parameters are telling me the p-adic valuations of the things in the subgroup killed by p. This is versal but not really universal because it has automorphisms.

There's a theorem due to Haynes Miller and myself and Paul Goerss in various pieces

Theorem 11.3. *There is a (more or less unique) functor from universal formal groups to E_∞ ring spectra. The one that sends the universal height n group that goes to E_n, the nth Morava E-theory.*

Sorry, that was a little bit brisk. I'm going to wrap this up. Chromatic homotopy theory is studying homotopy theory via these Morava E-theories. I didn't connect this to C_n. I was looking for something the cohomology of a higher Eilenberg–MacLane space than $\mathbb{CP}^\infty$. But I said that we should stay with $\mathbb{CP}^\infty$ and work with the other parameter that we can modify, and it turns out that that keeps track of height, the height of the formal group law.

I'm building up this theorem too much but I think this is a really striking result that I'll end with. This Morava E-theory is supposed to be analagous to C_n. Inside C_n are invertible objects, and for a cohomology theory, the notation is a little bit off, the invertible objects are $\operatorname{Pic} C_n$, and I could have called this $C_n^\times$. The group of units of E_n, there's different notation, $gl_1 E_n$ is a spectrum of units, so $[X, glE_1] = E^0(X)^\times$ [sic?].

So going back to C_1, my supervector spaces, E_1 was K-theory. The invertible elements $\operatorname{Pic} C_1$ is two lines, it's $\mathbb{Z}/2$-graded line bundles. But the units in K-theory consists of all virtual vector bundles of virtual dimension ± 1. So that's like 1 plus, there's a lot of those. If I was asking for, there's a lot of those. That's a much bigger space, there's a lot more to the units of K-theory. However, there's a surprising theorem, or there's a theorem that I find unreasonable but it's nevertheless a theorem

Theorem 11.4. *(Hopkins–Lurie) The strict units (that is* $\operatorname{Map}(H\mathbb{Z}, gl_1 E)$*) is* $K(\mathbb{Z}, n+1)$.

This is unreasonable because it lets you pick out a finite part of the units, picking out the $\mathbb{Z}/2$-graded line bundles from the K-theory units. You've done a lot, we just kind of guessed that this nth Morava E-theory corresponds to C_n, but it seems to have the one property we want, about its strict units.

This was a quick introductions to the animals we'll see in the next lecture. The next lecture will have more mathematics and less mathematical analogy. I'll try to describe the next series of results in a less technical way.

12. May 4: Diarmuid Crowley: Diffeomorphisms of discs: recent progress and open problems

Thank you very much to the organizers for organizing this very lovely conference. This is joint with Schick and Steinle. Numbers of you have heard about some of this and some have heard about none of it. I hope this won't be precisely the same as things you've heard.

Let me introduce the star of the show, D^k is the k-disk and f is a diffeomorphism. There's some exciting work, you can think about symplectomorphisms or contactomorphisms as well, and I'll require my diffeomorphisms to be the identity in a neighborhood of the boundary. So this is of course a very important object in differential topology and a number of subjects. We've effectively got a version of the little disks here, this is a topological group, with the C^∞ topology. This is not commutative but it's homotopy commutative, I'll say a little more later on. You could say it's a boring manifold but let's observe that we have an extension, the great thing about fixing on the boundary is that for any k-manifold I get an extension: For M a (closed) smooth manifold, I grab a disk wherever I like, and do my diffeomorphism there. You can map out of $\operatorname{Diff} M$ to a number of places, this acts on the metrics, it maps into the cobordism spaces that Mike was talking about, and one of the points of the talk is that we'll detect things by a map into real K-theory, and anything that you know with the α-invariant, we'll be able to pick that up anywhere in the middle.

Let me tell you how the talk is going to go. I'll give some selected history and then the α invariant and our main theorem. Then I want to talk about the proof, and all of this constitutes the progress. If time permits, there's some things we don't know about that I'll at least mention. If you have such a diffeomorphism you can take its derivative and get an element of the general linear group. When I differentiate such a map I get an obvious linearization map, I get a map to $GL(\mathbb{R})$ but that's homotopy equivalent, I get a map to $\Omega^k SO(k)$.

Here's a table of what we know and who to attribute it to.

k	$\mathrm{Diff}(D^k)$	Mathematicians
1	$\cong *$	folk theorem
2	$\cong *$	Smale
3	$\cong *$	Hatcher (hard!)

Let me show you an exotic thing. Look at $\pi_3 SO_3 \times \pi_3 SO_3$ and I'll construct a map due to Milnor which will end up in $\mathrm{Diff}_c(\mathbb{R}^6) \cong \mathrm{Diff}(D^6)$. So what's the idea? I think of α as a homotopy class of a map $(D^3, S^2) \to (SO(3), \mathrm{id})$. Given that I should give a diffeomorphism of $\mathbb{R}^6$ which I'll think of as $\mathbb{R}^3 \times \mathbb{R}^3$. So I get $F_\alpha(x, y) = (x, \alpha(x)y)$ which works in the strip $D^3 \times \mathbb{R}^3$ and extend by the identity. This isn't compactly supported. I can do the same sort of thing in the opposite direction, with F_β. Milnor's clever idea is to take the commutator $\sigma \coloneqq F_\alpha F_\beta F_\alpha^{-1} F_\beta$. We're churning up this piece of the six-disk in an interesting way.

So that's a concrete example of a wacky automorphism of the 6-disk that you can actually write down.

A little remark is that F_α is orthogonal with respect to the standard metric. Then with some clever rescaling you can see a copy of $D^1 \times D^5$, and it's preserving the D^5 slices [picture] and in fact σ lifts to an element of $\pi_1 \mathrm{Diff}(D^5)$.

Let me summarize, when I take α and β not divisible by 28, the Milnor pairing $\sigma : \pi_3 SO(3) \times \pi_3 SO(3) \to \pi_0 \mathrm{Diff}^6 \cong \mathbb{Z}_{28}$ factors through $\pi_1(\mathrm{Diff}\, D^5)$. The observation that it can be factorized is Antonelli–Burghelea–Kahn.

Now I want to talk about assembly, in the first case taking a map starting from $\pi_1 \mathrm{Diff}(D^k)$ and mapping it to $\pi_0 \mathrm{Diff}\, D^{k+1}$. I have some family of diffeomorphisms, and I just do this in sheets. I then get a diffeomorphism of the $k+1$ disk.

That's a simple idea but you can see there's nothing special about the 1-disk. In general I get a map from $\pi_i \mathrm{Diff}(D^k)$ to $\pi_0 \mathrm{Diff}\, D^{k+i}$ and I could factor through anything in the middle.

I can dub this group, anyway, Γ^{k+i+1}

Theorem 12.1. *(Smale–Cerf) There's an assembly map*

$$\pi_0 \mathrm{Diff}(D^k) \xrightarrow{E} \pi_0 \mathrm{Diff}(S^k) \to \Theta_{k+1},$$

the group of homotopy spheres. I take $[f]$ to $[E(f)] \mapsto [D^{k+1} \cup_{E(f)} D^{k+1}]$. The theorem is that this is an isomorphism for $k \geq 5$

Theorem 12.2. *(Kervaire–Milnor)* $0 \to bP_{k+2} \to \Theta_{k+1} \to \mathrm{coker}(J_{k+1})$

here bP_{k+2} is finite cyclic and the cokernel is finite; Θ_{k+1} is a finite Abelian group.

Now Novikov lets us, if we have a strange diffeomorphism of the disk, you try to pull it back along the assembly maps. HE found that many $\mathrm{Diff}(D^k)$ are not aspherical. We said this is a homotopy commutative H-space. If it were finite, then this would be a torus and thus aspherical. Once it's not aspherical, then, it is not homotopy equivalent to a finite CW complex.

Back to the table:

k	$\mathrm{Diff}(D^k)$	Mathematicians
1	$\cong *$	folk theorem
2	$\cong *$	Smale
3	$\cong *$	Hatcher (hard!)
4	?	
5	not aspherical	(our results: C–S–S) (3-primary)
6	not aspherical	(our results: C–S–S) (via α invariant, today.)
7	not aspherical	A–B–K (70s)

So I just want to say a few more things, advertise others' work

Theorem 12.3. *(Kupers)* $\mathrm{Diff}(D^k)$ *is of finite type for* $k \geq 5$ *(homotopy groups are finitely generated).*

Using ideas that Michael Weiss put into making his weird Pontryagin class, Sanders could exploit that to prove this theorem.

Theorem 12.4. *(Casels–Keating–Smith 2017)* $\pi_0 \mathrm{Symp}(D^4k, \omega_{\mathrm{ot}}) \to \pi_0 \mathrm{Diff}(D^{4k})$ *and this is nonzero and hits the Kervaire sphere if the Kervaire sphere is exotic.*

They improve this with assembly and give some homotopy groups for symplectomorphisms which I won't state.

Now I want to return to the work with Schick and Steinle and for that I need to tell you a little bit about the α invariant, which is defined in the first case for spin manifolds.

If I take a homotopy sphere in dimension $n+1$, this is in particular a spin manifold and there's an important map to K-theory which maps a spin manifold to the index of the Dirac operator. The homotopy spheres are a finite group and the KO groups we know from Bott periodicity, so we can't get the $\mathbb{Z}$ but

Theorem 12.5. *(Adams–Milnor) This* $\alpha : \Theta_{8j+\epsilon} \to KO_{8j+\epsilon}$ *is split onto for all* $j \geq 1$ *and* $\epsilon \in \{1, 2\}$.

