1. Batalin-Vilkovisky algebras

So I want to start today by talking about Batalin-Vilkovisky algebras, or BV algebras. I think the more usual way to approach this kind of algebra is either by introducing topological spaces called the framed little disks, and then describing these spaces' homology, or by presenting some operations and relations, and just going on from there. I don't want to do either of these; I want to motivate BV algebras by putting them in context in a whole family of types of algebras.

In my very first lecture, I talked about representations of the algebra of dual numbers. These were vector spaces with a square zero operator. You could ask for representations of that algebra not in vector spaces, but in chain complexes, and you'll need to specify the degree of Δ, let's say Δ has degree 1 for concreteness, and then a representation of the algebra of dual numbers is a chain complex with a compatible square zero degree one operator Δ. Here "compatible" means that it is a degree one chain map, that is, closed in the Hom complex, so $\partial\Delta-(-1)^{|\partial||\Delta|}\Delta\partial = 0$, so since these are degree $+1$ and -1, $\partial\Delta + \Delta\partial = 0$. When you have two square zero operators with this compatibility, it is called a mixed complex.

Now let's say that we have a space that is both a mixed complex and a commutative dga. I like to think of this geometrically, and think that the commutative dga is something like the k-valued functions near the point of a space, which you can multiply, and you can take different versions or variations on this idea, so this could be the $\mathbb{R}$-valued smooth or analytic functions near a point in a manifold, or the differential forms so that this is a proper dga with a differential, or the formal functions near a point in a supermanifold or dg manifold, or maybe we just have a sheaf of commutative dgas on our space that we got in some other way.

In any case, we have a commutative dga A and a mixed complex structure on the same space with the same ∂, and that gives us some sort of structure. This is a totally reasonable thing to consider, and I want to add one more requirement, which is, we're talking about a sheaf of local functions on a manifold, now, in our example, so I'd like Δ to be some kind of local operator that respects inclusions and restrictions. This is a natural thing to want to study.

However, it's also sort of wild. There is, a priori, no relationship at all between Δ and the product, and so you have to do a lot of processing to find meaningful structure here. But we can take finite approximations to this structure and they'll have a lot more structure. Specifically, a local operator, like Δ, is locally a *differential operator*, meaning it acts kind of like differentiation, like a derivative or a second derivative or a kth derivative. So the simplest version would be if Δ was a first order differential operator. Then you would have compatibility with multiplication, because of the way differentiation of functions works:

$$\Delta(fg) = \Delta(f)g + (-1)^{|f|} f\Delta(g)$$

This is the same compatibility you would have in a dga, so it's like a double dga. You could call it a dga in the category of chain complexes, or an algebra in the category of mixed complexes or something like that. This is maybe too simple because it doesn't involve anything we didn't have before. So the next approximation would be if Δ was a second order differential operator. This also has a compatibility with multiplication, but it's more complicated. You can't write $\frac{\partial^2}{\partial x^2}(fg)$ in terms of the second derivatives of f and g, but you can do it with a triple product, and if you write it out, you find the relation (I'm not going to put the signs in)

$$\Delta(fgh) = \Delta(fg)h + f\Delta(gh) + \Delta(fh)g - \Delta(f)gh - f\Delta(g)h - fg\Delta(h).$$

We could also draw this with trees like we did before.

So this is different enough from a dga that it's an interesting and complicated object of study, and surprisingly enough, this kind of structure shows up pretty naturally in a number of different fields, noncommutative geometry, vertex operator algebras, differential geometry, string topology. So now let's turn around and take this as a definition.

Definition 1. *A (dg)* Batalin Vilkovisky algebra *is a tuple (A, μ, Δ) where A is a chain complex with internal differential ∂, μ is a product that makes A a commutative dga, Δ is a square zero operator that makes A a mixed complex, and μ and Δ satisfy the "seven term relation" or "BV relation" of Δ being an order two differential operator (this time with signs):*

$$\begin{gathered}\Delta(fgh)\\ -\Delta(fg)h - (-1)^{|f|}f\Delta(gh) - (-1)^{|h||g|}\Delta(fh)g\\ +\Delta(f)gh + (-1)^{|f|}f\Delta(g)h + (-1)^{|f|+|g|}fg\Delta(h)\\ = 0.\end{gathered}$$

