CGP INTENSIVE LECTURE SERIES
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1. May 11: On entropies of autoequivalences on smooth projective varieties

I'd like to thank Jae-Suk but unfortunately he isn't here. Today I'll talk on this purpose. Today X will be always a smooth projective variety over $\mathbb{C}$ and by $D^b(X)$ I mean the bounded derived category of coherent sheaves on X.

To (X, f) (f is for simplicity an automorphism) we can consider a dynamical system, the set of maps $\{f^n\}_{n\geq 0}$. To this data we can consider the important invariant, the "topological entropy." I won't define this. But it's a real number which measure the complexity of the system. Instead we can consider a pair $(D^b(X), F)$, where F is a functor, for simplicity an autoequivalence. Then we get a dynamical system on the derived category, what we're interested in is $\{F^n\}$, the sequence of autoequivalences.

In 2013, Dmitrov–Haiden–Kartzarkov–Kontsevich introduced the notion of entropy of a functor $h(F)$, measuring the complexity of this system. Our main interest is this object, and one of our main theorems is the comparison of topological and categorical entropy with F induced by f.

We'll describe entropy, then our main theorems, and finally give a sketch of the proof of the main theorem.

For simplicity, I'll call this DHKK entropy, and it's defined, measuring the complexity of the following object. Take M and N in the derived category $D^b(X)$, and define a function $\delta_t : (M, N) : \mathbb{R} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by

$$\delta_t(M,N) \coloneqq \inf \left\{ \sum_{i=1}^{p} e^{n_i t} \middle| \right.$$

$$\left. \begin{tikzcd} 0 \arrow[rr] & & A_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & A_{p-1} \arrow[rr] & & N \oplus N' \arrow[dl] \\ & M[n_1] \arrow[ul] & & \cdots & & M[n_p] \arrow[ul] & \end{tikzcd} \right\}$$

where the triangle is exact, and this is called the complexity of N with respect ot M. If G is a split generator of $D^b(X)$, then $1 \leq \delta_0(G, M) < \infty$.

If X is a point, then $\delta_t(\mathbb{C}, M) = \sum_{\ell\in\mathbb{Z}} \dim_{\mathbb{C}} H^\ell(M) e^{-\ell t}$

Definition 1.1. If G is a split generator of $D^b(X)$, then

$$h_t(F) \coloneqq \lim_{n\to\infty} \frac{1}{n} \log \delta_t(G, F^n G)$$

and $h_t(F)$ is independent of G, and the "limit" exists.

Now I'd like another way to explain this entropy:

$$h_t(F) = \lim_{n\to\infty} \frac{1}{n}\delta_t(G, F^n G')$$

for G and G' split generators of $D^b(X)$.

To show this statement, we need the following properties of the complexity function.

$$\delta_t(M_1, M_3) \le \delta_t(M_1, M_2)\delta_t(M_2, M_3).$$

If $F : D^b(X) \to D^b(X)$, then $\delta_t(F(M), F(N)) \le \delta_t(M, N)$.

Then the proof is that $\delta_t(G, F^n G') \le \delta_t(G, F^n G)\delta_t(F^n G, F^n G')$ by the triangle property, which in turn is $\le \delta_t(G, F^n G)\delta_t(G, G')$, which is independent of n; the other direction is also easy.

Lemma 1.1. *(1) $h_t(F) = h_t(F')$ if $F \cong F'$.*
(2) $h_t(F^m) = m h_t(F)$
(3) $h_t([m]) = mt$
(4) If $F_1 \circ F_2 \cong F_2 \circ F_1$ (autoequivalences), then $h_t(F_1 \circ F_2) \le h_t(F_1) + h_t(F_2)$
(5) $h_t(F \circ [m]) = h_t(F) + mt$
(6) $h_t(F_1 \circ F_2 \circ F_1^{-1}) = h_t(F_2)$

and so $h_t : \mathrm{Auteq}(D^b(X)) \to \mathbb{R} \cup \{\infty\}$

This is similar to the conjugate invariant norms of the lunch seminar today. There is a degenerate locus and this commutativity, but it's close. Anyway. The original definition, complexity is quite difficult to calculate. But in some cases it's easy to calculate.

