

1. CHAIN LEVEL OPERATIONS

There's an equivariant story and a non-equivariant story, and this morning I'm only going to talk about the non-equivariant story. You remember, Chas-Sullivan operations were defined on the chain level, but only partially, but they induced a full structure on homology. Now I'll be defining full-level operations, but we'll pay the price for this in the sense that our chain model and construction will be technical.

Let M be an oriented closed d -dimensional complete Riemannian manifold. Let r be the injectivity radius (meaning balls of radius less than r are contractible). We'll define a chain model P_* for $H_*(LM)$ and a chain map $C_*(\overline{SD}(g, k, \ell))$ and we'll assign to this an operation in $\text{Hom}(P_*^{\otimes k}, P_*^{\otimes \ell})$, and this will be called the string topology construction ST . We could use Hom tensor duality, and then this is a map $C_*(\overline{SD}(g, k, \ell)) \otimes P^{\otimes k} \rightarrow P^{\otimes \ell}$.

In fact, the collection $C_*(\overline{SD}(g, k, \ell))$ form a properad. Let me describe the ideas, this is going to be a departure. Fix $\epsilon < r$

- (1) let me draw two loops in the manifold, with two points within an ϵ -ball of one another. What does it mean? They're really close, not necessarily intersecting, but close. Here's where I'll use the Riemannian metric. These two points can be connected by a unique geodesic arc that looks like a chord.

Before we'd have to collapse the tree or chord and get true intersection. Here I'll get a little arc. I won't detect real intersection. I'm detecting something relaxed, they're close. This is a unique geodesic arc. Call that idea "diffuse intersection."

- (2) Let μ be a d -dimensional cochain in $C^d(M \times M, M \times M - N_\epsilon)$, where N_ϵ is an ϵ neighborhood of the diagonal. Let this represent the Thom class of the diagonal. The class exists because we're using a manifold. I forgot the name of the McCrory paper. It's "On the classification of homology manifolds." I was going back and forth. Think of μ as a cochain on $M \times M$ supported on the neighborhood N_ϵ .

These ideas will go into the discussion. I'll try to do the construction to singular chains. The construction will produce something else, something weird. I'll use this to define my chain model, I'll say, okay, if I put in the weird thing, I get out the weird thing.

We'll start our first try with singular chains. Fix a cell c in $\overline{SD}(g, k, \ell)$ and a generator of singular chains $\sigma : \Delta \rightarrow LM^k$. When I've got this cartesian product, I'll take $\Delta \times \sqcup_k S^1 \rightarrow M$. So $\sigma(p) = \gamma_1, \dots, \gamma_k$, then (p, t) goes to $\gamma_i(t)$. This will be a simplex cross three circles. It's a family of three circles over my simplex, with distinguished points.

I have the evaluation map that goes from (c, σ) , what does it take me to? I have my cell and my family of three loops over a simplex mapping to M . For each chord, I will map to $M \times M$. If I were doing transverse intersection, I'd look at the preimage of the diagonal to see where chord endpoints coincide. Now I have my neighborhoods N_ϵ . Let me look at the chord endpoints. I can use the map of three circles over the simplex to tell me what to do to these endpoints. I map these into multiple copies of $M \times M$. The number of chords is the Euler characteristic of the graph.

If you land on the diagonal, you have true intersection. Near the diagonal, you get this diffuse intersection. I'm interested in the preimage of the product of these neighborhoods. I have $|\chi|$ copies of $M \times M$., and $ev^{-1}(N_\epsilon^{|\chi|})$ consists of diagram and map pairs so that the diffuse intersection is possible. Then I could map the diagram to M .

In the picture what you get, you have a cell and a diagram. Look at a preimage of a neighborhood of the diagonal, it's this spread out thing, it's diagrams and maps so that I can map the diagram into the manifold, in a whole family, I have a family of diagrams and a family of maps from circles. Think about this as one big space, and I've got the preimage of the diagonal, and I've got diagrams living over it that I can map into the space.

Now I have over the neighborhood a family of diagrams, I can restrict to the outputs. The hard step is going from the input circles to the diagram, the easy step is to restrict to outputs. What have I got? I've got the same thing again. I've got a neighborhood in the product with varying maps, I restrict to the outputs, and I get a map τ to M . This preimage $ev^{-1}(N_\epsilon^{|\chi|})$ gives a family of circles plus the map to M . This is trying to play the role of the base space of the output.

This is the output of the would-be construction starting with a singular chain. This is not exactly a singular chain, and remember, I want to do something, it's not the right dimension, you have some fuzz of base space. This doesn't look like what we started with.