So for example this works for $\Theta_9 \to KO_9 \cong \mathbb{Z}_2$

There's an important filtration, the Gromoll filtration, of $\Gamma^{n+1} \cong \Theta_{n+1}$, so $\Gamma^{n+1}_{(k)} = \mathrm{im}(A : \pi_{n-k} \mathrm{Diff}(D^k) \to \Gamma^{n+1})$.

Theorem 12.6. *(Cerf) You can always go one step back,* $\Gamma^{n+1}_{(n-1)} = \Gamma^{n+1}$.

Can you do better? Here's a scandal in differential topology. This filtration is not known for any value. We know Θ^k up to 62 but don't know this in any dimension.

So how far can we pull back the particular elements in the image of α?

Theorem 12.7. *(C.–S.–S.)* $\alpha(\Gamma^{8j+\epsilon}_{(6)}) \neq 0$. *That is,* $\alpha : \Gamma^{8j+\epsilon}_{(6)} \to \mathbb{Z}_2$ *is split onto.*

So for example, $\pi_2 \mathrm{Diff}(D^6) \to \pi_0(\mathrm{Diff}(D^8))$ but this is modest, I can do this for the million-and-eight disk.

I want to say something now about the proof. The key word I want to introduce here is Toda brackets. In the interests of time I won't tell you what these are. This is really a homage to Adams, who found an infinite family of elements that hit the α invariant, $\mu_{8j+\epsilon} \in \pi^s_{8j+\epsilon}$. So $\pi^s_7 \cong \mathbb{Z}/240$ which contains a unique element a with $2a = 0$ and $a \neq 0$. What's your favorite homotopy class which is of order 2? It's η, so take $S^8 \xrightarrow{\eta} S^7 \xrightarrow{2} S^7 \xrightarrow{a} S^0$ and coning these off [pictures] I get $\langle \eta, 2, a \rangle \subset \pi^s_9$ and the alpha invariant of this is $\{1\}$.

Then we get $f \in S^{8j-1-\epsilon} \to S^7$ so that $\alpha\langle f, 2, a\rangle = \{1\}$.

For this part of the talk I'll assume you know there's a space $\mathcal{G}$ so that $\pi_7^s \cong \pi_7\mathcal{G}$. There's something else $\pi_7 PL \to \pi_7\mathcal{G}$ and then you can map to $\pi_7 PL/O$, which is $\pi_7(PL_6/O_6)$. Why do we care? There's a theorem of Morlet, that $\operatorname{Diff}(D^k) \cong \Omega^{k+1}PL_k/O_k$. So with a shift the homotopy group elements correspond. So we found an interesting element a_{PL_6/O_6} to apply Today brackets on.

So we can do $\langle f, 2, a_{PL_6/O_6}\rangle$ and the α invariant of this is nonzero.

That was a little bit fast, but the real input is to figure out a way convert Adams' construction into the space of diffeomorphisms of disks. When I loop up I get this class in a way I can use.

That's something about the construction of these elements. Now in the remaining time I gave you one scandal already, that we don't know the Gromoll filtration. I'll present another scandal that we can use to resolve it.

The derivative or linearization map is what I'll turn to now. We take $\operatorname{Diff}(D^k)$, maybe symplectomorphism if you like, and you can differentiate it, and get $Tf_x : \mathbb{R}^k \to \mathbb{R}^k$ which gives us an element in $SL_k(\mathbb{R}) \simeq SO(k)$, and so up to homotopy I get this derivative map $D : \operatorname{Diff}(D^k) \to \Omega^k SO(k)$.

Let me say a little about Morlet in this setting. Let me pretend that you know how to differentiate PL things, then you get this square

$$\begin{tikzcd} \operatorname{Diff}(D^k) \arrow[r, "D"] \arrow[d] & \Omega^k SO(k) \arrow[d] \\ \mathrm{PL}\text{-}\mathrm{Homeo}(D^k) \arrow[r, "D_{\mathrm{PL}}"'] & \Omega^k SPL(k) \end{tikzcd}$$

and you can do a kind of Alexander trick, since the PL group is $*$, and you can prove this is a Cartesian square, and then $\Omega^k SO(k)$ is $\Omega^{k+1}PL_k/O_k$ and so the factorization from $\operatorname{Diff} D^k$ is the Morlet map.

So you can ask if this map $\pi_i \operatorname{Diff}(D^k) \to \pi_i \Omega^k SO(k) \cong \pi_{k+1} SO(k)$ is interesting? If you like, you can let $\pi_i \operatorname{Diff}(D^k) \cong \pi_{k+i+1}(PL_k/O_k)$ and ask if this map is *ever* nonzero and we have no known examples.

You could say that this boundary map is always zero because it's a dumb map. Using concordance theory, Milnor and Brumfiel, this map is essential, it's nontrivial on some Moore space. It's not that it would be zero for stupid reasons, but we can't compute it on homotopy.

Right. Why would you care? Let me give you two reasons for why you would care about this. So let's see. Let's look at this map on π_0 :

$$\pi_0 \operatorname{Diff} D^k \to \pi_0\Omega^k SO(k) \cong \pi_k SO(k) \cong \pi_k SO_{k+1}$$

and it's not hard to see that this takes a homotopy sphere in $\pi_0 \operatorname{Diff} D^k$ to its tangent bundle. It's a classical theorem that all homotopy spheres have the same tangent bundle as the standard sphere. How do you do this? You first have to show that this is stably trivial and work from there, I have to go and ask Adams for everything. This problem about the boundary map, as a differential topologist, you want to understand this fundamental fact in differential topology maybe better.

A second motivation is the Gromoll filtration in dimension 16, the group is $\Gamma^{16} = \mathbb{Z}/2$ so $\Gamma^{16}_{(15)} = \Gamma^{16}_{(14)} \overset{?}{\supset} \Gamma^{16}_{(13)}$ and [some missed] $\pi_{16}^s \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2(\eta_4)$ and we'd like to say that the Gromoll filtration stops there and this is zero.

Concretely in relation to D, we have $\pi_{16}PL_{14}/O_{14} \to \pi_{15}O_{14}$ and this is zero. But we know nothing about

$$\pi_{16}(PL_{13}/O_{13}) \to \pi_{15}O_{13} \cong \mathbb{Z}/2 \oplus \mathbb{Z}$$

but there's an inviting $\mathbb{Z}/2$ summand waiting to be hit.

There's a theorem of Burghelea–Lashof saying $D = 0$ on π_0 and π_1. Then A'Campo says that $D = 0$ on π_2. It would follow that this thing was nonzero, violating A'Campo. It's like Fermat, we wrote to A'Campo and he said he had lost his notes.

13. Bea Bleile: Fundamental Triples and Poincaré Duality Complexes with Highly Connected Universal Covers

Thank you very much and I wanted to thank the organizers, I've really enjoyed the conference so far, and the other speakers, it's also been good for the soul. The nice thing about this title is that Jonathan Hillman had a paper with a similar title and complementary content. One of the motivations to write down our part of the paper which is on proving some conjectures of Turaev was that Jonathan was using these things in his part, Jonathan wasn't quite game to use it until he'd seen it written down.

I'll start by introducing the background, then introduce homotopy classification, and fundamental triples, and then I'll talk about PD complexes with highly connected universal cover, proving Turaev's conjectures.

So've we've heard a lot about manifolds, and Poincaré duality complexes are just homotopical versions of manifolds. So we work in CW_0, where the objects are CW complexes with n-skeleton X^n and $X^0 = *$, with maps the basepoint preserving maps. We have the universal covering $\hat{X} \to X$, and fix a basepoint of $\hat{X}$ as well so we don't run into trouble and denote by $C(\hat{X})$ the cellular chain complex of $\hat{X}$. Sometimes I'll write π for the fundamental group $\pi_1(X,*)$, and let the group ring be Λ and consider $C(\hat{X})$ as a complex of left Λ-modules. A lot of things are going on at the chain level, so it's a mix of algebraic and topological things. Now I is the augmentation ideal and if we have a homomorphism $\omega : \pi \to \mathbb{Z}/2\mathbb{Z}$, this yields an anti-homomorphism denoted by a bar from Λ to Λ which takes $\sum n_g g \mapsto \sum(-1)^{\omega(g)} n_g g^{-1}$. When we don't want to restrict to orientable Poincaré duality complexes, this ω allows us to switch from left to right modules. So if you have a left Λ-module M, then you get the right Λ-module M^ω where $m\lambda \coloneqq \bar{\lambda}m$ and you do the same thing the other way around to go from a right Λ-module N to a left Λ-module ${}^\omega N$.

Then the homology and cohomology groups we work with, I'll just write them down, if M is a left Λ-module, then

$$H_n(X;M) \coloneqq H_n(M^\omega \otimes_\Lambda C(\hat{X}))$$
$$H^k(X;M) \coloneqq H_{-k}(\mathrm{Hom}_\Lambda(C(\hat{X}),M))$$

Then a Poincaré duality complex consists of a triple $(X, \omega_X, [X])$ where X is in CW_0 with finitely presentable fundamental group, ω is an orientation character $\pi \to \mathbb{Z}/2\mathbb{Z}$, and $[X]$ is the fundamental class in $H_n(X, \mathbb{Z}^\omega)$ such that capping with $[X]$ is a (Poincaré duality) isomorphism

$$H^k(X;M) \xrightarrow{\cap [X]} H_{n-k}(X;M^\omega)$$

for all M.

You might ask if there's anything new here, and there are Poincaré duality complexes that aren't manifolds, but you have to go to $n = 3$ for that.

So the question, also, that Kirby–[unintelligible]showed that every manifold is homotopy equivalent to a CW complex and so you can make PD complex. What about the other way? For $n = 1$ you get the circle. For $n = 2$, Eckmann–Müller, Linnell between 80 and 83 showed the answer is yes, you only get closed surfaces.