So let me say right away, a lot of people don't present BV algebras in this way. Let me show you another presentation of the same structure. You could ask, a BV algebra isn't quite an order one differential operator, it's not a derivation of the product, but you could ask about how badly it fails to be one. That is, you could define the deviation of being a derivation:

$$Dev(f,g) = \Delta(fg) - \Delta(f)g - (-1)^{|f|}f\Delta(g)$$

to be an operation itself, a bilinear operation. If you change the sign a little bit, define $\{f, g\} = (-1)^{|f|}Dev(f, g)$, then this bracket makes A into a dg Lie algebra with differential ∂ (with a modified degree, which I'm going to let you check as an exercise). The bracket and the product have "Poisson" or "Gerstenhaber" compatibility so that the bracket is a derivation of the product:

$$\{f, gh\} = \{f, g\}h + (-1)^{|g||f|+|g|}g\{f, h\}$$

The bracket also satisfies Jacobi, and the ∂ operator is a derivation of the bracket, as is the Δ operator. So let's summarize:

(1) the product is commutative and associative
(2) the bracket is skew and Jacobi (Lie) with modified degree
(3) the bracket and product have Gerstenhaber compatibility (the bracket is a derivation of the product)
(4) ∂ is a derivation of everything.
(5) the Δ operator is square zero
(6) Δ is a derivation of the bracket
(7) the deviation from Δ being a derivation of the product is the bracket

This is the other common way to describe a BV algebra. It has the benefit of involving some concepts that are more accessible (the Lie structure) but it's not *minimal* in the sense that this definition involves the bracket, which is a purely descendent structure. That is, any presentation of the operations involved must contain both the product and the BV operator in the span of the generators, because those are indecomposable operations, but the bracket is decomposable as an operation and so any presentation that contains it as a generator is redundant.

I'm not going to give any examples of BV algebras right now. There are several in the exercises and Kate is going to talk later today, I think, about how the string homology of a manifold forms a BV algebra. I will give an example of a simpler algebraic structure related to a BV algebra, namely a Gerstenhaber algebra.

Definition 2. *A (dg)* Gerstenhaber algebra *is a tuple* $(A, \mu, \{,\})$ *where* A *is a chain complex,* μ *makes it a commutative dga,* $\{,\}$ *makes it a dg Lie algebra (with modified degree), and the bracket and product have Gerstenhaber compatibility. In other words, a Gerstenhaber algebra satisfies conditions one through four of a BV algebra (the ones that don't involve* Δ*).*

The way that I have defined this, it should be clear that a BV algebra always has the structure of a Gerstenhaber algebra, by ignoring or forgetting the BV operator Δ. The properadic language that we haven't introduced yet makes this very clear by saying there is an injective map of properads from the Gerstenhaber properad to the BV properad. In any event, it's an interesting question, if you have a Gerstenhaber algebra, or a functor with values in Gerstenhaber algebras, whether it can be enhanced to a BV algebra or a functor valued in BV algebras. If the answer is no, the next question is what additional structure would allow you to enhance the functor in this way. Let me give an example of a Gerstenhaber algebra, and then we can pose these questions for you in the exercises.

Definition 3. *Let* A *be an algebra. The* Hochschild cochains of A with values in A*, which I will denote* $CH^*(A, A)$ *or* $CH^*(A)$*, consist of the spaces* $Hom(A^{\otimes n}, A)$*. We will give this two different gradings. The algebra degree of this component will be* n *but the Lie degree of this component will be* $n-1$*.*

There is a codifferential δ *on Hochschild cochains of degree* $+1$ *(not* -1*) of the form:*

$$(\delta f)(a_1 \otimes \cdots \otimes a_{n+1}) =$$
$$a_1 f(a_2 \otimes \cdots a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1})$$
$$+(-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}$$

Its cohomology will be denoted $HH^*(A)$*.*

Remark 1. *This is certainly not the most general definition. Typically,* A *could be a dga, and the values can be taken to be in any* A*-*A*-bimodule. We'll use this definition for simplicity.*

There is also a notion of Hochschild chains and homology, and there's one exercise involving those concepts, but I'm going to focus mainly on cochains and cohomology.

Now, let me define a couple of products on this space.

Definition 4. *The cup product on Hochschild cochains is a product (using the algebra degree)* $CH^m(A) \otimes CH^n(A) \to CH^{m+n}(A)$ *given by*

$$(f \cup g)(a_1 \otimes \cdots \otimes a_{m+n}) = f(a_1 \otimes \cdots \otimes a_m) g(a_{m+1} \otimes \cdots \otimes a_{m+n})$$

This is clearly associative and compatible with δ so this is a dga.

