1. Thom collapse and loop product

Yesterday we talked about the loop product and the associated Gerstenhaber and BV structures on the loops space of the manifold. I'm going to put the equivariant part of the story on hold for a little bit and come back to it.

Since the first papers of Chas-Sullivan, others have taken these ideas and made them more rigorous. To my knowledge, Cohen-Jones was the first one. This uses the language of spectra. There are sort of two camps, and the other perspective is chain complexes. I put myself in the chain complex camp. I want to use my perspective to say what is going on in spectra, but I won't ever say that word. If you know about spectra, you'll be able to see what I'm trying to say. It's kind of like a religion, and a different religion. I'm a Christian trying to explain Buddhism: "their god is named Buddha and they're all vegetarian." So if you're a Buddhist, I hope not to offend you.

I want to give a perspective on the Thom collapse and the loop product. If a and b are homology classes represented by submanifolds A and B, then I look at a map from $A \times B$ to $M \times M$. We're interested in where they intersect, so the diagonal map, and we want to look at the fiber product $A \times_M B$:

$$\begin{tikzcd} A \times_M B \arrow[r] \arrow[d] & A \times B \arrow[d, "a \times b"] \\ \Delta \arrow[r] & M \times M \end{tikzcd}$$

So we looked at a map $A \times_M B \to M$, dimension $|a| + |b| - d$. This let us make a wrongway map $H_*(M) \otimes H_*(M) \to H_*(M)$. Let N be a tubular neighborhood of $\Delta(M) \subset M \times M$. I'm talking about the total space of the disk bundle of the normal bundle in $M \times M$. The ingredients are going to be the Thom collapse, which, well, remember what we care about is the transversal preimage of the diagonal. We can get rid of things away from the diagonal by collapsing. I'll crush everything outside the neighborhood N of Δ to a point. This is $M \times M \xrightarrow{\tau} M \times M/M \times M - N$.

The next ingredient is the Thom class. I want relative homology and cohomology classes. Look at the map $(M \times M, \emptyset) \to (M \times M, M \times M - N)$. Then I get a map in the other direction $H^*(M \times M) \leftarrow H^*(M \times M, M \times M - N)$. The Thom class of M is a cohomology class u in $H^*(M \times M, M \times M - N)$, degree d, whose pullback $j^*(u)$ is Poincaré dual to the fundamental class of M pushed forward under the diagonal: $\Delta_*[M]$.

I'm thinking of u as supported on N. So evaluating this on a class that is supported off of N gives 0.

Theorem 1. *Given N as small as you like, there exists such a u.*

This isn't stated precisely. It's in McCrory, 1971, and Bott-Tu.

Before, you may have seen the Thom isomorphism for the cohomology of bundles. I'll state it for homology. If I look at $H_*(M \times M, M \times M - N)$, that's isomorphic to $H_{*-d}(M)$, where you should think you are capping with the Thom class.

Then we have what we need for homology intersection. Let's take two classes, put them in the tensor product $H_*(M) \otimes H_*(M) \cong H_*(M \times M)$, push forward along τ_* to $H_*(M \times M, M \times M - N)$, then to $H_{*-d}(M)$ by the Thom isomorphism, and this composition is $\bullet$.

In general, if you have an embedding of compact manifolds $M \overset{e}{\hookrightarrow}$, let N be a tubular neighborhood of $e(M)$, then you can define a map on homology. The usual induced map would go from N to X, but you can define a wrongway map $e_! : H_*(X) \xrightarrow{\tau} H_*(X, X-N) \to H_{*-d}(M)$. Here τ goes from X to $X - N$. I can draw the same basic picture.

Cool. So what we want to do is take this idea and apply it to the loop product.

I want to place what we did yesterday in the context we've been discussing, let me do that.

