1. Intro to Moduli Spaces

Remember, the Thom construction, if you had a Sullivan chord diagram of type (g, k, ℓ), there was a map $\mu_\Gamma : H_*(LM)^{\otimes k} \to H_*(LM)^{\otimes \ell}$

We fixed one graph and got an operation out of it. In fact, the construction in Cohen-Godin can be extended so that, well, we said that the set of Sullivan chord diagrams form a space. We considered edge lengths as continuous parameters. This is a connected space. This can be extended so that for all homology classes in $Sull(g, k, \ell)$ there exists an operation $H_*(LM)^{\otimes k} \to H_*(LM)^{\otimes \ell}$, and the degree is $|\alpha| - |\chi| d$. We saw that the operation μ_Γ was $-|\chi|d$. So the way that these things fit together is:

For a fixed Γ, μ_Γ corresponds to μ_α where α is a generator of $H_0(Sull(g, k, \ell))$. Cohen and Godin went beyond this, but let me just say, you can compose these operations, and there is some way of keeping track of this composition in the Sullivan chord diagram. You can imagine composing operations by identifying inputs and outputs in Sullivan chord diagrams.

An hour is not really enough time, a lifetime is not enough time, but this is supposed to be a really simple introduction to moduli spaces. I'm talking about Cohen-Godin to motivate why I'm doing this. Let me make few precise statements so that I can say something general.

Theorem 1. *There is an appropriate moduli space of Riemann surfaces, I'll call it* $\mathcal{M}(g, n)$*, such that the space of possibly marked metric fatgraphs* $(M)MFG(g, n)$ *is homotopy equivalent to this moduli space. You can modify a little on the right and a little on the left so there are lots of theorems like this: Strebel, Harer, Penner, Mumford, Thurston, Igusa, Godin.*

Given that this is true, there's a relation between the space of graphs and a space of surfaces. The second talk will be another version of this. This is how moduli spaces might come into the story. The conjecture that was totally reasonable (but false) was that $Sull(g, k, \ell)$ is homotopy equivalent to $\mathcal{M}(g, k + \ell)$. There's a counterexample due either to Godin or Bödigheimer. The first place you see it is $(1, 1, 1)$. You won't be able to see this, but if you look into it, the dimension of $Sull(g, k, \ell)$ is less than the homological dimension of $\mathcal{M}(g, k, \ell)$. Godin extended these string topology operations from the homology of the space of Sullivan chord diagrams to the homology of moduli space. This is a long, scary, paper called "Higher string topology operations" using spectra. There was work to do, since this was not homotopy equivalent. I want to give a brief introduction to the moduli space of Riemann surfaces. As a graduate student I talked a lot about moduli spaces, and I never knew what I was talking about. I want to give a bit of intuition for those of you who haven't heard of it before, or haven't been able to pick apart the definition to get an intuition.

Let's do a warm-up, configuration spaces. It takes a little bit of a leap. Let $C^n(\mathbb{C})$ be the set of configurations of n distinct labeled points in $\mathbb{C}$.

So $C^1(\mathbb{C})$, configurations of one point in the plane, how many places can you put one point, this is the plane, $\mathbb{C}$. I started with something that was a set, and already this is a space. It's reasonable that this is actually a space. Even the very first baby example, there's more structure going on here.

What about two points. If I ignore this condition for one second, I've got a plane's worth of possibilities for each point, so this is $\mathbb{C} \times \mathbb{C}$, but without the diagonal. If you don't see it yet, do a couple more, but a way to look at what this space is, you have a plane's worth of possibilities, so it's $\mathbb{C}^n - \Delta^{\text{pairwise}}$, which is the subspace of n-tuples of complex numbers where any two coincide.

So this is actually a really good place to start. A point in this space is a configuration of points in the plane. I might imagine a situation where two configurations are seen as equivalent. I want to place an equivalence relation on them. A function from the plane to itself is called conformal if it preserves oriented angles. Reflection is a nice thing we could picture, but it doesn't preserve oriented angles.

Here are some facts:

(1) A conformal automorphism of the plane can be written as a composition of translation, rotations, and dilations.
(2) Such an automorphism is completely determined by its value on two points.