Theorem 1.1. *(DHKK, 2.6)*

$$h_t(F) = \lim_{n\to\infty} \frac{1}{n} \log \delta'_t(G, F^n G')$$

where δ'_t is given by

$$\delta'_t(G, F^n G) \coloneqq \sum_{\ell\in\mathbb{Z}} \dim_{\mathbb{C}} \mathrm{Hom}(G, F^n G'[n])e^{-nt}$$

Remark 1.1. To prove this theorem or use properness, we need properness and smoothness of $D^b(X)$, and a dg enhancement. We always have this so we can just forget about it.

Now let me recall some notation. Consider the following. $\chi(M,N) \coloneqq \sum_{\ell\in\mathbb{Z}}(-1)^\ell \dim_{\mathbb{C}} \mathrm{Hom}(M, N[\ell])$, the Euler form. Introduce $N(X)$, the quotient of $K_0(X)$ by $[unintelligible]\chi$, the [unintelligible]Grothendieck group. In our case, this is a free Abelian group of finite rank.

The important point is that we have a group homomorphism from autoequivalences of X to $\mathrm{Aut}_{\mathbb{Z}}(N(X), \chi)$ which takes F to $[F]$, and

Definition 1.2. The *spectral radius* ρ of $[F]$ is

$$\rho([F]) \coloneqq \max\{|\lambda| | \lambda \text{ is an eigenvalue of } [F]\}.$$

From now on, $t = 0$, we let $\delta(M, N) = \delta_0(M, N)$, and $h(F) = h_0(F)$, and so on.

Theorem 1.2. *(Main theorem and conjecture, joint with Kituta) The conjecture says that $h(F) = \log \rho([F])$.*

The theorem says that if F is $\mathbb{L}f^*$ for $f : X \to X$, then $h(\mathbb{L}f^*)$ coincides with $h_\top(f)$.

Next,

Theorem 1.3. *If K_X or $-K_X$ is ample, then $h(\mathbb{L}F^*)$ is $\log\rho([F]) = 0$.*

Theorem 1.4. *(Kikuta) If X is an elliptic curve, then $h(F) = \log\rho([F])$*

So we can calculate entropy using linear algebra. In particular $h(F)$ is an algebraic number. Now I want to recall the famous theorem by Gromov and Yongdin which gives a motivation to consider such a conjecture.

Theorem 1.5. *The topological entropy $h_{\mathsf{T}}(f) = \log\max r_q(f)$ where $r_q(f) := \max\{|\lambda|\lambda$ an eigenvalue of $|f^*|_{H^{q,q}(X}$*

It is also known that $r_q(f) = \lim_{n\to\infty}\left(\int_X \omega^{d-q}\wedge(f^n)^*\omega^q\right)^{\frac{1}{n}}$ where ω is a Kähler form. Or it is also

$$\log\rho(f^*)$$

where f^* is the automorphism of $H^*(X)$. So this is a categorical version.

Now I explain how to prove this.

The point is to use projectivity. The key words are Kodaira vanishing and the HRR formula. Those are key.

So we write G as $\bigoplus\mathcal{L}^i$ where $\mathcal{L}$ is a very ample line bundle. This G is a generator by Orlov's theorem.

So first we show

$$h(\mathbb{L}f^*) = \lim_{n\to\infty}\frac{1}{n}\log|\chi(G,(f^*)^nG^*)|$$

which is similar up to sign with the Euler number.

The easy part is to show $\geq$; this is very easy because $|\chi(G,(f^*)^nG^*)| \leq \delta'(G,(f^*)^nG^*)$ and the important part, the key part, uses Kodaira vanishing. Here

$$Hom(G,(f^*)^nG^*[m]) = 0$$

if $m \neq \dim_{\mathbb{C}} X$, which follows from Kodaira vanishing.

Therefore $\delta'(G,(f^*)^nG^*) = (-1)^{\dim_{\mathbb{C}} X}\chi(G,(f^*)^nG^*)$.

Now we use Hirzebruch–Riemann–Roch, $|\chi(G,(f^*)^nG^*)|$ is

$$(-1)^{\dim_{\mathbb{C}} X}\int_X \mathrm{ch}(G^*)\,\mathrm{ch}((f^*)^nG^*)[\mathit{unintelligible}]$$

and then we obtain

$$\sum_{r=0}^{d}\sum_{q=0}^{d-r} c_{r,q}\int_X c_1(\mathcal{L})^{d-r-q}\,\mathrm{Td}_r(x)(f^*)^n c_1(\mathcal{L})^q$$

where $c_{r,q}$ is rational and $c_{0,q} > 0$

Then we use the following.