This construction works for something, just not for singular chains.

I have maps from $M^{2|\chi|}$ to my diagonal space $(M \times M)^\chi$, and I can pull back my Thom class representatives from each, getting a big class $U = \cup \pi_i^*(\mu)$, and I can pull back U under ev and get $ev^*(U) \in C^d(c \times \Delta)$. The support of $ev^*(U) \times \sqcup_\ell S^1 \rightarrow M$ via τ .

This is what the output looks like, let me take stuff like this and make it the input. I'll build my chain model based on what this did to a singular chain.

Let's go back and make the input look like this output. I'll replace, for a singular chain I had Δ an n -simplex and a map to M .

Δ an n -simplex \square an oriented n -dimensional polyhedron. This might be a Cartesian product of simplexes and a singular cochain in $C_{sing}^m(\square)$

$$\sigma : \Delta \times \sqcup_k S^1 \rightarrow M \qquad \sigma : \text{supp}(\alpha) \times \sqcup_k S^1 \rightarrow M$$

Let me say, the regular support isn't so nice, imagine I've done something to make my support nice, a sub-polyhedron of \square .

This construction is bigger. Let these guys generate a chain complex $P(k)_*$. This is in degree $n - m$ by its degree.

Define the boundary $\partial(\square, \alpha, \sigma)$. I have something I can take the boundary of and something I can take the coboundary of. This is $\pm(\partial\square, \alpha|_{\partial\square}, \sigma|_{\partial\square}) \pm (\square, \delta\alpha, \sigma)$. One thing that I should say about my "nice support" is that it is constructed so $\text{supp}(\delta(\alpha)) \subset \text{supp}(\alpha)$. Extend this linearly.

Proposition 1. *The homology of $(P(k)_*, \partial)$ is the homology of LM^k . In fact, I have Eilenberg-Zilber and Alexander-Whitney style maps between $P(K)_*$ and $P(1)_*^{\otimes k}$.*

Now I want to say how to take a cell c , and one of these funny things, and get a funny thing. So I want to construct a map $ST_\mu : C_*(\overline{SD}(g, k, \ell)) \otimes P(k)_* \rightarrow P(\ell)_*$. This is defined by taking c and \square, α, σ to $c \times \square, p^*(\alpha) \cup ev^*(U)$, and τ . The support

of $p^*)\alpha) \cup ev^*(U)$ plays the role of $ev^{-1}(N_\epsilon^{|\chi|})$. Then we extend linearly. With pre- and post-composition with my Eilenberg-Zilber Alexander-Whitney maps, I get the thing I said I was looking for:

$$C_*(\overline{SD}(g, k, \ell)) \rightarrow Hom(P_*^{\otimes k}, P_*^{\otimes \ell})$$

Notice that the construction ST_μ doesn't care about tree-like versus non-tree-like configurations of chords. You get a copy of $M \times M$ for every chord, I don't care if it lands in the diagonal, you have your support making things the right dimension.

Theorem 1. *ST_μ is a chain map, which commutes with the cellular boundary of \overline{SD} and the Hom D in $Hom(P_*^{\otimes k}, P_*^{\otimes \ell})$. For two choices of Thom class μ and μ' , the chain maps ST_μ and $ST_{\mu'}$ are chain homotopic.*

If you have questions, maybe we can talk privately not to make people wait longer.

2. EXAMPLES OF PROPERADS

Let's start with some easy examples.

The sort of canonical example, the reason we use this language, is the endomorphism properad of a chain complex. Let C be a chain complex; then define $End(C)(m, n)$ to be $Hom(C^{\otimes m}, C^{\otimes n})$. This has an obvious $S_m - S_n$ action by permuting the inputs and outputs, and the compositions maps are actual composition of endomorphisms of tensor powers of C , and the identity is the identity $C \rightarrow C$. Each of these is the Hom complex of two chain complexes, so each one of the spaces is itself a chain complex.

Probably the easiest combinatorial example is the Eulerian Frobenius properad. Let $EF(m, n) = \mathbf{k}$. Define all of the symmetric group actions to be trivial, and let the composition map be induced by the canonical isomorphism of tensor powers of \mathbf{k} with \mathbf{k} . This trivially satisfies all of the conditions to be a properad. A more common properad in the same vein is the commutative properad, where $C(m, n) = \mathbf{k}$ if $n = 1$ and $C(m, n) = 0$ otherwise. This is a subproperad of $EF(m, n)$.