This is the relation I want to put on my configuration spaces. Say we're only interested in configurations up to conformal automorphism. This is our equivalence relation; let's go back to our examples. Let me say something kind of stupid. Why is something like this a reasonable thing to consider? I give you my configuration and you turn it around to see it for you, that's a reasonable thing for me to consider, I think.

You can translate any one point to any other point if we do this in $C^1(\mathbb{C})$. We can translate any point to the origin. So $C^1(\mathbb{C})/\sim$ is one point, represented by a point at 0.

For $n = 2$, we have two points, I'm from Canada but I just started saying "zee," I try not to say "zed" or "zee." You can move that first point by translation to 0, rotate to get the second point to the positive reals, and dilate to 1.

What does that mean about the configuration of two points in the plane? That configuration space up to equivalence is again just a point.

Things get a little bit more interesting when you move up to three. We said that you could take the first point to the origin, the second point to 1, and then you have no control over the third point, it could be anywhere except those two points. So $C^3(\mathbb{C})/\sim$ is the plane minus two points. Now we have parameters in our space.

We started off with a set with the natural structure of a space, and this quotient also has the natural structure of a space.

Let me move to Riemann surfaces, and say what a Riemann surface is.

Definition 1. *A Riemann surface Σ is a one dimensional complex manifold. The transition maps are holomorphic.*

Equivalently, it's a two dimensional real oriented manifold equipped with a conformal structure (chart for which transition maps are conformal)

When should I consider two of these to be equivalent? With the second definition, Σ has an underlying topological surface.

Definition 2. *When are these the same? When they're conformally equivalent: Σ and Σ' are conformally equivalent if there is a conformal homeomorphism going from one to another.*

Don't freak out if you've never seen conformal before. To think about it, it's a homeomorphism that's locally conformal everywhere.

There are lots of things you can do. I said "there is an appropriate moduli space." You can make these surfaces more complicated. We'll want to

Definition 3. *A Riemann surface with boundary is, and I want to trail off and say that this is what you expect, instead of the plane and upper half plane, think of that as the complex plane and conformal. You also want to consider punctures, marked points, et cetera.*

We were considering configuration spaces up to equivalence, that's what we want to do now.

Definition 4. *Let S be an oriented topological surface. Then $\mathcal{M}_S$ be the set (eventually space) of conformal equivalence classes of Riemann surfaces with underlying topological surface S.*

There might be many way to turn S into a Riemann surface. We'll consider all possible ways to promote the topological structure to a Riemann structure. If this is your first time hearing the words "moduli space" think that these are the possible conformal structures on a topological structure.

Here's a major result that helps classify these.

Theorem 2. *Any simply connected Riemann surface Σ is conformally equivalent to exactly one of the following:*

(1) *the completed complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$, the Riemann sphere*
(2) *the complex plane, or*
(3) *the open disk in the complex plane.*

I wouldn't try to prove this now, it means that up to equivalence we know there are only three possibilities. If you know about the underlying geometry, this should be reminiscient of spherical, Euclidean, and hyperbolic.

What's a good thing about uniformization? Now we know all about every cover. This implies that the (conformal) universal cover $\tilde{\Sigma}$ of any Riemann surface Σ is one of these. All but finitely many have D as their universal cover. Especially if you're relating moduli spaces and fatgraphs, a lot of people make assumptions putting you in D. There are something like four other surfaces, so this isn't a strong assumption.

Recall, if S is compact it is determined up to homeomorphism by its genus and its number of boundary components. Let $\mathcal{M}_{g,n} = \mathcal{M}_S$. We know that $\pi_1(\Sigma)$ acts on $\tilde{\Sigma}$ by deck transformations.

I'll do an example, try not to rush it, and we do end up with a continuous family in this example.

Topologically, the torus is covered by $\mathbb{R}^2$. The fundamental group is $\mathbb{Z} \oplus \mathbb{Z}$, which acts on $\mathbb{R}^2$ by translations. If you've seen this in a topology class, I'm trying to say the thing you've heard before.

I can draw these as two vectors in $\mathbb{R}^2$, so a lattice. You can get the torus as a quotient of that action. The fundamental domain is this parallelogram, and the sides get identified, and you get the torus. Now I'm thinking about conformal classes above. What if I chose two different generators. Topologically, it's the same, and here is my fundamental domain, and conformally it's different. We'd expect that these two guys are not conformally equivalent. Given my choice of generators of π_1 I will get different structures on the torus.