Definition 5. *The $\circ_i$ product on Hochschild cochains is a product (using the Lie degree) $CH^m(A) \otimes CH^n(A) \to CH^{m+n}(A)$ given by*

$$(f \circ_i g)(a_0 \otimes \cdots \otimes a_{m+n}) = f(a_0 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+n}) \otimes a_{i+n+1} \otimes \cdots \otimes a_{m+n})$$

These are put together into the $\circ$ product as

$$f \circ g = \sum (-1)^{|g|i} f \circ_i g$$

I'll let you check that this is not compatible with δ in the sense that $\delta(f \circ g) - \delta(f) \circ g - (-1)^{|f|} f \circ \delta(g)$ is not zero, but it is (up to an overall sign) the commutator of the cup product $f \cup g - (-1)^{|f||g|} g \cup f$ (switching here to the algebra degree).

Then $\circ$ doesn't descend to the cohomology, but the commutator of $\circ$, $\{f, g\} = f \circ g - (-1)^{|f||g|} g \circ f$ does, and we get the following, which I will leave for you to prove to your satisfaction.

Theorem 1. *(Gerstenhaber)*
The Hochschild cohomology of an associative algebra with values in itself forms a Gerstenhaber algebra

2. Chas-Sullivan string topology

Yesterday I told you about algebraic things arising from intersections and concatenations of loops. The first paper was Chas-Sullivan, and people took it and ran with it. It was accepted to the Annals with revisions, and the revisions have not been made. Some people think parts of the paper aren't rigorous enough. People have made the arguments rigorous to their own satisfaction in different ways. If you're really good at technical points, you might not like some of the steps here, I might make it worse, but I'll blame it on them.

I'll start with the non-equivariant story, there is also an equivariant story, and here we'll let M be a closed oriented d-dimensional manifold, and LM be its loop space. That's where we're starting. What we're going to do is discover some algebraic operations on the homology of LM. My understanding of how Chas and Sullivan came across this structure was that it was totally obvious, and they were sure someone had come across it before them. So the thing to do is to combine:

(1) the transversal intersection of chains $C_*(M) \otimes C_*(M) \to C_*(M)$. We have a map at the level of homology. Let me invent new notation, so let me say $C_*(M) \pitchfork C_*(M)$ and not define it, to mean something transversal.
(2) The other thing is concatenation of loops sharing a basepoint, $\Omega M \times \Omega M \to \Omega M$ where $(\gamma_x, \gamma_y) \mapsto \gamma_x \cdot \gamma_y$.

Some ingredients are:

- The evaluation map $LM \to M$, ev, which takes γ to $\gamma(0)$
- I want to think about a singular chain in the loop space, generated by maps from the standard n-simplex Δ^n to LM. We have this adjunction, where we can view σ as a map $\Delta^n \times S^1 \to M$. The reason this is the best way to consider things is because there is a picture of what is happening in the finite dimensional manifold M. I can think of a generator as being a family of loops in M parameterized by Δ^n. I have the simplex as a base and then circles above it.

I'm thinking of Δ as playing two roles, it's the family of basepoints as well. I have a map from $\Delta \to LM$, and I can postcompose with ev to get the basepoints. For each point in the simplex I have a loop coming off it.

With this picture in mind I am going to take two of them and intersect them. The picture is the following:

[Picture and discussion]

The words are "intersecting families of basepoints and then concatenating loops along the intersection locus." I think this is the most important picture I will draw all week.

One thing I don't like about the paper is the notation, I've tried to make it better, maybe I've made it worse.

[Why do you want them to intersect transversally?] We want the intersection locus to be a chain in the manifold. Maybe you have something more complicated, but it's not a piece of garbage, it's a chain. That's not a submanifold, and in a similar way, if you don't have transversal intersection, you might not get a chain. That's why I changed the symbol from tensor product to something stupid. I'll only get this if the pair has a transversal intersection property.

Write $\Delta_x \overset{\sigma_x}{\to} LM$, and I always have the evaluation map. At some point I'll care about the dimension. The other one is $\Delta_y \overset{\sigma_y}{\to} LM$. A want to assume whatever I need. Everything is as nice as possible where I need it. The images of these two maps, $ev(\sigma_x(\Delta_x))$ and $ev(\sigma_y(\Delta_y))$ intersect transversally.