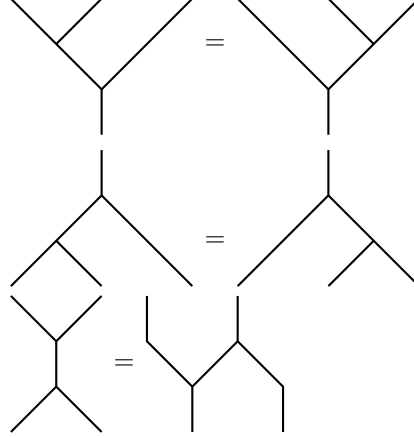
Let's do a more general construction. Let G be an \mathbb{S} -bimodule. Let's construct the free properad on G . This will be spanned by labelled directed graphs. We should number the vertices, number the incoming and outgoing half-edges at each vertex, and number the overall incoming and outgoing half-edges of the graph. Each vertex has a number m_v of inputs and a number n_v of outputs, and we want to take the sum over all of these graphs of the product over their vertices of $G(m_v, n_v)$, up to some equivalence relations.

$$F(G)(m, n) = \left(\bigoplus_{(m, n)\text{-graphs}} \mathbf{k}[S_n] \otimes \bigotimes_{\text{vertices}} G(m_v, n_v) \otimes \mathbf{k}[S_m] \right) / \sim$$

The equivalence relation should say a few things; it should say that acting by S_n or S_m on the labeling of the graph is the same as acting on $\mathbf{k}[S_n]$ or $\mathbf{k}[S_m]$. It should say that acting internally on the graph by relabelling edges at a vertex is the same as acting by the appropriate symmetric group on that vertex. It should say that changing the ordering of the vertices shouldn't change anything except maybe a sign, using the Koszul sign convention. All of these are technical details, and really shouldn't be worried about too much. You should concentrate on the picture. The free properad is made up of these decorated graphs, and then you

permute by permuting the inputs and outputs, and you compose by concatenating graphs vertically. The identity is the trivial tree with no vertices.

So let's describe a properad this way, the Frobenius (not Eulerian). Take the \mathbb{S} -bimodule G which has $G(2, 1)$ generated by μ and $G(1, 2)$ generated by Δ , both in degree 0 and with the trivial S_2 action. So this properad has a commutative product of some kind and a cocommutative coproduct. Then I want to impose some relations, so I'll say that I want them to be commutative and cocommutative and satisfy Frobenius compatibility. I can write all of this out with graphs:



Call this properad $Frob$. So you can do a little combinatorial argument and show that linearly, for $n, m > 0$, you have $Frob(m, n)$ is spanned by generators $\gamma_{m,n,g}$ with trivial S_m and S_n actions. The composition \circ_k takes $\gamma_{m,n,g}$ and $\gamma_{m',n',g'}$ to $\gamma_{m+m'-k, n+n'-k, g+g'+k-1}$.

I want to do another example, but maybe I'll save it until after we've talked about representations a little.

3. REPRESENTATIONS OF PROPERADS

So what is a representation? If you have a properad P and a chain complex C , then a representation of P on C is a map of properads $P \rightarrow End(C)$. So this means that for each operation in P , you get an endomorphism between the appropriate tensor powers of C , and these obey the same composition and symmetric group actions and identity constraints that occur in P , and this all also has to respect the differential. So, for example, what is a representation ρ of the Frobenius properad on C ? We have a generators and relations picture of this properad, and it's generated by this product and coproduct which are commutative, so the data of a representation are a product $\rho(\mu)$ that goes from $C \otimes C \rightarrow C$ and a coproduct $\rho(\Delta) : C \rightarrow C \otimes C$.

What about the relations? Since these have a trivial S_2 action, $\rho(\mu)$ should be commutative, $\rho(\mu)\sigma = \rho(\mu)$, and $\rho(\Delta)$ should be cocommutative. Furthermore, since this respects the differential, we have the following:

$$\begin{array}{ccc} \mu & \xrightarrow{\rho} & \rho(\mu) \\ \partial \downarrow & & \downarrow \partial \\ 0 & \longrightarrow & 0 = D(\mu) \end{array}$$

$$\begin{array}{ccccccccc}
\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 1 \quad 2 \end{array} & - & \begin{array}{c} 1 \quad 2 \\ | \quad | \\ \diagdown \quad \diagup \\ | \quad | \\ 1 \quad 2 \end{array} & + & \begin{array}{c} 1 \quad 2 \\ | \quad | \\ \diagdown \quad \diagup \\ | \quad | \\ 2 \quad 1 \end{array} & - & \begin{array}{c} 1 \quad 2 \\ | \quad | \\ \diagup \quad \diagdown \\ | \quad | \\ 1 \quad 2 \end{array} & + & \begin{array}{c} 1 \quad 2 \\ | \quad | \\ \diagup \quad \diagdown \\ | \quad | \\ 2 \quad 1 \end{array} = 0 \\
\end{array}$$

$$\begin{array}{c} | \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ | \end{array} = 0$$

And in the same way, a representation of this properad is a dg involutive Lie bialgebra.