For $n = 3$ the answer is no. In 1967, Wall showed that if and only if π is the fundamental group of a PD^3 complex and is finite, then and only then, π has periodic cohomology of period four. Milnor showed that there are such finite groups which are not the fundamental groups of a three-manifold (so this was 1957). The simplest one is S_3 and Swan constructed an example in 1960 with fundamental group S_3. For $n = 5$ there are counterexamples by Gitler–Stasheff, simply connected PD^5 complexes not homotopy equivalent to a 5-manifold.

Some useful facts. Take a Poincaré duality complex $X = (X, \omega_X, [X])$ in dimension n. Then

(1) (Wall) $X \cong X' \cup e^n$, where for $n > 3$, X' is the $n-1$-skeleton of X, and for $n = 3$, X' is three-dimensional but homologically three dimensional. If you take $e \in C_n(\hat{X})$ corresponding to e^n, then $[X] = [1 \otimes e]$.

(2) The map

$$\omega_{\mathrm{Hom}_\Lambda(C(\hat{X}),\Lambda)} \xrightarrow{\cap 1\otimes e} \Lambda^\omega \otimes_\Lambda C(\hat{X}) \cong C(\hat{X})$$

is a chain homotopy equivalence of degree n

(3) (Browder?) A map $f : X \to Y$ of PD^n complexes which is oriented degree one (meaning $\omega_X = \omega_Y \pi_1(f)$ and $f_*([X]) = [Y]$, then the induced map on fundamental groups is surjective.

There was a notion of a fundamental triple around earlier for three-manifolds and PD^3 complexes which we generalized, and now we come to fundamental triples, which give us the classification.

We want to look at degree one maps. Let PD^n_+ be the category of PD^n-complexes with degree one maps. Then there is a functor τ to the category of triples Trp^n_+, which is the category (T, ω, t), where T is an $(n-2)$-type in CW_0 and ω is a homomorphism $\pi_1(T) \to \mathbb{Z}/2\mathbb{Z}$ and $t \in H_n(T, \mathbb{Z}^\omega)$. An object in PD^n_+, X, we get the $(n-2)$-Postnikov section by attaching cells to kill higher homotopy groups, and then you send

$$(X, \omega_X, [X]) \mapsto (P_{n-2}X, \omega_X, p_*[X]).$$

Theorem 13.1. *(Baues–B. 2007) The functor τ is full and reflects isomorphism.*

Reflection of isomorphisms is easy. You just need Poincaré duality and Whitehead's theorem. For fullness, meaning that if we have a map $(T, \omega, t) \to (T'\omega', t')$ between two things in the image of the functor, that the map comes from one in Poincaré duality complexes.

So for fullness, take an isomorphism of triples, take the $(n-1)$-skeleton, take the chain complex, and take a chain map, and then use that the isomorphism preserves the ts to match up the fundamental classes.

Then you realize this as a topological map using Baues' homotopy systems.

As a corollary, you get that PD-complexes are orientedly homotopy equivalent if and only if the fundamental triples are isomorphic.

You hope to know something about Poincaré duality complexes when you know up to the middle dimension, and I was asked that by an audience member last time I talked about this, and I'm still no closer.

For $n > 3$, you could take $\left(P_{\lfloor \frac{n}{2} \rfloor}, \omega_X, p_{\lfloor \frac{n}{2} \rfloor *}([X])\right)$, a *pre-fundamental triple* and you could ask if this classifies PD-complexes. The answer is no, the corresponding τ still reflects isomorphisms but we don't get the corollary because, the counterexample is if you attach D^{2n} to $S^n \wedge S^n$ via $\alpha = [i_1, i_2] + i_1\beta$ where $\beta \in \pi_{2n-1}S^n$ with trivial Hopf invariant. A question is what information you'd have to add to detect homotopy type and if anyone has some ideas I'd love to hear them.

This brings us to the third part, on Turaev's conjectures. Originally [unintelligible]proved the classification theorem for PD^3 complexes and Turaev provided an alternative proof. If you take $(X, \omega_X, [X])$ a PD^n-complex with $(n-2)$-connected universal cover, then we can take $K = K(\pi, 1)$, which gives the $(n-2)$-Postnikov section. So Turaev showed that the classification, splitting, and [unintelligible]results extend to [unintelligible]. I want to report on the generalization of the realization and the splitting.

First formulate the conditions for realisation. We had some technical conditions because π is finitely presentable, you can convince yourself it's actually of FP_{n-1} type and because we have Poincaré duality we have that $H^i(\pi; {}^{\omega}\Lambda) = 0$ for $1 < i < n$. So for the main realisation condition, we use a functor from the category of chain complexes over Λ, call it G_r, to the stable category of Λ-modules.

If you have a map $f : C \to D$, you get a chain map $C_r \to D_r$, and the image of d^D_{r+1} sits in there, and we can take the cokernels of the differential maps d^C_{r+1} and d^D_{r+1}, and the induced map of cokernels is $G_r(f)$.

Then we need one more result, from Turaev, if $f : C \to D$ is a chain homotopy equivalence and C_r and D_r are projective, then $G_r(f)$ is a homotopy equivalence of modules, which is precisely what will give us our condition.

We apply G to the chain homotopy from Poincaré duality at the chain level, with some fudge because you have to shift on one side, and we get something from $C^{n-1}(\hat{X})/\mathrm{im} d^{n-1}$ (and I call that $F^{n-1}(C(\hat{X}))$) and that goes to to $C_1(\hat{X})/\mathrm{im} d_2 \cong I$, call this map η, it's a homotopy equivalence of modules.

Again, take $P = P_{n-2}X$, then because the Postnikov section is the identity lower down, this is $F^{n-1}(C(\hat{P}))$

Now the Turaev map is constructed, call it ν_C which works for C a complex of free Λ-modules, from $H_n(\mathbb{Z}^{\omega} \otimes_{\Lambda} C) \to [F^{n-1}(C), I]$, and it's constructed precisely so that $\nu_{C(\hat{P})}(p_*[X])$ is equal to η.

The ν_C is the composition of a connecting homomorphism with a suitable evaluation map.

Theorem 13.2. *(B.–Bokor) Take a finitely presentable group G satisfying the technical conditions and take $k = K(G, 1)$ with t and ω as before. Then (K, ω, t) is realized by a* PD^n*-complex if and only if $\nu_{C(\hat{K})(t)}$ is a homotopy equivalence of modules.*

So that's the condition that picks out, tells you about realisability. We're in the stable category, so a homomorphism is nullhomotopic if it factors through a projective.

So now to splitting or decomposition, the notion of connected sum for PD^n complexes also coes back to Wall. If we have two PD^n complexes, then we can write each of them as $X_i \cong X_i' \cup e_n$. Then we write $\hat{g}_i$ as the composition with the inclusion into $X_1' \wedge X_2'$. Then we attach an n-cell via $\hat{g}_1 + \hat{g}_2$ and define this as the connect sum $X_1 \# X_2$. You do things on the chain level to show this is a Poincaré duality complex. If you can write X a Poincaré duality complex, then $\pi_1(X) = \pi_1(X_1) * \pi_1(X_2)$. You can show that it goes the other way as well, if $\pi_1(X) = G_1 * G_2$, then the Turaev map applied to the fundamental class here gives you a homotopy equivalence of modules, and you see that $\nu_{C(\hat{K})}(p_*[X]) = \mu_1 + \mu_2$, and this is in the homology of a $K(G_1, 1)$ and a $K(G_2, 1)$, and so you realize these two and because of the classification theorem, then you can realize (G_i, ω_i, μ_i) by X_i and then by the classification theorem, $X \cong X_1 \# X_2$.

So a PD^n complex decomposes nontrivially as a connected sum if and only if its fundamental group decomposes as a free product. This applies for instance to closed manifolds.

There are some results with other connectedness assumptions but some of them are kind of yucky, making them nicer is yet to be done.

14. Yong-Geun Oh, Lagrangian Floer theory and mirror symmetry on toric manifolds III

So let's briefly recall, (M, ω) is symplectic with L a Lagrangian submanifold. In my talk I used the de Rham model, so $m_k : \Omega(L : \Lambda)^{\otimes n} \to \Omega(L : \Lambda)$. So you want to regard this as a correspondence using evaluation maps

$$\begin{tikzcd} & \mathcal{M}_{k+1}(L; \beta) \arrow[dl, "\mathrm{ev}_\dagger"'] \arrow[dr, "\mathrm{ev}_0"] & \\ L^k & & L \end{tikzcd}$$

and $m_{k,\beta}(\alpha_1, \dots, \alpha_k) = (ev_0)_!(ev_\dagger)^*(\alpha_1 \times \dots \times \alpha_k)$ and the m_k satisfy the A_∞ relations, which are the same thing as $\hat{d} \circ \hat{d} = 0$ where $\hat{d}$ is

$$\hat{m}_k : B\Omega[1] \to B\Omega[1]$$

where $\hat{m}_k$ is the coderivation induced by m_k. For each $b \in C^0[1] = C^1$, we can deform m_k to

$$m_k^b(x_1, \dots, x_k) = \sum m(b, \dots, b, x_1, b, \dots, b, \dots, x_k, b, \dots, b).$$