Back to the loop product: How did we define the loop product? We took two chains in the loop space, visualized them as chains of loops in the manifold, and on intersection loci we could concatenate loops. We really care about, for the loop product, intersection in M. It's not like we're trying to do intersection theory in the loop space. It might appear like that, and then it would be amazing, this is infinite dimensional, but we're really in M. Let's see how we can use this kind of perspective to say what we said yesterday. I'm thinking that we can make another fiber product

$$\begin{tikzcd} LM \times_M LM \arrow[r] \arrow[d] & LM \times LM \\ M \arrow[r] & M \times M \end{tikzcd}$$

We have two loops and a point in M, so $(\gamma, \gamma', p) \in LM \times LM \times M$ so that $(ev \times ev)(\gamma \times \gamma') = \Delta(p)$. So you're saying that the basepoint of both of these is p. We can identify this with the space $\{(\gamma, \gamma') \in LM \times LM : \gamma(0) = \gamma'(0)\}$ You can draw a picture of this. This is two loops that share a basepoint. This looks like a map of the figure eight into the manifold. Let me write this, then, as $Map(8, M)$. This is a circle where two points are identified. You can also think of it as a map from a circle where two points agree. This should end up with a point in the loop space.

The diagonal is a codimension d embedding. It's an exercise to show that what you have above is also a codimension d embedding. You have infinite dimensional stuff but a finite codimension embedding, and then you have these Thom tools at your disposal. I have a tubular neighborhood of $LM \times_M LM$ sitting in $LM \times LM$.

Lemma 1. *If $\tilde{N}$ is the preimage under $(ev \times ev)$ of N, this is a tubular neighborhood of the image of the pullback of Δ*

We'll do what we did before in the compact world, and things will work because of the finite codimension. So the Thom collapse will take us from $LM \times LM$ to the quotient of the complement $LM \times LM / LM \times LM - \tilde{N}$. The Thom isomorphism will relate the relative homology $H_*(LM \times LM, LM \times LM - \tilde{N})$ to $H_{*-d}(Maps(8, M))$.

What else do I need? Let's note: I've got $Maps(8, M) \to LM$, what is that map? It's the concatenation. It's useful to think of the figure eight as the circle with two points identified.

Now we're about ready to do it. Let's define the loop product as we did for the intersection product, replacing all the Ms with LMs.

So take $H_*(LM) \otimes H_*(LM)$, that's isomorphic to $H_*(LM \times LM)$, we can push-forward by τ_* to $H_*(LM \times LM, LM \times LM - \tilde{N})$, and this will give us something in $H_{*-d}(Maps(8, M))$, and then we concatenate to land in $H_{*-d}(LM)$. Then this composition *is* the product $\bullet$ from last time. There hasn't been any transversality. This is better if details are important for you, maybe.

What I'd like to do next is start introducing surfaces. We've pushed this product pretty far, but we're going to start generalizing these. Depending on how well I say this, well, let's look at surfaces. I'll draw one picture that makes everything completely transparent.

[Pair of pants with figure eight as a deformation retract]

Somehow, in terms of what we're trying to do, this pair of pants is better than the figure eight. There are two circles at the top and one at the bottom. These are showing us something about the map. I don't care how big the surface is, I care that it's a little bigger than the figure eight, and it shows me that I'm starting with two and ending with one.

I want to draw two diagrams that are kind of the same.

We have maps $LM \times LM \leftarrow Maps(8, M) \rightarrow LM$. I can say, if I have a pair of pants P, I can restrict P to the inputs or outputs:

$$\begin{tikzcd}
LM \times LM \arrow[d, equal] & Maps(P, M) \arrow[l, "\rho_{in}"'] \arrow[r, "\rho_{out}"] & RM \arrow[d, equal] \\
LM \times LM & Maps(8, M) \arrow[l] \arrow[r] \arrow[u, "\cong"] & LM
\end{tikzcd}$$

The Thom collapse let us make a wrong-way map on homology, from the homology of $LM \times LM$ to the homology of $Maps(8, M)$. So now this gives a map $H_*(LM) \otimes H_*(LM) \cong H_*(LM \times LM) \overset{\rho_{in!}}{\to} H_{*-d}(Maps(P, M)) \overset{\rho_{out*}}{\to} H_*(LM)$.

Maybe it's obvious to you guys. If we didn't look at this as coming from a surface, you wouldn't know what to do next. What if we replace this with some other surface with boundary. What we're going to want to do next is generalize. This diagram, if I replace it with a general surface, a surface with boundary components labeled as incoming or outgoing, k incoming and ℓ outgoing, if I had a map from Σ to M then I could restrict to the outputs and get a point in LM^ℓ, to the inputs and get something in LM^k, and what we want to do is realize, somehow, ρ_{in} as a finite codimension embedding, so we can reverse this arrow on homology. If we could do that, then we could go from $H_*(LM)^{\otimes k} \overset{\rho_{in!}}{\to} H_{*shift}(Maps(\Sigma, M)) \overset{\rho_{out*}}{\to} H(LM)^{\otimes \ell}$. So this naturally leads us to a k to ℓ operation coming from Σ. This is μ_Σ, say.