So, I am not going to be able to show you much of what you can do with this, but let me give you an open problem that there's lots of good evidence for, related to the duality between Frobenius and the involutive biLie properads.

Conjecture 1. *The properad structure on string topology that Kate is describing, generated just by the fundamental chains of the moduli spaces, form a free resolution of the involutive biLie properad.*

This would mean that the structure that you get at the level of the chains is the up-to-homotopy version of an involutive Lie bialgebra, and this fits in a class of results very much like this. If this is true, that's awesome, it shows that this example fits in with a bunch of already understood math. If it's false, that's just as interesting, it means that there are some string topology operations at the chain level that we don't actually know about yet on the level of homology, which would be very interesting to a lot of people.

That's all I wanted to say.

4. STRING TOPOLOGY PROPERAD, REPRESENTATION, AND MASTER EQUATION

Let's get started. This morning, I said I was going to start with the non-equivariant story. Now I'd like to continue with the equivariant story, which I didn't give you explicitly. The master equation will come out here.

So there exists an analogous chain model P_* for the S^1 -equivariant homology of LM relative to the constant loops.

There's an analogous construction that gives $C_*(\overline{SD}(g, k, \ell))$, and I want to take chains invariant with respect to rotation of marked points $C_*^{inv}(\overline{SD}(g, k, \ell))$, and we have a construction from one of these chains into $Hom(P_*^{\otimes k}, P_*^{\otimes \ell})$. We have that this is a chain map. If I took $End(P_*)$, we can put a properad structure on the other side, varying g, k, ℓ , call that $C_*^{inv}(\overline{SD})$. Is this a map of properads? The answer is "almost," which sounds bad, but \overline{LD} to the rescue, and that will let us make a new properad so that the answer is yes.

Some facts. The equivalence relation \sim on \overline{SD} induces one on $C_*^{inv}(\overline{SD})$. Next, if c and c' are equivalent in $C_*^{inv}(\overline{SD})$, the nice thing would be for them to give

the same operation. This isn't true here, but the difference $ST(c) - ST(c')$ is a boundary in $End(P)$. So $D(B) = ST(c) - ST(c')$.

We'll proceed for the special case, where $D(B) = 0$, so these relations hold on the nose. Then the string topology construction depends on the quotient. For every g , k , and ℓ , we have a map ST from my invariant chains to $Hom(P^{\otimes k}, P^{\otimes \ell})$.

Let's figure out the properad structure on $C_*^{inv}(\overline{SD})$. This is equivariant gluing. I'm only thinking in the easy setting. The data was the easy part and the conditions were hard. I'm just thinking about the pictures that he drew. The stuff you don't usually put into a talk, I'm ignoring. I want to take two equivalence classes. I want to draw the metric surface I'm getting from that construction.

I can rescale whichever one I need, I have a construction that produces a metric space. I'll rescale the lower one, take the different metric space, and glue the resulting metric space in. Maybe it will be obvious what to do next, I glue the two surfaces. There's an equivariant something that happens here. I don't have a marked point on the output. I want to turn one or the other into a family, marking in all possible ways. Then I do this gluing to produce an S^1 -family.

It's not clear that this is associative, but we have that equivariant gluing induces a properad structure on $C_*^{inv}(\overline{SD}/sim)$. This is the picture to have in your mind.

Now we have a properad structure, an endomorphism properad $End(P_*)$. If this were a map of properads, I would say this is a representation of this properad? Is string topology a properad map? Does it respect partial composition? The answer, as I said, is almost, and \overline{LD}/sim is going to save us. For example, $\overline{SD}(0, 2, 1)/\sim$ is a torus sitting over a point. Let me name the operation I get from the string topology of the top cell of this. This is an operation from $P_* \otimes P_*$ to P_* . I'll call this, suggestively, the bracket $[,]$. The answer that it's not a properad map is for the following reason. I really want to consider the string bracket. I have a two to one operation. The string bracket composed with itself, this is $[[,],]$, this is a three to one operation $P_*^{\otimes 3} \rightarrow P_*$, is not in the image of the string topology construction. However, it is homologous to a chain that is. That's what's going to save us.