The proposition is that this is still an A_∞ structure. For $k = 0$, then $m_0^b(1)$ is $\sum_{k=0}^\infty m_k(b, \dots, b)$ and we define $\mathcal{M}^{\mathrm{weak}}(L)$ as the set of b such that $m_0^b(1) \equiv 0 \pmod{()\mathbf{e}}/\sim$ where $\sim$ is gauge equivalence and $\mathbf{e}$ is the unit of (C, m). By definition, L is called weakly unobstructed if $\mathcal{M}^{\mathrm{weak}}(L)$ is nonempty. Then we look at the pair (L, b), where these form an object of our Fukaya category, where $b \in \mathcal{M}^{\mathrm{weak}}(L)$. Then by definition, we're given a potential function $\mathcal{PO}_L : \mathcal{M}^{\mathrm{weak}}(L) \to \Lambda_+$ where $\mathcal{PO}_L(b)$ has a nice form because $\mathbf{e}$ is the Poincaré dual of $[L]$. So then this is

$$\int_L \sum m_k(b, \dots, b) d\,\mathrm{vol}_L$$

I also need to deform the whole Fukaya category, using so-called bulk deformations. So $\mathbf{b}$ is an anmbient differential form on M. The way we'll deform this, we'll look at the moduli space of holomorphic disks with boundary and interior marked points $\mathcal{M}_{k+1,\ell}(L : \beta)$, where there are points $w, z_0, \dots, z_k, z_1^+, \dots, z_\ell^+)$ so that $\bar\partial_J(w) = 0$, the class $[w]$ is $\beta \in \pi_2(M, L)$, each z_i is in ∂D^2 and each z_j^+ is in the interior. [pictures]

Again you want to do a correspondence, a pullback and pushforward. Using this, you can define a so-called closed-open map here. This is a $\hat{q} : E(M)[2] \otimes B\Omega L[1] \to B\Omega L[1]$. Then what I'm saying is $q_{k,\ell}$ is $\Omega M[2]^{\otimes \ell} \otimes \Omega(L)[1]^{\otimes k} \to \Omega(L)[1]$ where this is the pushforward of the pullback;

$$q_{k,\ell}(\alpha_1, \ldots, \alpha_k, \omega_1, \ldots, \omega_\ell) = (\mathrm{ev}_0)_!(\mathrm{ev}_+, \mathrm{ev}_+^\dagger)^*(\alpha_1 \times \cdots \times \omega_\ell).$$

Proposition 14.1. *Define* $m_k^{\mathbf{b},b}$ *as* $q_{k,1}(\qquad; \mathbf{b})$.

Then this also satisfies the (curved) A_∞ relations. You can do this for any given L. By doing this over the whole Fukaya category, on each M, now, to deform the category you have to look at a chain of Lagrangians.

You can deform morphisms in a similar way. If you look at a chain of Lagrangians like this [picture] all you have to do is to look at punctured disks [pictures]. In this way you'll define $m_k^{\mathbf{b}}$ as

$$q_{k,1}^{\mathbf{b}} : CF(L_0, L_1) \otimes \cdots \otimes CF(L_{k-1}, L_k) \to CF(L_0, L_k).$$

Denote the resulting category by $\mathrm{Fuk}^{\mathbf{b}}(M, \omega)$.

This is the general story, I want to do some computations in the toric case. It looks intimidating but luckily we have a good example for which we can do all this structure explicitly.

When we go down to the minimal (canonical) model $(H, \{m\}_{k=0}^\infty)$ where $m_1 = 0$ and $H^*(L; \Lambda)$. Let me change the space to (X_Σ, ω), a toric manifold, and in this audience maybe the best way is to describe the moment polytope. Let me draw a picture. For example, [pictures] So P_Σ is the moment polytope and G_Σ is the set of inward pointing normal vectors at the facets. Σ is the so-called "fan" but just focus on this picture.

I want to reveal the essence of this computation by doing it slowly here.

What is it, by the way, this m_0, β? It involves $\mathcal{M}_1(\beta)$, the space of holomorphic disks with one marked point. The dimension here is given by the dimension of the Lagrangian submanifold plus the Maslov index of β plus one minus three so

$$n + \mu_L(\beta) - 2.$$

Then this is roughly a count of the number of holomorphic disks with Maslov index 2, passing through one generic point. This is the open Gromov–Witten invariant of one point.

Then this one has exactly dimension n, so ev_0 the domain and target have the same dimension. So under the assumption that $\mathcal{M}_1(\beta)$ has no boundary, then you can compute degree, and this is 1-point open Gromov–Witten invariant.

It turns out (this is work of Cho and myself) that in this toric case, for each facet, there is exactly one, a unique holomorphic disk of Maslov index 2. In the $\mathbb{CP}^2$ case there are three facets, you look at the Lagrangian submanifold, you look at the associated Lagrangian submanifold, and there are three holomorphic disks with boundary on this fiber with Maslov index 2. So you can compute $m_{0,\beta}$.

Let me do this more systematically. The consequence, furthermore, there could be other Maslov index 2 disks. In the Fano toric case, these are all such holomorphic disks. The result provides so-called Givental–Hori–Vafa potentials. On the other hand, for the non-Fano case, there are more, other hidden contributions, for example, for the Hirzebruch surface, there are four obvious holomorphic disks but there are more that are somewhere hidden, maybe holomorphic disks that are not smooth, singular disks. It makes computation of the potential function in general very complicated (unless the toric manifold is Fano)

Let me write down the potential function in general, $\mathcal{PO}_L(b)$. In the toric case you can replace with the harmonic forms, and in the toric case these are equivalent to [unintelligible]forms, which makes things nice, and so in this case you have

$$\mathcal{PO}_L(b) = \sum_{\mu(\beta)=2} T^{\omega(\beta)} \exp(b \cap \partial\beta) n_\beta$$

where $n_\beta = \deg[\mathrm{ev}_0]$ and $\beta \in H_2(X, L)$ with $[\partial\beta] \in H_1 L$.

In the Fano case, as I said, we know all the holomorphic disks of Maslov index 2, so then we say $W_{L(U)}$, the composition of the potential with the canonical embedding $\mathcal{PO}_L \circ \Phi_L$ where $\Phi_L : H^1(L : \Lambda) \to \mathcal{M}^{\text{weak}}(L(U))$. So the composition is $\sum T^{\ell_j(b)} e^{\langle b, v_j \rangle}$

Choose a basis for $H^1(L(U) : \mathbb{Z})$, $\{\mathbf{e}_i\}$. Then $b = \sum x_i(b)\mathbf{e}_i$ I should have said, ℓ_j is a linear affine function , $\ell_j(b) = \langle b, v_j \rangle - \lambda_j$ and the polytope is $\ell_j \geq 0$. So $y_i(b) \coloneqq e^{x_i(b)}$ and then the formula is

$$W_{L(w)}(b) = \sum y_k^v T^{\langle b, V_j \rangle}$$

So for example for $\mathbb{CP}^2$, our $W(b)$ is $y_1 T^{u_1} + y_2 T^{u_2} + \frac{T^{1-u_1-u_2}}{y_1 y_2}$.

At the moment it looks like it depends on u. This should just be a function on $\mathbb{C}^*$. A good way of doing this is to change the variable again, write $y_i = y_i(b) T^{u_i}$. Then this looks like $W = y_1 + y_2 + \frac{T}{y_1 y_2}$. Now how do you recover the fiber? All you have to do is, these polynomials, they have Novikov ring coefficients. Then the corresponding valuation points, there is the corresponding moment fiber.

So now this whole story can be deformed by an ambient cycle again. We can deform these potential functions (and this whole story) by a bulk parameter $\mathbf{b}$. So you'll be given a family of potential functions which I'll denote $W(y; \mathbf{b})$ and regard this as a family, a deformation of this potential function. We know that in general the deformation of a category gives rise to some class in Hochschild cohomology of the category, and in this way, the q_1, the q map, will define a map, let's say, the closed open map, which, let me write this.

In a symplectic manifold, $\mathbf{b}$ lives in $H^*(M)$ and there is some well-known cup product, a quantum product, and by considering the quantum product, with quantum cohomology of M, this deformation map q can be regarded as a map from the quantum cohomology class of M going to the Hochscihild cohomology of the Fukaya category $q_1 : QH^*(M) \to HH(\mathrm{Fuk}(M))$. This is folklore, but the precise version we wrote down was

Theorem 14.1. *(Abouzaid–Fukaya–O.–Ohta–Ono) This is a ring homomorphism and under the so-called semi-simpleness of quantum cohomology, this is indeed a ring isomorphism (under some generation conditions).*

We believe this is generally true, we are in the process of proving the same holds in general.

This involves some further understanding about homological mirror symmetry between matrix factorization and torics.

The proof of the ring homomorphism can be described by this picture, the quantum product is defined by inserting two elements, you look at the moduli space [pictures].

In general the Fukaya category is very abstract. To explicitly describe it it's useful to have a good class of Lagrangians. In the toric case, we have, I didn't mention, I want to mention, for a good description of the Fukaya category, it would be to choose to select a "good" collection of Lagrangian submanifolds. This is supposed to generate the whole subcategory. Cho looked at the punctured sphere, and chose [unintelligible]. He got something that generated the whole wrapped Fukaya category for a punctured surface. SO we need Abouzaid's generation criterion. I should say, sorry, the ring isomorphism is only true under the generation criterion, that the q_1 map to Hochschild, this becomes, at the end of the day, this is injective. In particular, it's a unital ring, so if you, well.

Finally, in the toric case, this collection is given by those pairs $(L(u), b)$ such that b is a critical point of the fiberwise potential function of u. There is some finite number of, in this case only one, there are some finite number of basepoints so that the associated potential function has a critical point. When you collect those, the number of such pairs is the Euler characteristic of this toric manifold, $\operatorname{rk} H^*(X_\Sigma)$. When you're given a toric manifold, we can explicitly find these pairs and it turns out that this collection satisfies the generation criterion. This proof needs to study so-called mirror symmetry between Saito's, Kodaira–Spencer map given by each toric divisor corresponding to the differential of the potential function, $\mathbf{b}_i = D_i \to \frac{\partial W_{\mathbf{b}}}{\partial \mathbf{b}_i}$.