I have so many choices I could make, and we could expect a continuous family. Let's try to figure out what $\mathcal{M}(1, 0)$ should be in stages. A first approximation would be two vectors in $\mathbb{R}^2$, linearly independent. But I'm ignoring some conformal equivalence. I could choose two pairs that give me the same Riemann surface structure. Let's try to get rid of those redundancies. These give equivalent Riemann surface structures if there is a conformal map taking $\{u, v\}$ to $\{u', v'\}$. I can multiply by any complex number. Since u is a complex number, this is the same thing as $\{1, \frac{v}{u}\}$. I could also take $\{u, v\}$ to $\{v, u\}$. So we can always arrange to have $\{1, w\}$ and we'll make the choice that w is in the upper half-plane.

Now our second approximation is that $\mathcal{M}(1, 0)$ is something like the upper half-plane $\mathbb{H}$. There are still too many structures. This may only be obvious if you've seen it before, but $\{1, w\}$ is equivalent to $\{1, w'\}$ if the change of basis is in $SL(2, \mathbb{Z})$. Each of these sets generate a lattice in the plane. When do two choices give me the same lattice? If the change of basis is in the lattice, then it's true. We can arrange for w to be in a specific set.

[picture]

This strip could be our third approximation. This is a fundamental domain of this $SL(2, \mathbb{Z})$ action, and we have identification along the boundary of this strip. The sides and the two edges get identified. So let me draw what we get. The moduli space $\mathcal{M}(1, 0)$ is homeomorphic to the open disk D. I drew it this way instead of like a regular disk is because folding up gives you these two weird points. There are two weird "pointy" points. The point in the middle looks, well, what does this correspond to? The lattice from $u = (1, 0)$ and $v = (0, 1)$. The fundamental domain is a for real square, which has more symmetry than other points, which have parallelograms. What you might want to think about, find some extra symmetry in the lattice. I should stop. I hope that if you knew nothing, now you know ϵ. This example is hard, and it's not that hard of a surface. We don't know a lot of the stuff. The moduli spaces come up in a lot of fields of math. People study different aspects of them. At 11 we'll take a graphical approach.

2. Moduli spaces and string diagrams

Let's take stock. We were talking about Sullivan chord diagrams. We have string topology operations from them, and there is an appropriate moduli space. This turned out not to be true. Godin took the Sullivan chord diagram construction and completely expanded it to get string topology operations from moduli spaces.

Let me tell you where we're going. If you look at the schedule, tomorrow morning is chain level string topology operations. The problem is that trying to make definitions there is hard, problematic, we had difficulties in Chas-Sullivan, we could only take transversally intersecting chains. Tomorrow I'll give a chain level version of this which will make things more complicated.

There are more spaces of graphs, with horrible names, you'll see why I gave them these names. There's a space of graphs $LD/\sim$ to produce string topology operations at the chain level. There's a map to the appropriately decorated moduli space $\mathcal{M}$, and this *is* a homotopy equivalence. I'll take a subspace of this, $SD/\sim$, and with appropriate modifications, this is the same thing as $Sull(g, k, \ell)$. The $SD/\sim$ will play the role of Sullivan chord diagrams. Not only that, but both of these spaces have, this is nice, we'll actually use natural compactifications of these spaces, $\overline{SD}/\sim \hookrightarrow \overline{LD}/\sim$. In general these spaces are noncompact, and we'll use a compactification coming from these spaces of graphs. When you compactify, you do get a homotopy equivalence.

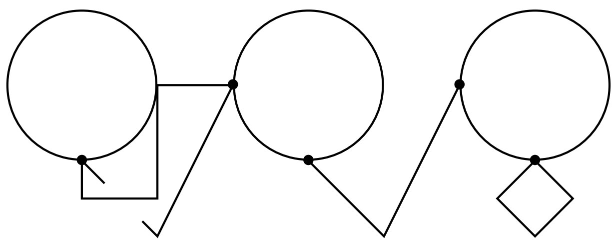
I won't talk about the string topology operations today. I will tell you about these spaces and how we might see a relation to moduli space. I don't know if anyone has looked at the poster, but that's a picture of $\overline{LD}/\sim$, the best space.

There will be a caveat in the last few minutes that says that this isn't literally true, that'll be in the last few minutes of the last talk. Hopefully things will still be clear then.