I'll consider the map that goes $\Delta_x \times \Delta_y \to M \times M$, the two compositions, the product, $ev \circ \sigma_x \times ev \circ \sigma_y$. Let $\Delta_{x \bullet y}$ be the transversal preimage of the diagonal $\Delta(M)$.

This is not necessarily a simplex. What dimension will it have? It will have dimension $|x|+|y|-d$, where d is the codimension of $\Delta(M)$ in $M \times M$, the dimension of the manifold.

We've got our intersection locus. Let's define a map from $\Delta_{x \bullet y} \to LM$ by $(\sigma_x \bullet \sigma_y)(p_x, p_y) = \sigma_x(p_x) \cdot \sigma_y(p_y)$. Because I'm doing it in $\Delta_{x \bullet y}$ the basepoints agree.

By definition, I'll extend $\bullet$ to chains, I've got a generator of $C_*(LM) \otimes C_*(LM)$ and get something in $C_*(LM)$. I can't do this at the chain level, really, because of transversality. I have $C_*(LM) \textcircled{$\pitchfork$} C_*(LM) \to C_*(LM)$, this is a partial product on the real tensor power, depending on a choice of decomposition of the result into simplices.

Even though this is not fully defined, I can look at how it interacts with the differential. The thing we get on homology will be perfectly good.

Lemma 1. *∂ is a derivation of $\bullet$, and $\bullet$ passes to a fully defined product on homology. This goes along with what we said yesterday. A cycle representative of two homology classes, we can take so that they intersect transversally. That is what we call the loop product on homology $\bullet$ of degree $-d$*

Let me make a small aside, so small that people in the back can't see it, you can compare this to $\cup$ on HH^* of an algebra. I'll say another aside like two or three times later.

I think I've given you an exercise, to verify that $\bullet$ is associative on homology. Maybe it's got even more structure, and the next question I'm going to ask is, is it commutative.

If you think about this geometrically, concatenation depends a lot on the order. So maybe it's surprising that the answer is yes. We'll show this by defining a product $*$ of degree $-d+1$. Let me draw the picture. Let's do it first for loops. If I have two loops γ_x and γ_y, let s be in the interval. What I want to see is the following intersection of the loops. Here is the loop γ_y and here it intersects the basepoint of γ_x at time s. I want to write down a formula that lets you go around y until you get to the basepoint of x and then goes around x, and then finishes going around y. So

$$\gamma_y *_s \gamma_x = \begin{cases} gamma(y)(2t) & 0 \leq t \leq \frac{s}{2} \\ \gamma_x(2t-s) & \frac{s}{2} \leq t \leq \frac{s+1}{2} \\ \gamma_y(2t-1) & \frac{s+1}{2} \leq t \leq 1 \end{cases}$$

Notice that $\gamma_y *_0 \gamma_x = \gamma_x \cdot \gamma_y$ and $\gamma_y *_1 \gamma_x = \gamma_y \cdot \gamma_x$. Think that the family of $*_s$ as giving a homotopy (this is not a precise statement). We want to extend $*_s$ to transversally intersecting chains. Again, I'm going to make the assumption about transversality wherever I need it in order to make my assumption make sense.

Consider two families $\Delta_x \times \Delta_y \times [0,1]$, and we'll define a map to $M \times M$. I'll take (p_x, p_y, s) to $\sigma_x(p_x)(0), \sigma_y(p_y)(s)$. Along the diagonal I get loops intersecting in the way I described. Let Δ_{x*y} be the transversal preimage of the diagonal again. The dimension is $|x|+|y|+1-d$. Define the analagous map $\Delta_{x*y} \overset{\sigma_x * \sigma y}{\to} LM$ by $(\sigma_x * \sigma_y)(p_x, p_y, s) = \sigma_x(p_x) *_s \sigma_y(p_y)$. These guys intersect in the appropriate way. We extend $*$ to chains and get something of degree $1-d$. I now get a map

$$C_*(LM) \pitchfork C_*(LM) \to C_*(LM)$$

So then, I can say,

Lemma 2. *$\partial(\sigma_x * \sigma_y) - \partial(\sigma_x) * \sigma_y + (-1)^{|x|}\sigma_x * \partial(\sigma(y)) = (-1)^{|x|}(\sigma_x \bullet \sigma_y - (-1)^{|x||y|} - \sigma_y \bullet \sigma_x)$.*

$*$ is fully defined on homology [sic].