I want to pick these things out in order to extend the construction to \overline{LD} .

I was jealous that Gabe got to hide things, I got to hide things too: [Reveal of picture, oohs and aahs]

If I look at $[\overline{SD}(0, 2, 1)] \circ_1 [\overline{SD}(0, 2, 1)]$, and take the string topology construction of that, it's homologous to the composition of the bracket itself. Let me look here. So starting here with one level, the diagrams look like three circles and two chords. I've given them the structure of a torus bundle over a base space, and the torus is keeping track of the input circle markings. I have an interval given by a parameter for the ratio of the two distances between the endpoints on the circle that has two points. There are three of these, so there are three intervals.

If I add levels, then the second chord, I can test its length, or the difference. Either one could be bigger, so I attach two squares to each of these intervals. On the poster everything is labeled and you can see why this is the identification. This is the space that I talk about all the time. I love it so much, it's really a compactification of moduli space. The green is where the string topology construction is defined. Here's what I'm going to do. I'm going to extend ST to \overline{LD}/\sim .

We have the string topology defined for \overline{SD}/\sim . Let me locate the composition on the left. Where is the composition? It's the fiber over this picture: [Picture]

This is the chain in the \overline{SD} chain properad, this is homologous to the other thing. The composition of the string bracket didn't appear in the image. This wants to give you the composition but doesn't, and I can extend the string topology construction. Then the string topology construction, I define at the boundary, to the real composition.

Now the blue and the orange are homologous. Here's the homology. Now I will send that chain, the homology, to the homology between the composition of the string bracket and the construction on the composition.

The main point of this part of the talk is being able to do this. Composition is not respected, but it's not that badly disrespected. I've made a choice, 123, so I have two other choices and I can do this on all three parts.

Notice what we've done, we've extended the string topology construction, not to every chain of \overline{LD} , but in particular to the fundamental chain. This is something that \overline{LD} has that \overline{SD} doesn't. It's a pseudomanifold. It has a fundamental chain (and its boundary), and that's where we've defined this extension. The procedure for $[\overline{LD}(g, k, \ell) / \sim]$ is completely analogous. The $[\overline{LD}(g, k, \ell)]$ themselves generate a free properad with a differential here. You can see the compositions at the boundary. I want to call this \mathcal{P} , since I had the other P for the chain model. I'll call it $\mathcal{P}_{\mathcal{M}}$. With you guys I can say that this is the properad from the compactified moduli space.

Then, let me look at the map $\mathcal{P}_{\mathcal{M}} \xrightarrow{ST} \text{End}(P_*)$, and this is a properad map. Aren't you so happy? That means we have a representation of this properad, it's an algebra over it. So $\mathcal{P}_{\mathcal{M}}$ governs the algebraic structure we have for equivariant homology of the loop space.

The example showed that we could get a properad map, but we get a little more out of it too, because this compactified moduli space has nice properties. Back to the example. Look at the string topology construction applied to the fundamental chain $[\overline{LD}(0, 3, 1) / \sim]$ I can look at the boundary of this in the hom complex, $D(ST([\overline{LD}(0, 3, 1) / \sim]))$ is the Jacobi. So the bracket satisfies Jacobi up to homotopy.

We can see that in higher g, k, ℓ we have a lot more. In general, each moduli space is giving us some relation like that, a relation among the operations whose compositions we send the boundary to. In general, let $X(g, k, \ell)$ be the operation from the top chain, $ST([\overline{LD}(g, k, \ell)])$, then $DX(g, k, \ell) = \sum X(g', k', \ell') \circ_n X(g'', k'', \ell'')$, and if I write that X is a formal sum $X = \sum X(g, k, \ell)$, and I can write this pretty succinctly. Write $\circ = \sum \circ_n$, and then all of the relations together are the same thing as saying $DX = X \circ X$. This is then a solution to the master equation. A solution to the master equation is giving us all the data at once, an infinite list of operations and an infinite list of relations.

I'm over, but this was the special case, and I can tell you what to do in the general case if anyone is interested. I stick in an extra cell, grout in the tiles, and I still have a little boundary. So I draw some more diagrams, stick something in all the holes. In the general case, I've filled most of the holes, not all of them, I think if I just sit down, it's done in the general case.

I was supposed to solve the master equation in my thesis. The term shouldn't be there, it doesn't matter, but that's where it is right now.

Thanks a lot. I hope you had a good week. We'll be around, we'll be at the bar. We'll find our way to Jupiter later. We have tea now.