I think I'll stop here for now. I went over yesterday and I'm slightly under, run and get your caffeine, and we'll come back and talk about how to do this. How do we realize this as a finite codimension embedding? The figure eight is what did it for us. What is the analog of the figure eight for such a Σ? We'll talk in a couple minutes.

2. Cohen-Godin string topology

Welcome back, where are my notes?

I forgot that I would be giving my talks back to back. Let me recall what we just did. We have a nice characterization of the loop product, consider the figure eight as a deformation retract of the pair of pants. We can see this as a two to one operation. We wanted to look at $LM^2 \leftarrow Maps(P, M) \rightarrow LM$, by restricting to the input and output boundary. We see that since the pair of pants contains the figure eight as a deformation retract, that we use $Maps(8, M)$ for $Maps(P, M)$.

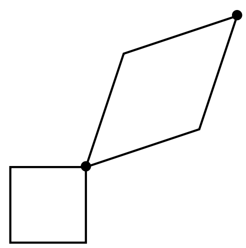
The construction gave us $\rho_{in!}$, the wrong way map on homology, and then we could use the usual induced map ρ_{out*}, and that composition is the loop product. The way we can generalize this is by taking k inputs and ℓ outputs, and doing the exact same thing. Now I can replace P with Σ, and the shift will be different. When I restrict to the inputs I have k maps from the circle, and at the end I have ℓ loops. No the full composition is

$$H_*(LM)^{\otimes k} \stackrel{\rho_{in!}}{\to} H_?Maps(\Sigma, M) \stackrel{\rho_{out*}}{\to} H_?(LM)^{\otimes \ell}$$

I'm going to leave string topology, but what our goal is to ask, what is the analogue for the figure eight when I want to generalize these things. That's what we're getting at here. The pair of pants is good because it tells us this is a 2 to 1 operation, the figure eight tells us about the actual intersections.

Let me introduce the tools to address this question, fatgraphs.

Definition 1. *A graph Γ is a 1-dimensional CW complex.*

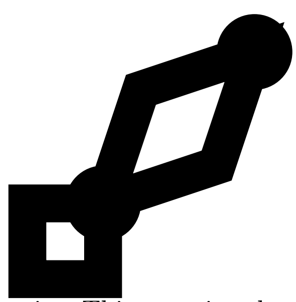


We call the 0-cells vertices and the 1-cells edges.

Definition 2. *A fatgraph Γ is a graph together with a cyclic order of the edges adjacent to each vertex*

I can draw a picture using the orientation of the chalkboard. Let me tell you why this is better than combinatorial. There is a construction taking a fatgraph and spitting out an orientable surface with boundary.

We take Γ to $\Sigma(\Gamma)$, the ribbon surface. I want to take this graph and fatten it to a surface. The thing that I can do, I can fatten each vertex to a little disk, and each edge to a band. I can use the cyclic order to tell me how to attach the bands to the disk.

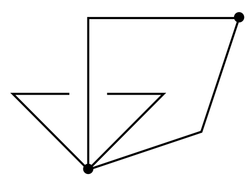


Make sure that you don't twist. This contains the graph as a deformation retract.

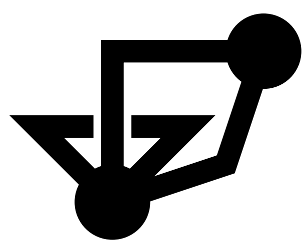
This contains Γ as a deformation retract or a spine.

Since we have this deformation retract, sometimes it's not always clear what the topological type of your surface, but we know that the Euler characteristic of a graph Γ is $\#V - \#E$, and since we also know that $\chi(\Sigma(\Gamma)) = 2-2g-\#\partial$ components, we can set these equal to each other. It's not too hard to count boundary components, this picture has three, and so you can set these two calculations equal and see (in this case it's not too hard, the graph is planar) that the genus is 0.

Let's see how this is different if we change the cyclic order: For me, I have to use the blackboard order, so I'll draw this:



If I fatten this I get



You can do the same calculation, see that $\chi(\Gamma) = 2-3 = -1$ and count we have only one boundary component, we get only one, so we calculate $-1 = 2-2g-1$ so $g = 1$. This is a pcunctured torus.