Definition 5. *A string diagram Γ of type (g, k, ℓ) is a metric fatgraph constructed from (a Sullivan chord diagram was circles and trees) k disjoint circles with total length 1 and some chords of length 1, so that the associated ribbon surface has genus g, k input boundary components corresponding to the k circles and ℓ output boundary components. I want the input circles to be marked.*

In Sullivan chord diagrams, we had inputs and outputs marked.

Let me make a drawing:



This can have non-tree parts.

Definition 6. *Let $\overline{SD}(g, k, \ell)$ be the space of string diagrams of type g, k, ℓ. Inside it, $SD(g, k, \ell)$ are those whose chord subgraph is a forest.*

Let me give you a construction, going from $\bar{SD}$ to metric spaces (with decomposition). I put a new vertex in the middle of each edge, and then add half-infinite edges coming down off of them. Then I turn this into a ribbon surface. I'm going to stick some things onto this. The total length of the circle is one. I'll stick a half-infinite cylinder on top. This will have circumference 1, an infinite hat.

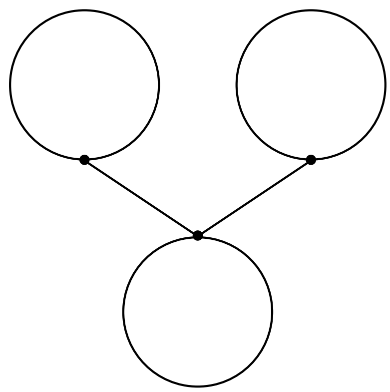
So then I will take strips, half infinite in height and of width given by the width of segments between consecutive vertices on the input circles, and I glue them in along the ribbon surface. It's hard to visualize, but I can tell you how to glue these pieces together. All of the pieces here inherit the metric. The ribbon surface only has a topological type. The ribbon surface retracts to its spine so we don't get a metric structure here. What do we get? It's a space, decomposed with the fatgraph living inside of it? It's a pair of pants, an infinite pair of pants, together with a copy of the string diagram sitting inside of it.

Generically, the output of the construction is a Riemann surface. This is easy to see in most of the places because I've cut the cylinder and the strips, they're cut out of the complex plane, there's a little bit going on at the gluing point that might not be clear, but you do have a Riemann surface structure everywhere.

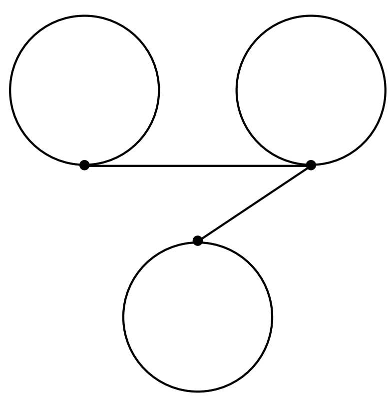
Hopefully you have an idea of the construction. Actually what I've said in this example doesn't cover anything. My only parameter is where the endpoints go. Here, let me take this graph. I'll still produce a pair of pants. The top will still be the same. The bottom will look different. The circumference of the legs will change. I'll get one big leg and one skinny leg.

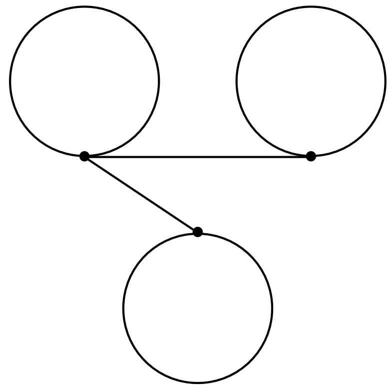
I don't have this tree-like condition in my space. What happens if the points come together? In the limit, I get one leg that's as fat as the waist and one leg that's infinitely skinny. It only happened because of this tree-like thing. Let me make a claim, you can ignore me for the rest of the day and verify it. The associated metric space is a Riemann surface if and only if Γ is in the open space. Yesterday there was a condition. Stuff really goes wrong here. The construction relates my space of diagrams to my space of Riemann surface structures. I'll call this a small output. It's not necessarily true that things that aren't treelike produce small outputs, but they all produce something bad.

Let me introduce an equivalence relation. I want to say two are equivalent if there's an isometry of the associated metric spaces preserving the decomposition. You might think, based on everything I've said, that you'll get a different metric space for different string diagrams. Let me take this: I have three inputs, I'll take two chords connecting them:



This gives a monkey saddle. My chord endpoints will have to be in the exact same place, and all three guys will produce the same thing.