The homology of the loop space, together with the loop product, this gives us graded commutativity of the loop product.

Definition 6. $\{x,y\} = x * y - (-1)^{|x|+1}|y| + 1y * x$

With these structures together, the homology of the loop space together with the loop product $\bullet$ and the bracket $\{,\}$ is a Gerstenhaber algebra.

That's kind of a lot, I think. The thing at the beginning isn't even an algebra, but then you get a product and bracket on homology. You don't necessarily have a BV algebra. That's something we can do here. We haven't exploited the circle action. This is where it always comes from for me, if I see a BV operator, I ask where the circle action is. So S^1 acts on LM, let me do it with a single loop. I can exploit this action by, if we let γ_x be a loop in the loop space, then we can describe an S^1 family of loops in LM and I mean, all the loops in the same orbit as γ_x under the action. These all have the same image in the manifold, I'm rotating the basepoint around. This might be more confusing, I'm starting with one loop and getting a one dimensional family. I get a family, a circle in the loop space. I've gone from a point to a one dimensional family. Let me go from a family to a $+1$ dimensional family. Define $\Delta_x \times S^1 \to LM$. I'm doing this operation to every point in the family. The BV operator is also called Δ, it takes $\Delta_x \times S^1 \to LM$ by $\Delta(\sigma_x)$. So $\Delta(\sigma_x)(p_x, s)(t)$ (and this is why I hate this notation). The first part is a loop in LM. So $(\sigma_x(p_x))(s+t \mod 1)$

Extend Δ to chains, and this is a little bit better than what we've done before, there hasn't been a transversality assumption at all. Then this gives us a map $C_*(LM) \to C_{*+1}(LM)$.

Lemma 3. *$\partial\Delta = \pm\Delta\partial$, so we have that Δ passes to homology.*

So to begin with, you might not see how this is connected with what we did before, but it's a theorem that they interact in the exact right way. We started with the homology being a BV algebra, it squares to zero,

Proposition 1. *$\Delta^2 = 0$ on homology.*

The nice way to think about this, I can mark every point on my loop. If I did it again, I'll get a two dimensional family, but it's degenerate, because it's only the same one-dimensional family. It's a degenerate 2-dimensional family.

Theorem 2. *$H_*(LM)$ with $\bullet$ and Δ is a BV algebra, and $\{,\}$ is the BV bracket. The Gerstenhaber algebra we got from $\bullet$ and $\{,\}$ is the Gerstenhaber algebra induced by this BV algebra.*

We had a Gerstenhaber algebra that we were able to promote. All of this, the hints of a relation to Hochschild, there's a real theorem, a family of theorems relating string topology to Hochschild homology in various ways. But

Theorem 3. *If M is simply connected then $A = C^*(M)$ then $HH^*(C^*(M)) \cong H_*(LM)$, and this is an isomorphism as BV algebras.*

You can't necessarily promote HH^* of an algebra to a BV structure, but you can if it's on the cochain algebra of a manifold, and I'd point you to Thomas Tradler's thesis.

The first reworking is Cohen-Jones, a homotopy theoretic realization of the loop product, and Thomas' thesis. This is the only thing I'll say about Hochschild the whole week. We'll see you at two.

3. Master Equations

I want to talk about deformations of a structure and how they relate to dg Lie algebras and the Maurer-Cartan equation.

Remember the metaphor that when we're talking about deformations, we're talking about automorphisms in some automorphism group (say, an infinite dimensional Lie group) near the identity automorphism. The little patch of automorphisms near the identity should be approximated by the tangent space at the identity, which you can give the structure of a Lie algebra.

Okay, that's all very well and good, but often the automorphism group is badly behaved. It might not be smooth or it might have singular points or directions, so we might not be able to do this at all. However, if you're in that situation, what you might try to do is relax your attitude a little bit and embed your problem into a smooth problem where you do have a Lie algebra. Once you've done this, that means that maybe not every vector in your Lie algebra is "good enough" to give you a deformation, and also, this problem was already there, maybe because this is the tangent space and it's linearizing something that might curve back on itself, maybe two vectors give you the same deformation. Probably the best known example of this is deformation of complex structures, which would take us a little too far afield, but maybe I'll just make the comment that automorphisms of complex structures do not form a very nice space, but automorphisms of *almost complex* structures do form a nice space, an infinite dimensional manifold, and so there is a Lie algebra for deformations of almost complex structures and you have the deformations of complex structures sitting as a subset inside of that.