Proposition 1.1. *(Dinh–Sibony)*

$$\left(\int_X (f^*)^n\omega^q\wedge\omega^{d-q}\right)^{\frac{1}{n}} \sim n^{\ell_q(f)}r_q(f)^n$$

We discussed r_q but n_q, well, $H^{q,q}$ can be decomposed into $H^{q,q}(X)_{\lambda_1,m_1}\oplus\cdots\oplus H^{q,q}(X)_{\lambda_s,m_s}$, and here f^* acts on $H^{q,q}(X)_{\lambda,m}$ by Jordan blocks $J_{\lambda,m}$. Then ℓ_q is m_1 (where r_q was $|\lambda_1$).

We know from what we've done that

$$\int_X c_1(\mathcal{L})^{d-q}(f^*)^n c_1(\mathcal{L})$$

for some leading term q. Then finally we obtain that h, well

$$\lim_{n\to\infty} = \lim \frac{1}{n} \log|\chi(G,(f^*)^*G^*)] = \log \max r_q f$$

Then define

$$h(\mathbb{L}f^*) = \log \max r_q f \geq \log \rho([F])$$

so

$$\lim_{n\to\infty} \log|\chi(G(f^*)^n G^*) \leq \log \rho[F] \leq \log \max r_q(f)$$

so $h(\mathbb{L}f^*) = \log \rho(F) = hopi$.

So

Remark 1.2. If $D^b(X)$ is coarser than D^bY then $X \cong Y$ implies $D^b(X) \cong D^b(Y)$ but not the counterpart.

Now I want to explain the proof of the second theorem.

Let $h(\mathbb{L}f^*) = 0$. Then choos G as $\bigoplus (K_X^{-1})^{mi}$ so that $\mathcal{L}$ is very ample. Since our construction doesn't depend on the choice of generator, because G is chosen in this way, $f^*G = G$, so $\delta(G,(f^*)^n G) = \delta(G,G)$, so the entropy is 0 since this doesn't depend on n.

For the second part, use the following famous theorem, by Bondal–Orlov

Proposition 1.2.

$$\mathrm{Auteq}(X) = \mathrm{Aut}\ X \ltimes (\mathrm{Pic}(X) \times \mathbb{Z})$$

So this group of autoequivalences always should contain this part but with our hypotheses it's as small as possible.

Then any autoequivalence F is given by

$$F(\quad) = \mathbb{L}f^*(\quad \otimes_X^{\mathbb{L}} \mathcal{L})[a]$$

for some $f \in \mathrm{Aut}\ X$, $\mathcal{L} \in \mathrm{Pic}(X)$, and $a \in \mathbb{Z}$.

Please recall the property that we can forget about $[a]$ because $h_t F \circ [m]$ is $h_t(F) + mt$ and now t is zero. Then we calculate

$$F = \mathcal{L}f^*(\quad \otimes_X^{\mathbb{L}} (\mathcal{L} \otimes K_X^m)) \circ K_X^{-m}$$

so that $\mathcal{L} \otimes K_X^m$ is very ample. So we can choose m in $\mathbb{Z}$ to do this.

This is possible because K_X and f^* commute. Here the important point is that we can decompose this as a composition, and these two parts commute, so we can evaluate the entropy in terms of the sum of the two functors,

$$h(F) \leq h(\mathbb{L}f^*(\quad \otimes_X^{\mathbb{L}} (\mathcal{L} \otimes K_X^m))) + h(\quad \otimes K_X^{-m}).$$

Lemma 1.2. *For $\mathcal{L}'$ in* Pic*, we have $h(\quad \otimes_X \mathcal{L}') = 0$.*

This is based on a famous fact in algebraic geometry that the image of a hom space of this kind, the growth is at most polynomial order. Since we consider limits like $\frac{1}{n}\log(\quad)$, which is zero since the growth is of polynomial order on the inside. So $h(\quad \otimes K_X^{-m})$ gives zero.