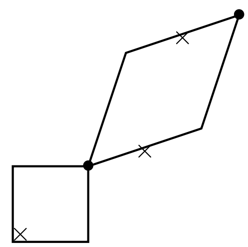
I think of the fat graphs and ribbon surfaces as being the same thing. This isn't literally true, but because of this I will conflate boundary cycles and boundary circles.

Next I want to define metric graphs:

Definition 3. *A* metric graph Γ *is a graph with a metric (edges have lengths).*

Now we have something continuous that we can define, edge lengths.

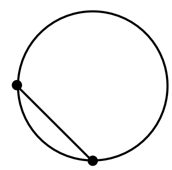
Definition 4. *A* marked metric fatgraph *is a metric fatgraph together with a distinguished point on each boundary cycle.*



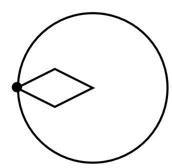
The next definition, possibly not due to Sullivan:

Definition 5. *(Cohen-Godin)*
A Sullivan chord diagram *of type* (g, k, ℓ) *is a marked metric fatgraph with no univalent vertices constructed from* k *disjoint circles (the* inputs*) and a disjoint union of* trees *of genus* g *with* $k + \ell$ *boundary components,* k *of which are isotopic to the* k *circles.*

Here is an example:

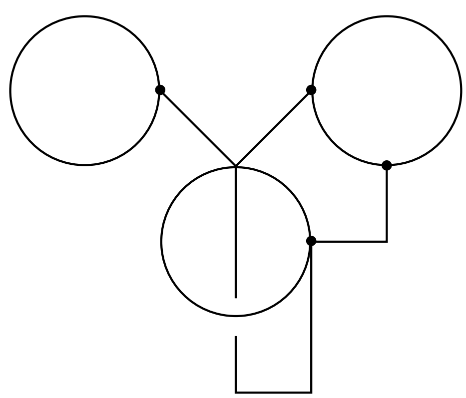


Here is a non-example:



The extra stuff isn't a tree. Well, the scandal is, Sullivan hates that they call it a Sullivan chord diagram because he thinks that these should be included, and Cohen says that Sullivan didn't ever say that, so, you can probably guess who I believe.

Here is a more complicated example



This graph wants to keep track of the intersection where the two points on the opposite side of the chord coincide. I want to pick up where those are equal. If I try to say that in a degenerate diagram, something will be overdefined. On Friday I'll be able to get rid of that condition.

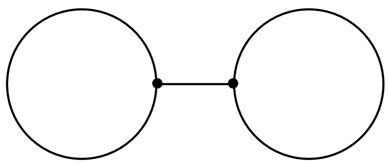
Let me make a remark. (Marked) metric fatgraphs of type (g, n) form a *space* with continuous parameters the lengths. Call this $(M)MFG(g, n)$. Then Sullivan chord diagrams are a subspace $Sull(g, k, \ell) \subset MMFG(g, k + \ell)$.

Remember, the diagram I want to reproduce in the more general setting is this one:

$$\begin{array}{ccc} Maps(8, M) & \longrightarrow & LM \times LM \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \times M \end{array}$$

Let's do this for a fixed Sullivan chord diagram Γ. We'll start with k input circles, so that'll be LM^k.

Then the idea is that the endpoints of the trees on the input circles tell you what you are trying to intersect. What is the Sullivan chord diagram that will give me the loop product? It's this:



So we want to identify the endpoints of this chord, say that they intersect at this point. So we can make a construction S from Sullivan chord diagrams to marked metric fatgraphs by collapsing trees.

[Example picture]

So $S(\Gamma)$ is what an intersection would look like. Let $V^{in}(\Gamma)$ be the vertices on the inputs. Let $M^V = Maps(V, M)$, which, I have two possible Vs in mind, $V^{in}(\Gamma)$, or $V(S(\Gamma))$. Here's a fact. The map S induces a map $V^{in}(\Gamma)$ to $V(S(\Gamma))$. It then induces a map on the mapping spaces in the opposite direction $M^{V(S(\Gamma))} \to M^{V^{in}(\Gamma)}$. If you think about them as ordered, this is just a Cartesian product, then everything is okay, and the map is just the diagonal. The codimension is $|\chi|d$.