So how can we capture this information? It turns out that a really common way to describe when an element of a Lie algebra satisfies whatever integrability conditions is in dg terms. You express your initial structure as a differential ∂ and then an element of your Lie algebra is a good deformation if you can use it as a supplement to the differential, that is, if $\partial + [\gamma, \quad]$ is still a differential. So the Jacobi identity in your Lie algebra means that this operator is automatically a derivation of your Lie algebra (remember, that was one of the ways we expressed Jacobi), but it doesn't automatically square to zero. To square to zero, we'd check that:

$$(\partial + [\gamma, \quad]) \circ (\partial + [\gamma, \quad])(x) = 0$$

for all x. If we expand this, we get

$$(\partial + [\gamma, \quad])(\partial x + [\gamma, x]) = \underbrace{\partial^2 x}_{0} + [\gamma, \partial x] + \partial[\gamma, x] + [\gamma, [\gamma, x]]$$

Now we can use the fact that ∂ is a derivation of the bracket, and use the Jacobi relation to further take this apart. I should have set the degree of γ to be -1 so that bracketing with it matched with the differential, and so with signs, there's cancelation and we get

$$\partial\gamma, x] + [\frac{1}{2}[\gamma, \gamma], x] = [\partial\gamma + \frac{1}{2}[\gamma, \gamma], x] = 0$$

So in general, in order for γ to give a differential, you need the quantity

$$\partial\gamma + \frac{1}{2}[\gamma, \gamma]$$

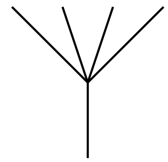
to be in the center of the Lie algebra. We'll simplify this just a little bit further to the stronger but easier condition that this is zero, not just central, and call this the Maurer-Cartan equation, or master equation, in a Lie algebra:

$$\partial\gamma + \frac{1}{2}[\gamma, \gamma] = 0$$

I don't want to go any deeper than that into the general theory of deformations, I want to get to an example, but let me say one other thing first: so far I've been focusing on this in terms of, we have some structure, and to deform that structure, we end up looking at solutions to a Maurer-Cartan equation. Another thing that happens, and this is more clearly in line with what Kate's eventually going to show you in string topology, on Friday, I think, is the idea that if you have a Lie algebra that comes from some place, you can look for solutions to the Maurer-Cartan equation in it, and this will *define* a kind of structure. This probably isn't completely clear, and I want to move to my example. Hopefully you'll be able to see both of these at the same time: the master equation governing deformations of a structure and also the master equation defining a structure.

I want to work again with the Hochschild cochains. Last time I did this with A, an algebra. Today I want to do something simpler, and look at the Hochschild cochains of a *vector space*. This is going to seem a little bit degenerate.

Using the Lie grading, $CH^{n-1}(V)$ is the space $Hom(V^{\otimes n}, V)$, an element of which we can draw, again, as a tree:



but we don't have the codifferential δ or the cup product $\cup$, both of which used the product of the algebra in their definition. In fact, for the purposes of this example, let's pretend that we've never even heard of an associative algebra, we live in a world where we know all about Lie algebras but there's no such thing as an associative algebra, I'm doing this so that I can emphasize the "reveal" at the end. Anyway, even though we don't have $\cup$ or δ, we can still define $\circ_i$, $\circ$, and the bracket, and verify that the Hochschild cochains form a graded Lie algebra, with the Lie grading. There's no differential, it's just a graded Lie algebra.

So let's look at what solutions to the master equation look like in this Lie algebra. Since there's no differential, the master equation simplifies even more, and just looks like:

$$\frac{1}{2}[\gamma, \gamma] = 0$$

We were looking in degree -1 in my general framework to match the differential, and so here, even though we don't have a differential, we know these are cochains so we expect the differential to go up, so let's look at the degree 1 solutions to this equation, which means they are in $Hom(V^{\otimes 2}, V)$. So they are products of some sort. What is the bracket of a product with itself? Well, since the bracket is the commutator of the $\circ$ product, we know that $[\gamma, \gamma] = \gamma \circ \gamma + \gamma \circ \gamma$, so the equation becomes

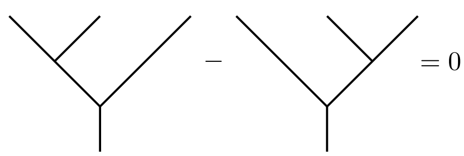
$$\gamma \circ \gamma = 0$$

A side note: this is a common thing to have, a bracket arising as the commutator of a product, and in cases like this it's perfectly legitimate to rewrite the Maurer Cartan equation using this product in more generality:

$$\partial \gamma + \gamma \circ \gamma = 0$$

I may say a few more words about this at the end.