Now for the calculation of f, we should calculate just the left factor. I'll write F_1 for $\mathcal{L} \otimes K_X^m$, and then $h(F_1)$ is

$$\lim_{n\to\infty} \frac{1}{n} \log \delta(G, G^* \otimes \mathcal{L} \otimes K_X^m \otimes f^*(\mathcal{L} \otimes K_X^m) \otimes \cdots \otimes (f^n)^*(\mathcal{L} \otimes K_X^m))$$

which is

$$\lim_{n\to\infty}\frac{1}{n}\log\delta'(G,G^*\otimes\cdots)$$

and by Kodaira vanishing again, this is

$$\lim_{n\to\infty}\frac{1}{n}\log|\chi(G,G^*\otimes\cdots)| = \lim_{n\to\infty}\frac{1}{n}\log|\chi(G,(F')^nG^*)|$$

and again by linear algebra this is at most

$$\log\rho([F_1])$$

and because $\mathcal{L}'$ acts on the numerical Grothendieck group by an uppper triangular matrix with identity on the diagonal blocks. Then the spectral radii coincide to make this $\log\rho([\mathcal{L}f^*])$ and this is 0, so this is the second theorem.

Now I want to comment Kikuta's result. Assume $\dim_{\mathbb{C}} X = 1$. For genus 0, this is $-K_X$ ample, and genus 1 is elliptic curve. If $g > 1$ then K_X is ample. So we already gave a calculation for $g = 0$ and $g > 1$. So the elliptic curve case is the only one remaining. The theorem says if X is an elliptic curve, then $h(F)$ for F an autoequivalence is $\log([F])$.

The key point is the following (for details ask him): first consider $\mathrm{Auteq}(X) \to \mathrm{Aut}(N(X),\chi)$, which is just $SL(2,\mathbb{Z})$, this is the point, and $h : \mathrm{Auteq}(X) \to \mathbb{R}_{\geq 0}$ is a class function, this h has the same value along conjugacy classes. Also, we need the following, we have a short exact sequence

$$0 \to \mathrm{Aut}(X) \ltimes \mathrm{Pic}^0(X) \times 2\mathbb{Z} \to \mathrm{Auteq}(D^b(X)) \to SL(2,\mathbb{Z}) \to \{1\}$$

and $h(F)$ is zero on the first term, because, well, $2\mathbb{Z}$ is generated by $[2]$ translation, and Pic^0 is line bundles of degree 0. Anyway, the entropy vanishes, we can check, on the normal subgroup, so we should calculate the value of each conjugacy class of $SL(2,\mathbb{Z})$.

If $[F]$ is in $SL(2,\mathbb{Z})$ is of finite order, then $\tilde{h}([F]) = 0$. Then we can check equality easily.

So the problem is the rest of $SL(2,\mathbb{Z})$, the essential part, then you use the classification of conjugacy classes by Karpenkov, known as LLS period, and calculate an autoequivalence of the following form: $S^2T^{m_{2n}}ST^{-m_{2n-1}}\cdots ST^{-m_1}S$, then S and T are the "standard" generators of $SL(2,\mathbb{Z})$, and for this kind, you can calculate these, which correspond exactly to conjugacy classes of $SL(2,\mathbb{Z})$, and the last part is somehow, some technical thing.

This is in some sense expected because, I want to explain in two ways.

Remark 1.3. S is known as "Fourier–Mukai transform" which interchanges rank and degree (very naively speaking). So this takes a structure sheaf to skyscraper sheaf, and vice versa. So this mixes H^0 and H^2. Therefore, in that way, $S : D^b(X) \cong D^b(X)$, but on the left side $\mathbb{L}f^*$, if we move it to the other side, $S^{-1}\mathbb{L}f^*S$, this is not of the form $\mathbb{L}g^*$ for any g. But always for automorphism we have a [unintelligible]theorem, so it's natural to expect this kind of statement. In some sense this is independent or invariant under this kind of transform. But because of this picture, for this functor we can expect the analog of the theorem.

Also we can consider mirror symmetry. The remark then is that $D^b(X) \cong D^b(\mathrm{Fuk}(Y))$. Then if this holds, then very roughly speaking, this $N(X)$ should correspond to middle homology groups of Y but if this holds, then this is a categorical equivalence, the autoequivalence group should coincide. So if some good automorphism on Y, if we had one, then it would define an autoequivalence on X, and in this case, because this Y is a symplectic manifold and we can define topological entropy, and we have that kind of theorem on Y, so it's natural to expect a similar statement on X. This explains the more general conjecture.

Anyway, so if we could have an equivalence between the algebraic and geometric one, based on the algebraic calculation we could calculate the geometric invariants on the geometric side. Unfortunately so far we have not calculated nontrivial invariants on the geometric side. But that's for next time.