15. May 5: Craig Westerland: Structure theorems for braided Hopf algebras

I want to tell you about a project I've been working on for the last year or so, and in many respects it has nothing to do with the talk that TriThang gave on Monday, although it grew out of that work. If you remember his talk, he said that if you want to study the homology of these Hurwitz spaces of branched covers of configurations in the plane, I can compute it with something called a quantum shuffle algebra, $\operatorname{Ext}^{*,*}_{A(V_\epsilon)}(\mathbf{k}, \mathbf{k})$. But this comes with a caveat, writing this down doesn't mean you understand either side. You don't understand really anything about this right hand side. But also the multiplication in the shuffle algebra is a delightful combinatorial gadget but it doesn't help you write things down in terms of generators and relations. And except in the simplest cases, I have no idea how to write things down for this algebra. The goal today is about writing down a nice presentation for these algebras, and we're not going to get there but we'll get some other things of interest.

These quantum shuffle algebras are braided Hopf algebras, you might hope there are like Milnor–Moore or Poincaré–Birkhoff–Witt theorems in that context and there aren't. But I'd like this to be like the start of that.

I'll make this a pure algebra talk for a while. Let me start with some setup. Let H be a Hopf algebra over a field $\mathbf{k}$, and I want to remind you of a couple of words, some probably familiar and some probably not familiar. An element g is *grouplike* if $\Delta(g) = g \otimes g$. An element x is *skew-primitive* if there exist grouplike elements g and h such that $\Delta(x) = x \otimes g + h \otimes x$. It is *primitive* if g and h are the identity.

From the grouplike guys, I can produce a group $G(H) \subset H$, the group of grouplikes, and the fact that this is a Hopf algebra makes this into a group. Maybe less familiar is the *coradical* of H, which is the sum of all simple subcoalgebras, I'll notate this as H_0. It's maybe hard to think about. This is dual to the radical. The line generated by a grouplike is a subcoalgebra. A word I'm guessing almost none of you is familiar with is that H is said to be *pointed* if each simple comodule is 1-dimensional. For a topology audience this is an awful word. Trying to imagine what this means is hard. An example of a simple comodule is a grouplike; it's the same as saying the coradical H_0 is the same as $\mathbf{k}[G]$, the group ring on the grouplike elements.

We've got all the language now to state the theorem, which is something called the Andruskiewitsch–Schneider conjecture.

Theorem 15.1. *(W.) If H is a pointed finite dimensional Hopf algebra over a field of characteristic zero, then it is generated by grouplike and skew-primitive elements.*

So this, previous work, this has previously been established, lots of special cases by Andruskiewitsch and Schneider, in the setting, where, let's see if I can intimidate you, when the infinitesimal grading is Cartan-type. Recently, Angiono (2013) did it in the Abelian case, when the group of grouplikes G is Abelian.

This is part of the classification of finite dimensional pointed Hopf algebras using what is called the lifting method. They have a very large program for trying to describe these things and part of that is about exploring all the possible relations that could occur. One starting point to classify them is the question of how many generators you need. In most of the cases you were interested in, you knew this already, and I'm saying that it works more generally.

I want to spend a couple of minutes telling you how to reduce this theorem to a theorem about braided Hopf algebras. That's, let's see, this might be going a little bit deeper into the weeds of Hopf algebras than we're comfortable with, but I promise you it will pay off in a moment.

The method starts with the coradical filtration. My Hopf algebra H, I can define a filtration on it by saying H_n is the set of things whose diagonal lies in $H \otimes H_0 + H_{n-1} \otimes H$. So H_1 is things whose diagonal lies in $H \otimes H_0 + H_0 \otimes H$. So we have a name, these are skew-primitives. So H_0 are grouplike and H_1 the skew-primitives. It has all the properties you need to say that, let H^{gr} be the associated graded thing, $\bigoplus H_n/H_{n-1}$, and this becomes a graded Hopf algebra.

The conjecture is saying that H is generated in H_0 and H_1, which is the same as showing that the associated graded is generated in H_0 and H_1. But there's something cool which is, we can kill off the things coming from H_0. There is a projectino from H^{gr} to $H_0 = \mathbf{k}[G]$, which makes H^{gr} a coalgebra for $k[G]$. So we can define $R \coloneqq H^{gr} \square_{\mathbf{k}[G]} \mathbf{k}$, the cotensor product. This is the dual to the tensor product with the ground field, killing off a subalgebra.

What can I tell you about this thing? This construction R is a graded connected braided Hopf algebra in the category of Yetter–Drinfel'd modules for G, YD_G^G. It's connected because we killed off degree zero. The Hopf algebra structure on the associated graded gives one on R, but it's not Hopf in the usual sense but in a braided monoidal category. I can reconstruct the original Hopf algebra as the *Radford biproduct* of the group ring with R, so we have generation by grouplikes and R. As an algebra it's a twisted tensor product but writing it down precisely will take a lot of work.

Let's just sort of put over here a reminder that to show the theorem, it's now equivalent to saying, to showing that R is primitively generated.

Let me tell you about Yetter–Drinfel'd modules. There's a more general setting but let's just work over the group ring. These are modules over the group ring $\mathbf{k}[G]$, let's make the right-modules M, which have a decomposition

$$M = \bigoplus_{g \in G} M_g$$

so that $(M_g)h = M_{g^h}$ where $g^h = h^{-1}gh$.

This is a category, in fact a braided monoidal category. If I have two guys, $M \otimes N$ is the tensor product over $\mathbf{k}$ in the usual way, and the action of g is diagonal, if you've got a more general Hopf algebra the action involves the diagonal of the Hopf algebra, and the grading is $(M \otimes N)_g$ is the sum of $M_n \otimes N_k$. The braided part is the most important part, the isomorphism from $M \otimes N \to N \otimes M$, which takes $m \otimes n$ to $n \otimes mg$ if $n \in N_g$. If I do this twice and braid again, I get $mg \otimes nh$ if $mg \in M_n$. This is only going to be symmetric monoidal if these actions are trivial.

There's a more general condition, but a *braided Hopf algebra* is a Hopf algebra object in this category of Yetter–Drinfeld'd modules.

That's the sort of thing that this R is, a Hopf algebra object in this category. Let's look at a couple of important examples. The first is, if V is a Yetter–Drinfel'd module then $T(V)$ is a braided Hopf algebra. You know how to multiply, and you make v primitive for all $v \in V$. This construction looks harmless. It's a little misleadingly so. The quantum shuffle algebra that TriThang talked about, $A(V)$, if V is a Yetter–Drinfel'd module, if I take $T(V)^*$ (if V is finite dimensional at least) then this is $A(V^*)$. This tells me that the coalgebra structure on $A(V)$ is the tensor coalgebra, the deconcatenation coproduct. Then this description setting the generators to be primitive tells me that the coproduct is dual to the quantum shuffle product, which was a complicated gadget.

A third example is something called the *Nichols algebra*, called BV, the image of the natural map $T(V) \to T^{co}(V)$ induced by id_V, the subalgebra of the shuffle algebra generated by V. There's a remarkable characterization, that $B(V)$ is the unique braided Hopf algebra with, well $P(A)$ the primitives maps to $Q(A)$, the indecomposables. It's the unique such one where this is an isomorphism and these in fact are V.

So if the word braided weren't there and we were just talking about Hopf algebras, like assuming that the group was trivial, then a braided Hopf algebra is the same as a Hopf algebra. Then what is $B(V)$? Its primitives are isomorphic to its indecomposables, so it's primitively generated, so it's cocommutative. The same is true for the dual Hopf algebra, so the dual Hopf algebra is primitively generated, so it's commutative. So if I use the Milnor–Moore theorem, I don't even need to be in characteristic zero. Being cocommutative means I'm the universal enveloping algebra on my primitives and being commutative tells me that the primitives are Abelian, and so I'm the symmetric algebra on P. So symmetric algebras are the analogues of these Nichols algebras.

To ground this more, for specific G and V you can produce, say, the Borel part of the enveloping algebra, for like a Kac–Moody case.

Let's go back to the sketch of the aim of the proof. The Hopf algebra R, we'd like to show that it's primitively generated. We know that R is a connected graded Hopf algebra, and in fact, the filtration was such so that, the skew-primitives are the primitives. That will map to indecomposables as before. This is in fact an injection. I"m saying $P(R)$ is indecomposable because I can only add up to 1 by adding up 0 and 1. We've got the awkward thing that, we know the primitives are indecomposable, but we want to say the converse is true. But we have a name for this being an isomorphism. So equivalently, we want to show that $R = B(V)$ for V the primitives. The theorem that's getting that all to work is that if S is a connected, primitively generated finite dimensional braided Hopf algebra over $\mathbf{k}$ of characteristic zero, then S is a Nichols algebra. If $S = R^*$ then the primitives of S will surject onto the indecomposables of S, it's the dual of my injection. I can switch to a question about primitively generated ones in this way, and I still want a Nichols algebra.

Let's take a moment to breathe and wonder why this is true. I'm saying something like, if I have a finitely generated primitively generated Hopf algebra, then it's a symmetric algebra. How to I do that? I use Milnor–Moore and Poincaré–Birkhoff–Witt. That's the thing I want to prove. But aiming for Poincaré–Birkhoff–Witt, or Milnor–Moore, I try to do something involving the enveloping algebra of the primitives and then relating this to the symmetric algebra. But you get off the ground by knowing that the primitives are a Lie algebra. But $P(A)$ is not necessarily a Lie algebra. What goes wrong? If I take $\Delta[a,b]$, I should get $\Delta[a,b]\otimes 1 + 1\otimes\Delta[a,b]$. So let's do it:

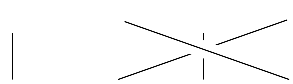
$$\begin{aligned}\Delta([a,b]) &= \Delta(ab-ba)\\ &= (a\otimes 1 + 1\otimes a)(b\otimes 1 + 1\otimes b) - (b\otimes 1 + 1\otimes b)(a\otimes 1 + 1\otimes a)\\ &= [a,b]\otimes 1 + 1\otimes[a,b] + (a\otimes b - \sigma(b\otimes a)) + (\sigma(a\otimes b) - b\otimes a)\end{aligned}$$

and these cancel if σ is just a swap but otherwise you're hosed.