You might also notice by looking at diagrams that I could say this combinatorially.

Proposition 1. *$\overline{SD}(g,k,\ell)/\sim$ is a connected cell complex of dimension $4g-4+2k+2\ell = 2|\chi|$. There are two parameters for each chord. You attach $|\chi|$ chords to get something of the right Euler characteristic. So then $SD(g,k,\ell)$ sits inside this as a union of open cells which is dense. This is a compactification. I've deleted illegal things which amounts to taking some cells out.*

I can say all of these things with or without the equivalence, but I care about the quotient space. $\overline{SD}(g,k,\ell) = SD(g,k,\ell)$ if $(g,k,\ell) = (0,k,1)$.

In some sense, that's why the first Chas-Sullivan paper is easier, because these two spaces are the same.

Now I want to make the definiton more complicated. I said for the purposes of string topology, I have $\overline{LD}/\sim$. Let me give you a definition of a string diagram with levels.

Definition 7. *A string diagram with levels of type (g,k,ℓ) is:*

I'm going to draw a picture. It'll end up with a metric fatgraph of the appropriate type. I have k disjoint circles, total length one, each with a marked point. I'll attach chords, and in the end I want things to be connected. I attach that chord with level one. I have more chords. I want to remember to attach them later. Level one chords have length $\ell_1 = 1$. I'll keep going. I'll attach longer ones. My attaching rules are as flexible as ever. Level two chords have lengths ℓ_2 which is greater than 1 and at most 3. Level 3 chords are longer than ℓ_2 and shorter than $\ell_2 + 2$.

Definition 8. *$\overline{LD}(g,k,\ell)$ is the space of string diagrams with levels of type (g,k,ℓ). I have an analogous $LD(g,k,\ell)$. For SD and $\overline{SD}$ I had a combinatorial condition. There's an analogous condition that's harder to state, so I won't state it that way.*

I take one of these guys, what is the picture I get. I'm not going to actually draw the picture, and you can't make me. You have the same construction, except you put little marks at $\frac{1}{2}\ell_1$, $\frac{1}{2}\ell_2$, and $\frac{1}{2}\ell_3$. I'll have a lot more stuff to go around. In the end I'll still have a topological type g,k,ℓ. It's actually kind of neat to see how it works.

I want to tell you why $\overline{LD}/\sim$ is great. $\sim$ is analogous on that on $\overline{SD}$. They're equivalent if they give you the same metric space. There's a combinatorial way to say this using chord slides; it's again, more complicated.

I should say, $\overline{SD} \subset \overline{LD}$. Likewise, the same is true when you mod out by equivalence.

A string diagram is in LD, this is a definition, if the construction makes a surface as the associated metric space.

The general definition, you might already know,

Definition 9. *A pseudomanifold of dimension n with boundary is a cell complex so that there exists a simplicial decomposition satisfying:*

(1) *Any simplex is the face of a n-dimensional simplex.*
(2) *Any $n-1$-simplex is the face exactly 1 or 2 n-simplices.*

Singularities of these are of codimension 2 or greater.

Proposition 2. *$\overline{LD}(g,k,\ell)/\sim$ is a pseudomanifold with boundary of dimension $6g-6+3k+3\ell-1$. A point in the boundary, and if you're at the boundary, you have to satisfy at least one of two conditions. One is that the associated metric space has a small output. The other (the last talk cares about the second condition, tomorrow), some ℓ_i is maximal. In my head, I'm thinking the spacing is maximal.*

More things are good, the boundary I can describe, I have a simpler subspace given by $\overline{SD}/\sim \subset \overline{LD}/\sim$, and this is a deformation retract, and one of the best things, for the appropriate moduli space $\mathcal{M}(g,k,\ell)$, there is a map to $\overline{LD}(g,k,\ell)/\sim$ and the image is a union of open cells, in fact, $LD/\sim$, and also it's dense. I'll use as a definition, $\overline{M}$ is $\overline{LD}/\sim$. I want to factor the definition through here. I want to factor my definition through here. This gives me a compactification of moduli space. With this definition, $\overline{SD}/\sim$ is in $\overline{M}$ as a deformation retract. It's so great. I've kept you long enough. I can draw a version of this if you want to stick around.