2. May 12: On orbifold Jacobian algebras for invertible polynomials

I have completely changed subject, and this is related to my talk in January. The object I'm considering is rather different. First I prepared some notation. Let f be a polynomial in N variables $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that the Jacobian ring $\mathbb{C}[x_1, \ldots, x_n]/(\partial_1 f, \ldots, \partial_N f)$ is finite dimensional $\mathbb{C}$-vector space. This defines an isolated singularity at the origin. We consider another object associated with f, called Ω_f, this is a globally defined N form $\Omega^N_{\mathbb{C}^N}$, divided by $df \wedge \Omega^{N-1}_{\mathbb{C}^N}$. Let me say, if you fix a nowhere vanishing N form ω, then $\mathrm{Jac}(f) \overset{\cong}{\to} \Omega_f$. One of the most important properties is the existence on this ring of the symmetric bilinear form J_f which is non-degenerate $\Omega_f \times \Omega_f \to \mathbb{C}$, defined by, well, write

$$\omega : [\phi dx_1 \wedge \cdots \wedge dx_N]; omega' : [\psi dx_1 \wedge \cdots \wedge dx_N]$$

then

$$J_f(\omega, \omega') = \mathrm{Res}_{\mathbb{C}^n} \begin{bmatrix} \phi\psi dx \wedge \cdots \wedge dx_N \\ \partial_1 f \partial_2 f \cdots \partial_N f \end{bmatrix}$$

and then $\mathrm{Jac}(f)$ is a Frobenius algebra, that is, there exists a symmetric non-degenerate pairing $\eta : \mathrm{Jac}(f) \times \mathrm{Jac}(f) \to \mathbb{C}$ such that

$$\eta(XY, Z) = \eta(X, YZ).$$

Now we can consider the generalization $(Jac(f), \Omega(f)) \rightsquigarrow (HH^{\cdot}(T), HH_{\cdot}(T))$, where this is an example, when T is $HMF(f)$ (the triangulated category of matrix factorizations of f) or $(\mathbb{C}[\underline{x}], f)$.

We want to study the pair $(\mathrm{Jac}(f, G), \Omega_{f,G})$ for a "Landau–Ginzburg orbifold."

First, Kaufmann considered this kind of object where $\mathrm{Jac}(f, G)$ is in correspondence with "2-cocycles," introducing the notion of the G-twisted Jacobian algebra of f. Later Krawitz gave a construction of $\mathrm{Jac}(f, G)$ but ist is not $\mathbb{Z}/2\mathbb{Z}$-graded.

We want to show the existence and uniqueness for some pair (f, G) and to do this we'll restrict to invertible polynomials. Our definition, though, is quite general, so I'll start by writing the condition for this pair.

So first I want to introduce a group of symmetries of f. First introduce

Definition 2.1. The largest Abelian group, the maximal Abelian symmetry, is

$$G_f = \{(\lambda_1, \ldots, \lambda_N) \in (\mathbb{C}^*)^N | f(\lambda_1 x_1, \ldots, \lambda_N x_N) = f(x_1, \ldots, x_N)\}.$$

Definition 2.2. We call a pair (f, G) where G is a finite subgroup of $G_f \cap SL(N, \mathbb{C})$ a *Landau–Ginzburg orbifold*

Today we'll restrict to the special case where G is a subspace of $G_f \cap SL(N, \mathbb{C})$, which gives us a G-invariant holomorphic N-form ω. Then we get the existence of the isomorphism between $\mathrm{Jac}(f)$ and Ω_f. So I should give some more notation.

For $g \in G$, I'll use $\mathrm{Fix}(g)$ to denote the fixed locus $\{\underline{x} \in \mathbb{C}^N | g \cdot \underline{x} = \underline{x}\}$, a linear subspace, and f^g is $f|_{\mathrm{Fix}(g)}$ and this is again an isolated singularity, and we have a surjective algebra homomorphism

$$\mathrm{Jac}(f) \twoheadrightarrow \mathrm{Jac}(f^g).$$

I'll also need the *age* of g for a $\mathbb{Z}/2\mathbb{Z}$ supergrading, but I'll omit it. As an example, if

$$g = (e^{2\pi\sqrt{-1}\frac{a_1}{r}}, \dots, e^{2\pi\sqrt{-1}\frac{a_N}{r}})$$

with $0 \le a_i < r$ (where r is the order of g), the age of g is $\sum_{i=1}^N \frac{a_i}{r}$

Now I can introduce $\Omega_{f,G}$.