So that raises the question, what structure *does* $P(A)$ support? There are a lot of people in this room good at thinking of algebraic structures that this supports. The right framework is operads. Is there an operad that governs this structure? The answer is yes. Let me talk briefly about braided operads.

A *braided operad* $\mathcal{C}(n)$ is a collection of objects with two pieces of data, an action of the nth braid group on $\mathcal{C}(n)$ along with substitution maps $\mathcal{C}(m)\otimes\mathcal{C}(n)\xrightarrow{\circ_i}\mathcal{C}(m+n-1)$, which is the analogue of plugging the n-ary operation into the ith slot of an m-ary operation. These should be associative, equivariant (you have to be a little careful here) for the cabling map from $B_m\times B_n\xrightarrow{\circ_i}B_{m+n-1}$. Let's do an example.

is



with the two strands from the braid on the right cabled along the overstrand of the braid on the left. So braided operads should govern things in braided monoidal categories

(1) $\operatorname{Br}\operatorname{Ass}(n) = \mathbf{k}[B_n]$ and an algebra for $\operatorname{Br}\operatorname{Ass}$ is an associative algebra in a braided monoidal category.

(2) Unfortunately I actually need the completion $\widehat{\mathrm{Br\,Ass}}$, which is a *pro-operad*, and the nth space

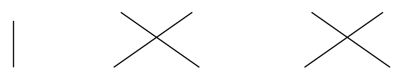
Just as the associative operad contains the Lie operad, and this governs the primitives, we want to find something here to govern our braided primitives.

Let me make a quick definition

Definition 15.1.

$$\mathbb{S}_n = \sum_{\tau \in S_n} \tilde{\tau}$$

where this is the lift, the Matsumoto lift of S_n the symmetric group to the braid group B_n, e.g., the lift of



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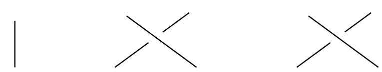

This is called the *quantum symmetrizer* and there's also

$$\mathbb{S}_{p,q} = \sum_{(p,q)\text{-shuffles}} \tilde{\tau}$$

Now the *braided primitive operad*

$$\mathrm{Br\,Prim}(n) = \bigcap_{p+q=n} \ker \mathbb{S}_{p,q} : \widehat{\mathrm{Br\,Ass}}(n) \circlearrowleft$$

and

$$\mathrm{Br\,Prim}^{\infty}(n) = \ker(\mathbb{S}_n : \mathrm{Br\,Ass}(n) \circlearrowleft)$$

and everything you need to go through goes through in this context. Sorry to have run out of time just when we were getting to the fun bit.

16. Byeongho Lee: Hypercommutative operads and topological vertex operator algebras

Thanks for the invitation and for inviting me. These algebras were first introduced by Lian–Zuckerman in 1993. My title could have also been different: hypercommutative algebras are the same as formal Frobenius manifolds. But I wanted to attract some attention from the operads people. Now topological vertex operator algebras are a formulation of topological conformal field theory. The theorem-to-be (this is a work in progress) is that given a topological vertex operator algebra, then by forgetting something we can get a Frobenius manifold. The idea is from physics. So my job here is to translate the physics literature. If this is done, then the first question that we should ask must be to describe the inverse image of Frobenius manifold.

This is going to be my genuine contribution in this regard, if this is done. We have many questions we want to ask here, and we want to lift the question to questions about topological vertex operator algebras where there is rigid structures.

So my outline, first, since I put operads in my title, I'll start with hypercommutative algebras. Then I'll talk about vertex operator algebras, so I'll talk about vertex operator algebras. Not a lot of people are familiar with them so I'll start with this. Then I'll briefly mention how to get a Frobenius manifold from a topological vertex operator algebra. Then finally I'll talk about possible applications and further questions. I cannot say everything very precisely but the first two parts will be more precise and the last two parts much less precise.

This is my outline. Let's start with hypercommutative algebras. These are algebras over the hypercommutative operad. What is an operad? There are people who are not familiar with operads, I cannot give you all the precise definitions but I can give some descriptions. The first thing that comes to mind when I heard about operads is the picture of a corolla with n inputs and one output, and some compatibilities. So what are algebras? This structure should be preserved on algebras, so you have, we're working on the category of vector spaces, so you have an n-ary operation $H^{\otimes n} \to H$ with compatibilities. This is what algebras over operads will look like, these maps and these compatibilities.

What is a hypercommutative operad? The title should have been singular, like "…hypercommutative operad", not "…operads" sorry about that. This is $H_*(\overline{\mathcal{M}}_{0,n}, \mathbb{C})$ with compatibilities given by gluing map and forgetting.

The gluing maps, if you have a sphere with a bunch of points and another, you number them, and you can glue them and get a nodal sphere. If you have four points you can forget to three points. The compatibilities are given by these maps.

Then what is a hypercommutative algebra? We're working in complex vector spaces, so H is a finite dimensional vector space over $\mathbb{C}$ and we have a non-degenerate symmetric bilinear form g, which we need because this operad is actually a cyclic operad. In an ordinary operad, you have n inputs and one output, but here you can't tell the difference between inputs and outputs. Then the n-ary operations go $\alpha_n : H^{\otimes n} \to H$ and using g we can go back and forth to $\beta_{n+1} : H^{\otimes n+1} \to \mathbf{k}$. How do we do this? We go from α to β by taking $a_1 \otimes \cdots \otimes a_{n+1}$ to $g(\alpha_n(a_1 \otimes \cdots \otimes a_n), a_{n+1})$ and in the other case, we can invert the matrix of g and get $g^{ab}\Delta_a \otimes \Delta_b$, and then $\alpha_n(a_1 \otimes \cdots \otimes a_n) = \beta_{n+1}(a_1 \otimes \cdots \otimes a_n \otimes \Delta_a) \otimes \Delta_b$ and this is independent of the choice of basis. So we won't work with α_n but with β_{n+1}.

I said this is a cyclic operad so we can think of this thing as a compatibility: that β_n are symmetric n-linear forms, for $n \geq 3$. I want to describe the compatibilities here, there are actually a nice way to write down the compatibilities of this algebra. To do this, let's make another identification. We have symmetric n-linear forms, which correspond to homogeneous polynomials of degree n, in a basis for the dual space of H. Call this Y_n, so $x \otimes y + y \otimes x$ corresponds to $2xy$. Then we set $Y \coloneqq \sum_{n \geq 3} \frac{1}{n!} Y_n \in \mathbb{C}[\underline{x}]$ where $\underline{x}$ is the basis.

So what is the compatibility condition? This is expressed using a famous differential equation, the WDVV equation, which is

$$Y_{abk} g^{k\ell} Y_{\ell cd} = Y_{adm} g^{mn} Y_{nbc}$$

where the subscripts on Y are partial derivatives and we are using Einstein conventions. The proof is to compare term by term and you get this exact geometric data. This is a hypercommutative algebra, also known as a formal Frobenius manifold. The simplest example, is A_n-singularities, comes from minimal model of topological conformal field theory. So here, see "G-Frobenius manifolds", advertising my paper, so if you want to play with a concrete example you can look at my paper. I wrote down the details about the structure very explicitly.

So this was the first part. The second part is about vertex operator algebras. There are two key ideas, I would get lost, but the two key ideas, are Hilbert spaces and correlation functions. I write these down to not get lost in my talk. So this is supposed to be a quantum field theory, a conformal field theory, so we should build a Hilbert space. Conformal here means we have an action of the Virasoro algebra, I'm just talking about the simplest example. The Hilbert space should be a representation of this algebra, so this is an infinite dimensional Lie algebra over $\mathbb{C}$ generated by L_n for n integral and a central element Z with $[Z, L_n] = 0$ and the commutation relations $[L_m, L_n] = (m-n)L_{m+n} + \frac{Z}{12}(m^3 - m)\delta_{m+n,0}$.

We construct a representation using Verma modules, so how do we construct such a module? We choose $c, h \in \mathbb{C}$ and call this module $M(c,h)$ with a basis that looks like $L_{-n_1} \cdots L_{-n_k} v_0$ for $n_1 \geq \cdots \geq n_k > 0$ and $k \geq 0$. Here c is the *central charge* and h is the conformal weight. Here we take $Zv_0 = cv_0$ and $L_0 v = hv_0$ along with $L_n v_0 = 0$ for $n \geq 1$. We need to be able to take inner products. So let's define the expectation value for $v \in M(c,h)$ with $v = kv_0 + \cdots$ and define $\langle v \rangle$ as k.

Using this we define the Hermitian form $H(L_{-n_1} \cdots L_{-n_k} v_0, L_{-m_1} \cdots L_{-m_n} v_0)$ as

$$\langle L_{n_k} \cdots L_{n_1} L_{-m_1} \cdots L_{-m_k} v_0 \rangle.$$

So we call $M(c,h)$ *unitary* if H is positive semi-definite and

$$H(L_n v, w) = H(v, L_{-n} w)$$

so that L_n and L_{-n} are adjoint to one another.

Theorem 16.1. *$M(c,h)$ is unitary for $0 \leq c < 1$ and $h > 0$.*

There is a finite number of allowed values for c and h, given a central charge, the allowed ones are $1 - \frac{6}{m(m+1)}$, there are a finite number of h that are allowed, $h_{p,q} = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}$. You want this to be positive semidefinite and look at some determinant to conclude this.