3. Properads

[Note: these notes are from an earlier version of the talk and are not reflective. Also, the definition of "connected" doesn't even make sense—not even wrong. You've been warned]

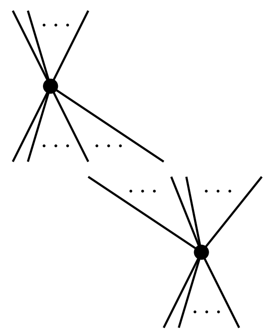
So I've spent a long time describing, in the abstract, different kinds of algebras, coalgebras, and gebras. I want to move up one level of abstraction and talk about all of these gebras as representations of a more general structure, a properad, as I promised in the first lecture. I don't want to give a formal definition at first because it will be sort of too much and too technical, but let's work toward one, little by little, keeping as our motivating example the spaces $Hom(V^{\otimes m}, V^{\otimes n})$ of m to n operations on a vector space. We might not get to any examples today, so just hang on if it's too heavy and I promise that the examples won't be horrible, when we get to them.

So if we want to model a space of multilinear operations, we probably want to keep track of the number of inputs and outputs. So a properad will be a collection $P(m,n)$ of "m to n" operations. Generically, we will allow these to be nonnegative integers, but often we will want to restrict to the case when one or both of m and n are positive.

What can we do with these? Well, first, before we do any composition, we can act on the left by the symmetric group S_n and on the right by S_m, and these two actions commute. So $P(m,n)$ should be a $k[S_n]-k[S_m]$-bimodule.

Now, let me say, this is not really my favorite way to describe this, I'd rather do something a little more categorical, where for any two finite sets S and T you have a space $P(S,T)$ and for any isomorphism of finite sets you get an induced isomorphism $P(S,T) \to P(S',T')$, but this uncategorified version will certainly work.

What's next? Well, the main structure that multilinear operations have is that you can compose them. So in a first pass, if you have two operations f_i^j and g_m^n, where the notation means that f has i inputs and j outputs, then you could compose the last k outputs of g into the first k inputs of f, I'll repeat the picture I drew in the first lecture:



and you get a composition $P(i,j) \circ_k P(m,n) \to P(i+m-k, j+n-k)$. This is not a particularly nice way of writing this because it doesn't interact nicely with the symmetric group actions, and this is not how I'll describe it when I'm giving a formal definition, but you can certainly work with this as a definition, just the constraints on the data will be really messy to write down. Now, you could imagine doing this with any nonnegative k, but for properads we will always restrict to positive k for technical reasons; doing this makes properads fit into a general framework of monoidal categories in a way that PROPs, where we include $k=0$, do not.

So what else? Well, there is a special element in $P(1,1)$ which is the identity map, so we'll say that there is a unit map $\eta : k \to P(1,1)$ so that $\eta(1) \circ_1 \psi = \psi$ and $\psi \circ_1 \eta(1) = \psi$.

This is the data of a properad, and it has to satisfy basically two things. First of all, composition has to be associative. This is kind of stupid the way we have described things because there are, I guess, three different associativity constraints depending on the pattern of composition, and they involve both $\circ_k$ and the symmetric group actions. There are also equivariance constraints, that composition has to respect the symmetric group action. You could write down a list of these as well, in terms of different induced actions. I don't want to do that.

So let me take a step back for a second and say, defining things like this makes the data relatively simple, but the relations are kind of a headache to write down. In my formal definition, I'm going to do the opposite, write it down in a way so that the data is a little more difficult to process but the relations can be written simply.

This is going to take a little bit of work, so bear with me.

Definition 10. *An* $\mathbb{S}$-bimodule *is a collection* $P(m,n)$ *of* $k[S_n]-k[S_m]$ *bimodules, for nonnegative* m *and* n*.*

Definition 11. *Let $\bar{n}^{i,N}$ or just $\bar{n}$, denote a partition of $\{1,\ldots,N\}$ into disjoint sets $n_1,\ldots,n_i$.*