Here's my claim, we've discovered the analogue of the figure eight. $S(\Gamma)$ is the analogue of the figure eight.

Putting it together, we get

$$\begin{array}{ccc} Maps(S(\Gamma), M) & \longrightarrow & LM^k \\ \downarrow & & \downarrow \\ M^{V(S(\Gamma))} & \longrightarrow & M^{V^{in}(\Gamma)} \end{array}$$

The vertical map on the right, I have k inputs and I want points in M, these should be in correspondence with $V^{in}(\Gamma)$. These points on the graph tell you where the evaluation maps should occur. It's about as much work to see that $Maps(S(\Gamma), M)$ is the fiber product or pullback.

I'm going to go a couple of minutes over. Over here before, we had a codimension d embedding that let us get a wrongway map on homology. I have a codimension

$|\chi|d$ embedding, so that was what I wanted so that I could apply the Thom collapse. What does that diagram look like?

Instead of the figure eight, I have $S(\Gamma)$, so I get

$$LM^k \xleftarrow{\rho_{in}} Maps(S(\Gamma), M) \xrightarrow{\rho_{out}} LM^\ell$$

It's a finite codimension embedding so I can reverse the arrow on homology and get

$$H_*(LM)^{\otimes k} \xrightarrow{\rho_{in!}} H_{*-|\chi|d}(Maps(S(\Gamma), M) \xrightarrow{\rho_{out*}} H_{*-|\chi|d}LM^{\otimes \ell}$$

Why are we working with Sullivan chord diagrams? The figure eight doesn't show us the inputs. Here we know what the input circles are, and that everything else is an output circle. So the first sale is that we have this but it's hard to see, later I'm going to show you a construction that allows you to do something with the chords. That's a couple of reasons to think that the chord diagrams are a little bit better.

3. Gebraic algebra

So far, basically every algebraic structure we have described has been an "algebra," meaning that it has operations which take many inputs and have one output. We talked briefly about coassociative coalgebras, which have operations with one input and many outputs. Next I want to expand the world of the discussion some to talk about "gebras" in general, which can have operations that are many to many. Let me start with the example of Frobenius algebras.

Recall that the cochains of a space are a dga, and the cohomology are a commutative dga, while the chains are a dg coassociative coalgebra and the homology a cocommutative dg coassociative coalgebra.

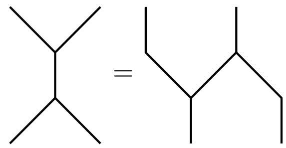
If the space is a closed n-manifold, you can use Poincaré duality to identify the degree k homology with the degree $n-k$ cohomology, which means that for an n-manifold M, the homology H_*M (or cohomology, but whatever) is both a commutative algebra and a cocommutative coalgebra; because this isomorphism changes the degree, the product map is degree $-n$ instead of degree 0. Let me say as a side note, you don't need to use the duality isomorphism, then the cup product, then the duality isomorphism to describe the product; you can instead choose representatives of the homology classes you want to multiply that intersect one another transversally, and then the product will be the homology class of the transversal intersection of these representatives. Kate has talked about this.

So we have an algebra structure and a coalgebra structure on $H_*(M)$. But these are not just an algebra and coalgebra that know nothing about one another; you can check that they are compatible in a very specific way, so that together they form the structure of an *open Frobenius algebra*, which I'm about to define.

I want to be a little careful here, because different people would call different things a Frobenius algebra. I know something like nine different nonequivalent versions of what a Frobenius algebra is, depending on whether it has a counit, a unit, an inner product, a co-inner product, whether you assume nondegeneracy, and so on. So for me,

Definition 6. *A Frobenius algebra is a triple (F, μ, Δ) where F is a chain complex, (F, μ) is a commutative dga, (F, Δ) is a cocommutative dg coalgebra, and μ and Δ*

satisfy the "Frobenius compatibility condition:"



Because the product is commutative and the coproduct is cocommutative, this implies three other equations by acting on the left and on the right by $\sigma \in S_2$. I won't write them down, but they all have the same left hand side and are similar.