Then to have an irreducible module, we define $W(c,h)$ to be the quotient of $M(c,h)$ by the kernel of H. Then this will be positive definite and will be an irreducible representation of the Virasoro algebra.

The Hilbert space is constructed as: the Hilbert space of central charge c is the sum

$$\bigoplus W(c, h_i) \otimes W(c, h_i').$$

We want eventually to put a vertex operator algebra structur on this Hilbert space.

I'm running late here. So the vertex operator algebra relevant for the minimal model is V_c where c is one of the values over there. Then as a vector space this has a basis that looks like $L_{-n_1} \cdots L_{-n_k} \Omega$ and the Virasoro algebra acts as usual. At this point this is just a vector space, and we should put some quantum fields on it, $T(z) := \sum L_n z^{-n-2} \in \operatorname{End} V_c[[z^{\pm 1}]]$. Then the condition of being a quantum field is that for any v, we have $L_n v = 0$ for large enough n. This is a condition of being a quantum field. A nice thing is that you can touch and play with these here. A very important property is the *state-field correspondence*, a very nice property of two dimensional cfts. How is this expressed in vertex operator algebras? If you look at that quantum field, the stress energy tensor, you can see that this $T(z)\Omega$ is in $V_c[[z]]$, and if you evaluate at $z = 0$, well, z can have a negative power, but one of the conditions of being a quantum field is that you don't have negative powers, and then this evaluation should be $L_{-2}\Omega$ in the vector space, and $Y(L_{-2}\Omega, Z) = L(Z)$ is the state field correspondence.

Then we need to give something for other vectors in the Hilbert space, so $Y(L_{-n_1}\cdots L_{-n_k}\Omega, Z)$ is

$$\frac{1}{(n_1-2)!}\cdots\frac{1}{(n_k-2)!} : \partial_z^{n_1-2}L(z)\cdots\partial_z^{n_k-2}L(z) :$$

where this is the normally ordered product which you need because otherwise you get ill-defined expressions. I don't have enough time to give you details.

An important property of quantum fields is that they should satisfy this locality contiion

$$(z-w)^*[L(z), L(w)] = 0.$$

For any field $A(z)$ and $B(w)$ and any u and v in V_c we have $H(u, A(z)B(w)v) = H(u, B(w)A(z)v)$. Because of the state field correspondence you can put vectors instead of fields here. And anyway you get something symmetric again.

In this case this depends only on the homology class of $\overline{M}_{0,n}$, so it doesn't depend on the position of the insertion. This is called a *topological* conformal field theory. I used too much time so far.

So for a *topological* vertex operator algebra, the Hilbert space is built in a similar way but we use the topologically twisted $N = 2$ superconformal algebra. The generators are L_n, the $U(1)$-current J_n, then two supersymmetries $Gr^{\pm}$ and the central element Z. I won't write down the relations. We have to change the stress energy tensor, this is a Eguchi-Yang twist (not an A or B twist). So we have $T(z) = T + \frac{1}{2}\partial J$.

Then $G^+_{-\frac{1}{2}}$ is the BRST charge, and the cohomology class with respect to this operator will be finite dimensional and this homology will be the Hilbert space. Let me just mention a few things that I want to apply this to.

I'd like to consider this, I have a problem about orbifolding Frobenius manifolds. This question is relevant, for example, to orbifold quantum cohomology. If you lift this problem to a problem on vertex operator algebras, then this was studied, this was already used in the early days of vertex operator algebras, for example to construct moonshine modules. So I think the natural way of answering this question is to lift this question to one about topological vertex operator algebras. That was my motivation. There are a couple of other applications but I'm already past the time. Thank you.

17. Mike Hopkins: Brauer groups in chromatic homotopy theory

Thank you for coming back to the third and last talk of the conference, the last talk you have to sit through (for me that was the one before this). This does connect to my last two talks, but I realized in preparing my notes that I was going to spend most of the time setting things up. So I'm instead going to jump to the results and connect to the previous ones if there's time. This is joint work, today, with Jacob Lurie, inspired by a remark, let me just say, work, of Vigleik Angeltveit.

Last time I talked about Morava E-theories, these are cohomology theories, E_∞ ring spectra that come out of algebraic considerations, functorial, they come to us for free, and regulate the world of homotopy theory. They have some properties. One is that $E_n^0(*)$ which is $\pi_0 E_n$ is a formal power series ring $W[[u_1, \ldots, u_{n-1}]]$ where W is the ring of Witt vectors of some field of characteristic p which you can think of as being the p-adics although sometimes you want further properties, separability and so on. It's two-periodic like K-theory. So $E_0^*(X)$ is ordinary cohomology, well, cohomoly with coefficients in $\bar{\mathbb{Q}}_p$, but that's not quite right, because that's not two-periodic, so it's really in $\bar{\mathbb{Q}}_p[u^{\pm 1}]$ where u is in degree 2, so that's really made 2-periodic. So then $E_1^*(X)$ is roughly K-theory, but it's really an E_∞ ring spectrum. But the philosophy is that E_n is a decategorification of n-$\mathrm{Vect}_{\mathbb{C}}$, which passes a remarkable number of reality checks.

Hopefully what I'll tell you you'll find interesting even if I don't get to the details of the connections. So E_n is supposed to generalize the ring of vector spaces, and over a ring there are some other things I'd like to do. So one question is whether there are residue fields? That would be something we might call K_n and we might hope that the K_n cohomology of a point, or $\pi_0 K_n$ sholud be $\pi_0 E_n/\mathfrak{m}$, it's a local ring, and then the maximal ideal is $(p, u_1, \ldots, u_{n-1})$. There are no obstructions (or choices) for E_n, it's given to us universally. But modding out an ideal in homotopy theory is something that, that's not something you can do canonically. Let me just tell you some facts about this.

(1) There are E_n-modules K_n with *no* multiplicative structure.
(2) Another fact, among those that do, K_n can *never* have a commutative, it can never be commutative. That's unlike ordinary algebra. In homotopy theory, reducing $\pmod{()p}$ is not something you can do and keep commutativity. Thomas Niklaus was saying recently that reducing modulo p somehow corresponds to shifting n.
(3) A third property, this depends how you define the moduli problem, but there are uncountably many different associative algebra structures, and that's a theorem, a result, due to A. Robinson and also to Vigleik.

The thing that got this project kicked off is this remark due to Vigleik that sometimes these K_n are Azumaya algebras. That gives you a way of approaching studying and classifying these with a little bit more structure. If you think about classifying simple algebras over a field, how do you do that? You introduce this equivalence relation of Morita equivalence, identify them if one is a matrix algebra over the other, then tensor product gives you a lot more structure, you can study these as an Abelian group. So my goal is to describe the classification of these Azumaya E_n-algebras.

So there's various reasons why that's an interesting question and I'll put that off to the end. There are some constructions in the middle that I think are of general interest and I'm already feeling pressed for time. So let me define this term for you.

Definition 17.1. An E-algebra is *Azumaya* if the map $A \otimes_E A^{\mathrm{op}} \to \mathrm{End}_E(A)$ is an equivalence (and A is dualizable as an E-module) There's a missing condition, that the map from E-modules to A-bimodules that sends M to $M \otimes A$ is an equivalence of categories, but let's indulge me that this condition, right.

In the context of field theory, this is saying that A is a dualizable object.

Definition 17.2. We say A and B are Morita equivalent if $A \otimes B^{\mathrm{op}} \cong \mathrm{End}_E(V)$ for some E-module V, (which is again probably a generator of some kind).

Definition 17.3. Then the *Brauer group* of E is the Morita classes of Azumaya algebras under tensor product.

The connection to my previous talks I"ll postpone to the end.

There are things I'm sweeping under the rug, the tensor product (I'm in K_n-local E-modules, for the expert), but this could be an arbitrary symmetric monoidal structure, the generator condition, but let's ignore that.

I want to, so for example, this depends on the category of E-modules. Let me give you a couple of examples. If E is $\mathbb{C}$ and by module I just mean module, so this is $\mathrm{Vect}_{\mathbb{C}}$, then $Br(E) = 0$, the usual Brauer group. If E is $\mathbb{C}$ and by E-module I mean $\mathbb{Z}/2$-graded vector spaces, then this Brauer group is called the Brauer–Wall group and is cyclic of order 2. It's generated by the Clifford algebras, so Cl_1 is a generator and Cl_n is equivalent to Cl_m if $n \equiv m \pmod 2$. Maybe in the interests of time I won't do another example (say, over the reals).

There's another construction I want to mention, then I want to generalize to Morava E-theories. So let me see if I can, right, so here's another construction, or let me just say, more generally, suppose L/K is a Galois extension of fields with Galois group G. Then $H^2(G; L^\times)$, that's the kernel of the map $Br(K)$ (really I should say K-modules but I mean usual ungraded K-modules) to $Br(L)$. Okay, so I want to calculate, describe, the classification of Azumaya E_n-algebras, so the Brauer group of E_n.

There are two things I need to do in order to do this.

(1) I need to construct invertible algebras
(2) I need to classify them.

I want to start by constructing them. I want to look at the examples of the Clifford algebra and this Galois cohomology and find something common to them that works in the general case.

Serre has a beautiful series of lectures in the Cartan seminar on Galois cohomology, and Serre's article on the Brauer group gives an explicit construction from $H^2(G, L^\times)$ to a Brauer element. Let me describe that construction.

What does α in $H^2(G; L^\times)$ classify? It's a group extension P of G by units in L. So the free Abelian group on P, that's the group ring on P, that's an algebra over $\mathbb{Z}[L^\times]$, which maps back to L, so you can look at $\mathbb{Z}[P] \otimes_{\mathbb{Z}[L^\times]} L$. This is Azumaya over K.