Definition 2.3. Introduce a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-module

$$\Omega'_{f,G} \coloneqq (\Omega'_{f,G})_{\overline{0}} \oplus (\Omega'_{f,G})_{\overline{1}}$$

each one of these itself $\mathbb{Z}$-graded, they are

$$(\Omega'_{f,G})_i = \bigoplus_{\substack{g \in G \\ N - Ng \equiv i \pmod 2}} \Omega'_{f,g}$$

where N_g is $\dim_{\mathbb{C}} \mathrm{Fix}(g)$ and $\Omega'_{f,g} \coloneqq \Omega_{f^g}$ if $\mathrm{Fix}(g)$ is nontrivial and for $g \in G$ where $\mathrm{Fix}(g) = \{0\}$, we have $\Omega'_{f,g} \coloneqq \mathbb{C}|_g$ with this space generated by the symbol $|_g$.

There is a pairing $J_{f,G} : \Omega'_{f,G} \times \Omega'_{f,G} \to \mathbb{C}$ by $J_{f,G} \coloneqq \bigoplus J_{f,g}$ where

$$J_{f,g} : \Omega'_{f,g} \times \Omega'_{f,g'} \to \mathbb{C}$$

where $J_{f,g}$ is $(-1)^{N - N_g - \mathrm{age}(g)} |G/K_g| |K_g|^{-1}$ and this gives a nondegenerate bilinear form. Here $K_g \subset G$ is the maximal subgroup fixing $\mathrm{Fix}(g)$.

For these vector spaces $J_{f,g}(|_g, |_{g^{-1}})$, we define this as $(-1)^{N - \mathrm{age}(g)} \frac{1}{|G|}$.

Definition 2.4.

$$\Omega_{f,G} \coloneqq (\Omega'_{f,G})^G.$$

As a $\mathbb{Z}/2\mathbb{Z}$-vector space, this is isomorphic to Hochschild homology of the category of $\mathbb{Z}$-graded G-equivariant matrix factorizations of f (up to a shift by N). We also have the Hochschild cohomology of this category, but it's quite difficult to get the product structure.

So we want to propose the algebra that would satisfy some conditions, and we want to show the existence and uniqueness of such an algebra structure compatible with this module. This is an open problem for, in the theory of Landau–Ginzburg orbifolds, up to twisted sectors much work has been done, but the product structure in twisted sectors is difficult, quite different.

The next object will be key in our story.

Definition 2.5. Let $\mathrm{Aut}(f,G)$ be the automorphisms $\{\varphi \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[\underline{x}] | \varphi(f) = f, \varphi \circ g \circ \varphi^{-1} \in G$ for all $g\}$ where here we identify G and the subgroup of $\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[\underline{x}])$.

Remark 2.1. $\mathrm{Aut}(f,G)$ is isomorphic to $\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[\underline{x}] \star G)$, where $*$ is the skew group ring.

Then $\varphi \in \mathrm{Aut}(f,G)$, well, $\varphi : Fix(\varphi \circ g \circ \varphi^{-1}) \to \mathrm{Fix}(g)$

This subgroup does not appear in Kaufmann's or Krawitz' work, and it will help with uniqueness of the product later.

Now we can define:

Definition 2.6. The *G-twisted Jacobian algebra of f* is, well,

(1)
$$\mathrm{Jac}'(f,G) \coloneqq (\mathrm{Jac}'(f,G))_{\overline{0}} \oplus (\mathrm{Jac}'(f,G))_{\overline{1}}$$
where each component has the same parity, like before:
$$(\mathrm{Jac}'(f,G))_{\overline{i}} \coloneqq \bigoplus_{\substack{g\in G\\ N-N_g\equiv i \pmod 2}} \mathrm{Jac}'(f,g)$$
and
$$\mathrm{Jac}'(f,g) \cong \Omega'_{f,g}$$
for all g in G as $\mathbb{C}$-vector spaces.