Some people will describe this relation in different ways, with tensors or by saying something like "the coproduct is a module map over the algebra. Frobenius algebras are also closely related to two-dimensional topological field theories. Not this version of Frobenius algebras, but one of the other, closely related ones are in bijection with functors from the 2-dimensional cobordism category to the category of chain complexes. This is now a little bit of an old-fashioned point of view, we should be using higher categories,

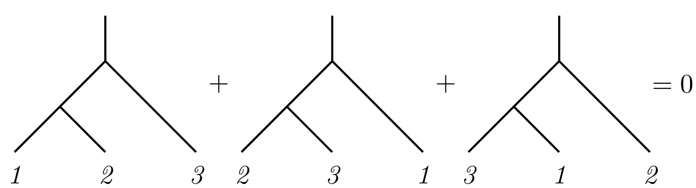
So this is the first gebra we've talked about.

Here's another example that I won't go into in any detail. Suppose that G is a topological group. Then the multiplication $G \times G \to G$ induces a multiplication on chains and homology, so that both of these are dgas. At the same time, G is still a space, so chains and the homology form a dg coalgebra. You could do the same thing on cochains with which one was which reversed. In any event, these do NOT have Frobenius compatibility, but have something else, called bialgebra or Hopf compatibility, that I'm not going to write down. If G is a Lie group then the homology is both a Hopf algebra and simultaneously a Frobenius algebra, and maybe there's even more compatibility.

The other example that I do want to go into in a little more detail is the example of a Lie bialgebra. First, let me describe a Lie coalgebra. Just as in the case of a coassociative coalgebra, we get the definition of a Lie coalgebra by reversing all of the arrows.

Definition 7. *A* dg Lie coalgebra *is a chain complex C along with a cobracket $\Delta : C \to C \otimes C$ which*

1. *is skew-symmetric: $\sigma\Delta = -\Delta$, and*
2. *satisfies the coJacobi relation that we get by turning Jacobi upside down:*

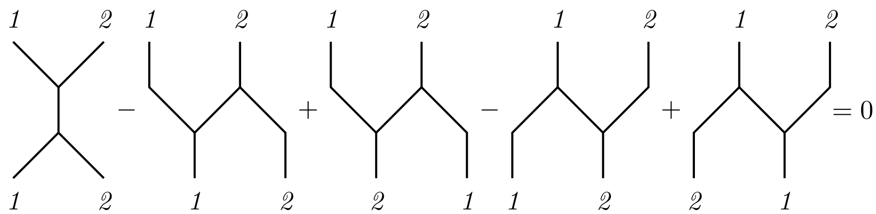


Just like we combined a dg algebra with a dg coalgebra to get a Frobenius algebra or a bialgebra, we can imagine a structure that has a Lie bracket and a cobracket.

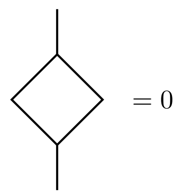
Definition 8. *A* Lie bialgebra *is a chain complex C along with a bracket $\{,\}$ and cobracket Δ so that*

1. *$\{,\}$ makes C a dg Lie algebra,*
2. *Δ makes C a dg Lie coalgebra, and*

(3) *the bracket and cobracket satisfy* Drinfel'd compatibility*:*



We say that the Lie bialgebra is involutive *if the bracket annihilates the cobracket:* $\{,\} \circ \Delta = 0$ *or*



Kate has shown you that this is the easiest structure that you get on the equivariant homology of the loop space, an involutive Lie bialgebra, and then I think she's aiming toward describing a richer structure on the equivariant chains that induces the involutive structure on homology.

4. Equivariant Chas-Sullivan String topology

There is also an equivariant version of Chas-Sullivan, it may be that the stuff we did yesterday made you unhappy, I might be about to make you mad again. There are two ways I'm going to make you mad. I shouldn't project. I'm going to use the loop product as we defined it yesterday. I'm going to do another thing in order to draw the pictures I want to draw. This is a really bad thing.

The naive thing is what I'm actually going to do. We said that naively, the equivariant story was, naively, look at the S^1 action on the loop space, and say that the H^{S^1} was equal to the naive quotient $H_*(LM/S^1)$. This can be a badly behaved space if there are fixed points, where we construct the homotopy quotient and take the homology of that. I'm going to live in the naive world so that I can connect back to things that we know. The things I'm going to say can be made precise in the actual world.