Anyway, back to our example, what is $\gamma \circ \gamma$? Well, remember, $\circ$ is $\circ_0 - \circ_1$, so we can write this in trees as follows:



This should look familiar. This is the associativity relation for the product γ. So we just discovered that a Maurer-Cartan solution, a solution to the master equation, this is the same thing as an associative algebra. We just used a master equation in my second way, to define a "new" algebraic structure.

Okay, so now what? Let me remind you that one thing we can do with γ is perturb the differential of our complex. We started with a vector space so we didn't have the Hochschild differential, but now we can put this in and consider $CH^*(V)$ with the differential $[\gamma, \quad]$. I'm not going to show this on the board, but I asked you to verify as an exercise that this differential is exactly the differential I defined for the Hochschild cochains of an algebra.

So now let's look at that situation, we already have a product, γ, we have the Hochschild cochains $CH^*(A)$ (I'll call it A because it's got an algebra structure), and so I have a codifferential δ (and the cup product, which I'm not going to use) along with the bracket or the $\circ$ product. So now we can ask for degree one solutions to

$\delta\eta + \frac{1}{2}[\eta, \eta] = 0$.

We could take this apart in terms of trees again, but I want to do something a little more abstract. We know a couple of things: I've claimed that $\delta\eta = [\gamma, \eta]$, and we showed that γ being associative was the same thing as $\gamma \circ \gamma = 0$, so we can put this together, expanding the brackets in terms of the $\circ$ product, and get

$$\gamma \circ \gamma + \gamma \circ \eta + \eta \circ \gamma + \eta \circ \eta = 0$$

which we can factor as

$$(\gamma + \eta) \circ (\gamma + \eta) = 0$$

and we know what this means, this means that $\gamma + \eta$ is an associative product. So if we start with nothing, no structure on our vector space, solutions to the master equation give us associative algebra structures. If we start with a product, then solutions to the master equation give us deformations of that product, that is, give us infinitesimal products η which are probably not associative themselves, but you can add them to your given associative algebra product γ and get an associative structure back. So here's a master equation being used in the first way I mentioned, giving deformations of a structure you already have.

Okay, so let me just say a few very vague things in more generality. We started with a Lie algebra but I was working almost exclusively with the $\circ$ product. Depending on your setting, it can make a lot of sense to look at a master equation like the one I wrote down eventually:

$$\delta x + x \circ x = 0$$

or one in any number of other contexts. Often, since the Lie algebra is only giving you a first order approximation, you want to go further, which you can do with the L_∞ master equation, where you use the up to homotopy version of a Lie algebra that has higher and higher brackets. If you're in non-commutative geometry, you probably want to use an associative or A_∞ master equation instead of a Lie one. It makes a lot of sense if you're doing BRST quantization, you have a BV algebra and you probably want to look at a BV master equation that involves ∂ and Δ. Some people think that there are Lie bialgebras governing certain deformation situations (I don't know very much about this) and that there should be a bialgebra master equation.

Let me say two more general things. Here's some kind of connection to geometry that I like. Remember the idea of the automorphism group of a structure in the deformation apporach. You could imagine that you have a moduli space of structures and you have the automorphism group above each structure, and so that gives you a bundle with fiber the automorphism group. Then if you look at the Lie algebra which which is the tensor product of the differential forms on the moduli space with the Lie algebra of the automorphism group, this Lie algebra is the connections on the bundle that we're talking about, and the left hand side of the Maurer-Cartan equation is the curvature form of this connection. So what you're saying in this case is that Maurer-Cartan solutions are local flat connections at this point in moduli space.

So then the last thing is, we can write this equation down not just in algebra, but in topology. I don't know what a Lie group is, topologically, but we can look at the unsymmetrized version, $\partial X = X \circ X$, and we have to define what $\circ$ means, and the general idea is that X is a bunch of spaces, moduli spaces, say, and the boundary of moduli space is made up of gluings or products or whatever of the lower level pieces of the moduli space. This is the sense, I think, in which Kate will be using the master equation idea at the end of the week.