(2) Then there is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-algebra structur $\circ$ on $\mathrm{Jac}'(f,G)$ where $\mathrm{Jac}'(f,g)\circ\mathrm{Jac}'(f,h) \subset \mathrm{Jac}'(f,gh)$ for g and h in G. Further, the subalgebra $\mathrm{Jac}'(f,\mathrm{id})$ is isomorphic to $\mathrm{Jac}(f)$. Further, $\Omega'(f,G)$ is a free $\mathrm{Jac}'(f,G)$-module of rank 1. For example, we can chose a holomorphic volume form $\omega = [dx_1\wedge\cdots\wedge dx_N]$, always G-invariant. Then this induces an isomorphism $\vdash$ between $\mathrm{Jac}'(f,G)$ and $\Omega'(f,G)$.

(3) This module satisfies that
   - $\mathrm{Jac}'(f,g)\vdash\Omega'_{f,h}\subset\Omega'_{f,gh}$. The $\mathrm{Jac}'(f,\mathrm{id})$-module structure is the same as the module structure from $\mathrm{Jac}(f^g)$. This is a compatibility condition.

(4) Also, maybe, by using the $\vdash$ correspondence, we have the $\mathrm{Aut}(f,G)$-action defined by
$$\varphi^*(X)\vdash\varphi^*(\omega) \coloneqq \varphi^*(X\vdash\omega).$$
We can define the pullback using the invariance property. Then we get an action like this, and the condition is that this is $\varphi^*(X)\circ\varphi^*(Y) = \varphi^*(X\circ Y)$, which is natural but a very strong condition. Also $X\circ Y = (-1)^{\overline{XF}}g^*(Y)\circ X$ for X in $\mathrm{Jac}'(f,g)$. So we should have a G-twisted commutative algebra here.

Now we can talk about compatibility with the bilinear form.

(5)
$$J_{f,G}(X\vdash\zeta,\zeta') = (-1)^{\overline{X\zeta}}J_{f,G}(g^*(\zeta)X\vdash\zeta')$$
for invariance of $J_{f,G}$ to later give a Frobenius algebra.

Finally,

(6) we have a universality property on G, which I'll omit because I didn't check the full detail.

The problem is that there might be more than one such twisted algebra for the pair f and G.

Conjecture 2.1. There exists a G-twisted Jacobian algebra for (f,G), in particular the G-invariant part is uniquely determined (up to isomorphism).

If this is defined, then

Definition 2.7. $\mathrm{Jac}(f,G)$ is the G-invariant part of $\mathrm{Jac}'(f,G)$, and we call this the *orbifold Jacobian algebra of* (f,G)

Theorem 2.1. *(Basalaev–T.–Werner) The conjecture is true for f an invertible polynomial and $G\subset G_f\cap SL(N,\mathbb{C})$ and $N\le 3$; in this case conditions* 1 *through* 5 *are necessary.*

Theorem 2.2. *("theorem", Basalaev–T.–Werner) The conjecture is true for f invertible and $G \subset G_f \cap SL(N, \mathbb{C})$. In this case condition 6 is also necesary.*

Theorem 2.3. *(Basalaev–T.–Werner) Let f be an invertible polynomial in $N = 3$ defining an ADE type singularity, so $G \subset G_f \cap SL(3, \mathbb{C})$. Then we can consider a crepant resolution $\hat{f}$ of $\mathbb{C}^3/G$, and inside the resolution $\widehat{\mathbb{C}^3/G}$ we have a chart, $\mathbb{C}^3$ with a singularity, call this $\bar{f}$. Then $\mathrm{Jac}(f, G) \cong \mathrm{Jac}(\bar{f})$.*

This is expected from the geometrical point of view.

I don't have enough time, but I should give a definition:

Definition 2.8. Since f is a polynomial, we can expand it

$$f = \sum_{i=1}^{N} a_{ij} \prod_{j=1}^{N} x_j^{E_{ij}}$$

so the number of variables and the number of monomials agree. We say f is *invertible* if it has this form and $E = E_{ij}$ is invertible over $\mathbb{Q}$.

A typical example, well, we have a classification, $x_1^{a_1} x_2 + \cdots + x_{N-1}^{a_{N-1}} x_N + X_N^{a_N}$ (*chain type*) or $x_1^{a_1} x_2 + \cdots + x_{N-1}^{a_{N-1}} x_N + X_N^{a_N} x_1$ (*loop type*), I guess I should say, these are just examples.

By rescaling we can throw a_{ij} away, so any such f is a direct sum of these ones.

For these invertible polynomials, we have this uniqueness statement. This is based on elementary combinatorial methods. In the general case we can directly write down a product structure, once we fix a basis, and by using something like the Hessian of f, we can write down the product, up to a (quite complicated) sign.