So, let's think about what this space is. How does S^1 act on it? What is the orbit of $\gamma \in LM$? It's a family of loops parametrized by S^1, so that they all have the same image, I'm just rotating it. You have a set, an equivalence relation, you can think of all the elments in an equivalence class, or an equivalence class itself. I might also think about this, represent the orbit by picturing an unmarked loop. The construction I'm going to show later takes one notion as its input and one as its output. A point in the space is an unmarked loop, Chas Sullivan call this a string space. A loop has a basepoint, a string doesn't.

I'll describe the real thing in a not-real way, now, but don't worry. You can think about the fibration given by taking the actual quotient or homotopy quotient by fitting homology into the exact sequence. I'm describing a real thing in a not-real way. One map goes $H_*(LM) \to H_*^{S^1}(LM)$. The other one goes from $H_*^{S^1}(LM) \to$

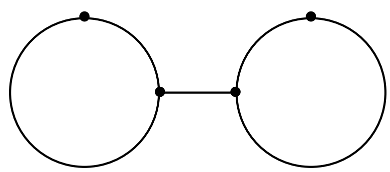
$H_{*+1}(LM)$, let's call them E and M, standing for "erase" and "mark." The picture is to take a loop and forget the marking, and you've got an unmarked loop, a string in the string space. If you have a marked loop, you have no canonical marking, you mark in all possible ways, so you go up by one. One remark: the BV operator is the composition of these: $\Delta = M \circ E$. That's actually your BV operator. The other composition $E \circ M = 0$.

Using the loop product defined on ordinary homology, I can define something on equivariant homology.

Definition 9. *Let x and y be equivariant homology classes. I can define $[x, y] = \pm E(MX \bullet MY)$.*

This gives me an $|x| + 1 + |y| + 1 - d$ equivariant homology class, so this has degree $2 - d$. This is called the string bracket. If I do this in two dimensions, this should be degree 0. This satsifies the Jacobi identity. The equivariant homology with the string bracket is a Lie algebra. This agrees with the Goldman bracket.

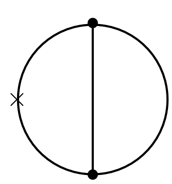
Now that we've gone through the construction with Sullivan chord diagrams, I might think about unmarked loops or families of loops. Remember, for the non-equivariant version, if we mark in all possible ways, and intersecting in the marking, I have a family of diagrams that looks like this:



The markings at the top move around, and that's an $S^1 \times S^1$ family, so that's T^2, and that's the 2 of $2 - d$. We can use these similar diagrams to discuss equivariant homology operations, moving back and forth between the two ways of looking at things. Mark in all possible ways on the input, do the non-equivariant operations, and then forget the markings on the output. This is a little more abstract because there is a whole family of diagrams, and this family that I have written leads to the string bracket.

Because we did have that discussion, Gabriel brought up the Frobenius thing here, this Lie algebra is something I am kind of okay with, there is some compatibility here, it's Frobenius (Cohen-Godin). Then we have another diagram, chord diagram, which gives $\Delta : H_*^{S^1}(LM) \to H_*^{S^1}(LM) \otimes H_*^{S^1}(LM)$.

We cut and reconnected for the Goldman bracket, and then there was the Turaev cobracket. We're focusing on what's happening at the endpoints of the chord, so it's secretly basically the same thing. The picture is:



Theorem 2. *The equivariant homology of LM with the string bracket and cobracket form an involutive Lie bialgebra. When $d = 2$, this agrees with Goldman-Turaev.*

Thinking about Sullivan chord diagrams, we've got algebraic structures generalizing the bracket and cobracket on equivariant homology. Given what we did earlier, I think it should be easy to imagine generalizing this to other structures. When it comes to surfaces, our fatgraphs giving us operations and surfaces, this is, well, here's the difference, Sullivan chord diagrams are a marked metric fatgraph, there's a marked cycle for each boundary component. For the non-equivariant case, we have these surfaces Σ and if I have a map from Σ into M, I can restrict to the inputs and outputs, the thing that was important, in order to land in the k-loop space, there was a distinguished point. The ribbon surface for the chord diagram has the marked point. You can do the obvious thing to parameterize with the circle. Equivariantly, one way to look at it is by taking unmarked diagrams and like I said, the annoying way of mixing things, I can mark the inputs but not the outputs. That's not a great punchline but I'll stop there, take T^k families of these. It's probably easier to think about non-equivariant stuff. I'll go back and forth a little bit.