Let me translate this maneuver to topology. A group is a group but the free Abelian group corresponds to the suspension spectrum Σ^∞. Now we can just do the analogous construction. I need to turn all these things into spaces. I'm not going to take all the features of the example, but I want to do this for E_∞ rings. Suppose R is an E_∞ ring. It's a spectrum, so it consists of a sequence of spaces, and equivalences $\underline{R}_n \cong \Omega \underline{R}_{n+1}$. Then R corresponds to a cohomology theory and $R^n(X) = [X, \underline{R}_n]$. So there's a map from $\underline{R}_0 \to \pi_0 R$, the components, then among those is $(\pi_0 R)^\times$ and the pullback is $GL_1 R$. An element is a unit if and only if it is if you restrict to every point of the space. So $[X, GL_1 R]$ is $R^0(X)^\times$. This has a group structure and when [unintelligible], this $GL_1 R$ is actually an infinite loop space. I wanted some principal bundle over G whose fiber was the units in some ring. That's classified by some (twisted) map into the classifying space. I'll get to G in a minute but I want to imitate this without the group structure. So imagine I have a space X (eventually G) mapping to $BGL_1 R$. This classifies a principal bundle P over X with fiber $GL_1 R$. So now I can copy Serre and hefine X^ζ as $\Sigma^\infty P_+ \otimes_{\Sigma^\infty GL_1 R} R$ and this is a Thom spectrum. In fact, when R is the sphere spectrum, then $BGL_1 R$ is the classifying space for spherical fibrations, and X^ζ is the usual Thom complex. This is the most twisted up cohomology.

This isn't yet a group.

Definition 17.4. If ζ is a group homomorphism thne X^ζ is an A_∞ ring spectrum.

These are twisted forms of group algebras just like Serre's construction is a twisted form of the L group algebra of G.

Let me do an example. Suppose X is the circle. A map from $S^1 \to BGL_1R$, that's $\zeta \in \pi_1 BGL_1 R = \pi_0 GL_1 R = \pi_0 R^\times$ and you can work out that this is the cone on the map from R to itself given by $1 - \zeta$.

More generally, if $X = S^1 \times \cdots \times S^1$, and I had elements $\zeta_1 \times \cdots \times \zeta_n \to BGL_1R \times \cdots \times BGL_1R$ and then I can go to BGL_1R and that composition is ζ, then X^ζ is the tensor product on the cones of the maps $1 - \zeta_i$, so I get the Koszul complex for killing all of these. This gives me a way to mod out by the elements in a regular sequence.

This formula also connects to the other way to construct Azumaya algebras, the Clifford algebra construction. Let's look at one other example. Let me say one other thing. If $X = G$ then the group homomorphisms from G to BGL_1R is the same as space maps from $BG \to B^2GL_1R$. So I want to connect this, I had two sources of invertible algebras. One was the Serre construction from Galois cohomology or suitably twisted group rings and the other was Clifford algebras, so let me explain the Clifford algebra construction from this point of view.

Now, so, for this, I want to take my ring R to be cohomology with complex coefficients, but I need to make it two-fold periodic. I'm glad I get to do it because when, Clifford algebras are $\mathbb{Z}/2$-graded algebras and that bugs topologists, you multiply things in degree one and you get to degree two, and you say that's the same as something in degree zero, and that involves a choice, say of u. There's a potential for running into things, like in K-theory the Adams operation don't respect the periodicity, they change the choice of the elements u. So it's better to see things as making this 2-periodic, but not specify the element, so $R = H\mathbb{C}[u^{\pm 1}]$ with $|u| = 2$. What's GL_1R? Well $\pi_0 = \mathbb{C}^\times$. Then $\pi_1 = 0$, $\pi_2 = \mathbb{C} = \pi_4 = \cdots$, so it looks like it's probably a product of Eilenberg–MacLane spaces. That's true, there are no possible ways to connect these. We're interested in BGL_1R, which has $\pi_2 = \mathbb{C}^\times$ and $\pi_4 = \pi_6 = \cdots = \mathbb{C}$, and the odd ones are zero. I'm interested in homotopy classes of maps from X (which is BG) to B^2GL_1R, and from the description of B^2GL_1R, this looks like it might be $H^2(X, \mathbb{C}^\times) \oplus H^2(X, \mathbb{C}) \oplus H^4(X, \mathbb{C}) \oplus \cdots$ and that's true, but it's not a canonical isomorphism. This wouldn't be true any more over $\mathbb{Z}/p$; these would be connected in some odd way. I shouldn't write this in this way. I should remember that there's an exponential map

$$\exp : \mathbb{C}[[u]] \to \mathbb{C}[[u]]^\times$$

and if I use that exponential map, this gives me an isomorphism from $[X, B^2GL_1R] = H^2(X, \mathbb{C}^\times) \oplus \cdots$

This doesn't affect the result but it affects the way I think about the result. These should be thought of as the exponential of some cohomology classes. But we'll get to that in a minute. Okay.

So I want to take X to be BG where G is a torus. So what is homotopy classes of maps from X to BGL_1R in this case? Let me write Λ for $\pi_1 G$. Then the component in degree two is a map $\Lambda \to G^\times$ and $j \in H^4(BG)$ is a quadratic form q on Λ, and the Thom spectrum, we loop this, since time is running out let me just tell you that the Thom spectrum is the Clifford algebra of q. This was one of the things I wanted to convey, you might like to know this outside the world of chromatic homotopy theory. All of these are part of this one Thom complex construction.

Now I want to take E to be the nth Morava E-theory and look at this map, X will be BG for G the n-torus, and I want to look at maps from BG to $B^2GL_1E_n$. This has lots of pieces associated to it just as in the cases of the complex numbers. The first part is a linear map $\zeta_1 : \Lambda \to W[[u_1, \dots, u_{n-1}]]^\times$. One fact is that $G^\zeta \sim *$ if ζ_1 has image outside $(1+\mathfrak{m})^\times$. So I want to think that this sits in $(1+\mathfrak{m})^\times$ and from there I can go to $\mathfrak{m}/\mathfrak{m}^2$ by $1+\alpha$ to $\alpha \pmod{()\mathfrak{m}^2}$.

Then a second fact is that G^ζ is a K_n if and only if $\Lambda \to \mathfrak{m}/\mathfrak{m}^2$ extends to an isomorphism $\Lambda \otimes k \to \mathfrak{m}/\mathfrak{m}^2$ for $k = \pi_0 E/\mathfrak{m}$.

Now the next piece of *zeta* is a quadratic form $q : \Lambda \to \pi_2 E$

Then the third fact is that G^ζ is Azumaya if and only if q is non-degenerate.

Definition 17.5. A *polarized* n-torus is a torus $(S^1)^n$ equipped with a map from $BT \to B^2GL_1E_n$ satisfying my conditions, two and three, so that T^ζ is a K_n and an Azumaya algebra.

Theorem 17.1. *(H., Lurie) Every K_n Azumaya algebra comes from a polarized torus.*

So that's one construction and this leaves me making a decision on the fly. There's only about forty minutes left, no just kidding, only ten minutes left. There are experts who want to hear about the classification. But there's more things I can say about the relationship to physics. The topologists who wants to hear the details are my friends so if I disappoint them they'll still be okay, I can apologize later. There's a nice scheme for classifying these structures, but I want to step back and say something about what this classification is supposed to *mean*.

So I'll just say that once we work this out, the classification, you get the group, which comes filtered, you get the associated graded, you get a construction of every element of the group, but we don't quite have the group structure. I want to describe a map from this group to something else that should be an isomorphism. I'll try to be somewhat quick about this. There's a funny thing that these chromatic homotopy theories have. I started with Clifford algebras to emphasize, among other things, that there's a logarithm or exponential. In chromatic homotopy theory there's a logarithm that has no right to be there, there's a formula for it due to Charles Rezk, and it's an infinite loop map from $GL_1E \to E$. For reasons I can't get into this gives a map from the Brauer group of E to $\pi_{-2}E$, which is supposed to be a logarithm.

Some version of this is supposed to be an isomorphism, and I'll have to tell that to my friends later. But the problem is to describe the composition that goes from polarized tori to elements in the Brauer group of E to $\pi_{-2}E$, that's something we know, a free [unintelligible]over the formal power series ring. There's some modifications that make this look like a monomorphism.

I want to end by talking about what is going on in the question, reminding you that this is some sort of logarithm. I'll try to wrap this up sort of quickly, but this is a more philosophical note that it might be interesting to know about. I have this map from BG to B^2GL_1E? What does a map into this classify? A point in GL_1E is a unit, a point in BGL_1E is a torsor, a unit in E-modules. So this second one will classify categories non-canonically equivalent to E-modules. So this classifies a bundle of E-linear categories all of rank 1. Because I said B^2GL_1E, they're non-canonically equivalent to E-modules. If I take this category and take its space of sections. So G^ζ, that's the space of sections of this bundle. If you want that says that G^ζ is the integral over BG of whatever ζ was, which is like a quadratic exponential e^q. This is the part that's canonically picked out. You can make this totally rigorous, that the Morita class of G^ζ, it's really an integral of this family of E-linear categories. But it's a quadratic exponential, an exponentiated classical action. So this first map is an integral of a quadratic exponential, and the map from Brauer groups to $\pi_{-2}E$ is a logarithm, so this is supposed to look like the thing that you do, a logarithm of an integral of an exponential, a Feynman path integral formula, that's all, that's what we're seeking now, that's supposed to be a decategorification of something that, remember, well maybe I'd better stop, this is some decategorification of some construction of a topological field theory by an exponential action. I'll stop there and say thank you everybody for bearing with me and it's been a great conference.