Definition 12. *Let $\bar{m}^{i,k}$ and $\bar{n}^{j,k}$ be partitions so that the partitions m_a and n_b are all nonempty. A* path *in k between two elements ℓ and ℓ' of $\{1,\ldots,k\}$ is a sequence $\ell=\ell_0,\ldots,\ell_N=\ell'$ so that for the pair ℓ_s,ℓ_{s+1}, either they are both in m_i for some i, or they are both in n_j for some j. A permutation in the symmetric group S_k is said to be $\bar{m},\bar{n}$-*connected *if there is a path between any pair $\ell,\ell'\in\{1,\ldots,k\}$. The set of $\bar{m},\bar{n}$-connected permutations is a subset of S_k denoted $S^{c}_{\bar{m},\bar{n}}$, or just S^c_k if $\bar{m}$ and $\bar{n}$ are clear.*

If there is an empty partition, then the only connected permutation is the identity, and then only when $k=0$ and $i+j\leq 1$.

Definition 13. *For a partition $\bar{n}$, let $S_{\bar{n}}$ denote the product $\prod S_{n^i}$. Define a product on $\mathbb{S}$-bimodules as follows:*

$$P\boxtimes_c Q(M,N)=$$
$$\sum \mathbf{k}[S_N]\otimes_{S_{\bar{n}}} P(k_a,n_a)\otimes_{S_{\bar{k}}}\mathbf{k}[S^c_K]\otimes_{S_{\bar{k}'}} Q(m_b,k'_b)\otimes_{S_{\bar{m}}}\mathbf{k}[S_M]/S_i\times S_j$$

where the sum is over all numbers K, all pairs of partitions $k^{i,K}$ and $k'^{j,K}$, and all partitions of M and N into matching sizes $n^{i,N}$ and $m^{j,M}$.

I need to explain the $S_i\times S_j$ action, which is sort of by conjugation. An element σ of S_i acts by σ^{-1} on the left on the $\mathbf{k}[S_N]$ factor as a block permutation with blocks of size n_i, on the partitions k_i and n_i by permutation, and on the right on $\mathbf{k}[S^c_k]$ (which changes as a subset with the change in the permutations by acting on the right as the block permutation σ with blocks of size k_i.

This is too complicated a way to describe this, basically, it's just saying that we can permute a partition as long as we "undo" the permutation by acting correctly on the S_N and S^c_K factors.

This is a complicated way to present a simple idea, which is that you take a bunch of Ps on the bottom and a bunch of Qs on the top in a connected way, and then you make sure that everything respects all the symmetric group actions. This product has nice properties:

Theorem 3. *(Vallette)*
$\boxtimes_c$ is associative up to a canonical isomorphism and has a unit up to isomorphism which is the $\mathbb{S}$-bimodule I which has $I(1,1)=\mathbf{k}$ and $I(m,n)=0$ otherwise.

Definition 14. *A* properad *is a* monoid *in the category of $\mathbb{S}$-bimodules. That is, a properad is an $\mathbb{S}$-bimodule P equipped with two things:*

(1) *A product map $m: P\boxtimes_c P\to P$ which is associative:*

$$\begin{tikzcd} P\boxtimes_c P\boxtimes_c P \arrow[r, "m\boxtimes_c \mathrm{id}"] \arrow[d, "\mathrm{id}\boxtimes_c m"] & P\boxtimes_c P \arrow[d, "m"] \\ P\boxtimes_c P \arrow[r, "m"] & P \end{tikzcd}$$

A unit map $\eta : I \to P$ for the product:

$$\begin{tikzcd}
P \boxtimes_c I \arrow[r, "\mathrm{id}\boxtimes_c \eta"] \arrow[dr, "\cong"'] & P \boxtimes_c P \arrow[d, "m"] & I \boxtimes_c P \arrow[l, "\eta\boxtimes_c \mathrm{id}"'] \arrow[dl, "\cong"] \\
& P &
\end{tikzcd}$$

So you can see that with this definition, it was kind of hard to define the product, but then the definition of what a properad is was easy, it's just an associative algebra with this weird product.

Let's relate this back to the intuitive notion and that's the last abstract thing we'll do for a while, we'll start with some examples next. The intuitive notion had these products $\circ_k$ and to recover a $\circ_k$, what you do from this is use η a bunch to get a bunch of copies of the identity. So $f \circ_k g$ is the same thing as

$$m((\eta(I) \otimes \cdots \otimes \eta(I) \otimes f) \boxtimes_c g \otimes \eta(I) \otimes \cdots \otimes \eta(I)).$$

I'll let you work out the details as an exercise.