1. Introduction

General. I want to give you an introduction to what I'll be talking about. We'll start with Chas and Sullivan's original construction. We'll go on to Cohen and Godin's reformulation. We'll see how moduli spaces get involved, and finally, on Friday, we'll talk about compactified moduli spaces in string topology, my work. I want to give a brief introduction to the kinds of problem that Chas and Sullivan were talking about when they stumbled into string topology. The stuff that I'm talking about right now will be generalized quickly later.

The brief introduction will be on the Goldman bracket and the Turaev cobracket.

To begin with, let Σ be an oriented compact topological surface with or without boundary. Consider free homotopy classes of curves on the surface. Let $\hat{\pi}$ be the set of free homotopy classes of oriented curves on Σ.

If you know about the first fundamental group, this should be reminiscient. Here free means we don't care about a basepoint. I'll apologize right now, but I'm going to conflate free homotopy classes of curves and representatives. Things that I say will be defined in terms of representatives but will be independent of that choice. Things like α will mean one or the other. Let V be the vector space generated by $\hat{\pi}$ (over rational coefficients).

We have a giant set, and so the vector space that's infinite dimensional.

What I want to do is perform a construction to give us an algebraic relation on V, the Goldman bracket. Let α and β be in $\hat{\pi}$ or be representatives of these classes. Assume α and β have representatives that intersect only in transverse double points. The picture of such an intersection is like this.

Now, let $p \in \alpha \cap \beta$, and we'll define $\alpha \cdot_p \beta \in \hat{\pi}$. I'll cut the strands at the intersection and reconnect them in the only other way that keeps the orientations locally.

[Picture]

I'll use this local move to define an operation called the bracket of α and β. This will be the sum over all the intersection points of a signed version of $\alpha \cdot_p \beta$, where the sign comes from the orientation of Σ:

$$\sum_{p \in \alpha \cap \beta} \pm \alpha \cdot_p \beta$$

So in my picture I'll get a negative sign, and I'll get a positive sign in my other intersection.

Theorem 1. *(Goldman) This is well-defined, and then we can extend the bracket linearly to get an operation $V \otimes V \to V$. This is the Goldman bracket.*

Now we have on this big infinite dimensional vector space a two to one operation. We can ask what properties this satisfies.

Theorem 2. *The vector space V together with the bracket is a Lie algebra.*

Some of you may not have seen the definition of a Lie algebra yet, this is the kind of thing that Gabriel will be talking about. The local move where we cut and reconnected, we could picture doing this where the strands come from the same curve. That's where Turaev's cobracket comes in.

It's exactly the same kind of idea. We'll take an element of $\hat{\pi}$, perform the cutting and reconnecting at each intersection point, and get an operation that's completely analogous. So here's α, and I've drawn a curve with only one self-intersection point

p, and here's the first use of Δ, Gabriel and I are going to use Δ for a bunch of different things. The Goldman bracket took two curves to one, and here we'll start with one and get two. I'll get these pictures for $\Delta(\alpha)$:

[Picture]

What's the difference between these two pictures. For the Goldman bracket we had a first curve and a second curve. We have two ways of ordering the curves, first and second or second and first. The sign we get will depend on whether we have local agreement or disagreement with the orientation.

Theorem 3. $\Delta(\alpha)$ *is well-defined.*

We can do the same kind of extension linearly, and define an operation $V \to V \otimes V$. That's your Turaev cobracket. Just like the vector space V with the bracket is a Lie algebra, this satisfies some things too.

Theorem 4. *V together with the cobracket is a Lie coalgebra. If you've seen a Lie algebra, maybe you've seen a Lie coalgebra before, but maybe not, this is one of the structures that Gabriel will talk about. If I look at the two structures together, this is an involutive Lie bialgebra.*

I want to take just a couple more minutes and say, "what have we done?" We've taken a construction with pictures and ended up in algebra, this is algebraic topology, but this remembers something about the intersections of the curves that we started with.

If you know something about the representatives of the classes you're talking about. Here's a fact. If α and β have disjoint representatives, then their bracket is zero, $[\alpha, \beta] = 0$.

Theorem 5. *If α has a simple representative, which has no self-intersections, and assuming $[\alpha, \beta] = 0$, then α and β have disjoint representatives.*

What would be the best thing? The bracket would know all about intersections. The best thing would be that if the bracket is zero, there are disjoint representatives. That's not true, and it's one of your exercises.

Another fact, if α has a simple representative then its cobracket is zero. That's basically the same statement, you have no intersections to sum over. Is there an analogue to Goldman's theorem, but for the cobracket? This is the question that Chas and Sullivan were thinking about. If $\Delta(\alpha) = 0$, does α have a simple representative?

The answer is no, or, not necessarily. Imagine taking a simple curve, and going around it twice. That curve, move it in its homotopy class, and so it has transverse double points, then its cobracket is zero. So the power of a simple curve has cobracket zero. Does α have a representative which is a power of a simple curve. This was not obvious for a long time. Much later, Chas made significant headway.

Theorem 6. *Except possibly for a sphere with 0, 1, or 2 boundary components or the torus, there is a curve with zero cobracket that is not the power of a simple curve.*

Let me draw an example to end. It's a good exercise, not one I assigned, to check that $\Delta(\alpha)$ is zero.

Two tangents in terms of work, Chas has entirely combinatorial descriptions of these things, and because it's combinatorial, she can do computations and calculate intersections of curves. Since the first string topology papers, that's where her work has gone. The other tangent is, string topology is another generalization of this kind of stuff.

Algebraic Introduction. Mathematics is full of binary operations, like multiplication, that have "multiple inputs" and "one output," and also binary cooperations, like the diagonal map of a set, that have "one input" and "multiple outputs." There are plenty of examples of structures that have many to one operations, one to many operations, and many to many operations. My favorite are in the flavor of "field theories," but there are lots of examples.

There are many frameworks to talk about such structures; I'm going to use my lecture series to describe one general yoga for doing so. This way of looking things started with what are called PROPs, for "Products and Permutations." In a PROP, you have spaces of k to ℓ operations with k inputs and ℓ outputs, and you have ways of concatenating them horizontally and composing them vertically which satisfy various identities that you would expect if you think about the way that many to many operations generally work. PROPs turned out to be too hard to work with, generally, and so over time people analyzed different simplified versions of them: operads, cooperads, dioperads, $\frac{1}{2}$-PROPs, and so on. At the same time, there were various additions made to the theory to account for different structures like traces and nondegenerate inner products that weren't well-handled by the many-to-many framework, like cyclic and modular operads and wheeled PROPs.

I'm going to be talking about a particular version, properads, that just use the vertical composition of many to many operations, because this is a sufficient tool to describe what's going on algebraically, at the chain level, in string topology, and because the theory for properads is sufficient that we can make statements and conjectures and perform analysis, we have a toolkit, let's say.

So this is the goal, and although we won't be able to get into the nuts and bolts of the application to string topology, I hope that by the end of the week you'll have the ability to look under the hood yourself.

I want to show how that works in a toy model by talking not about representations of properads, but representations of dgas.

If you think about the history of the notion of a "group," the first groups that were studied with rigor, Galois groups, symmetric groups, Lie groups, and so on, were all viewed as the groups of automorphisms of some structure. That is, one did not study groups but rather representations of groups. Only in the late nineteenth century did the "group" as an abstract notion, free from specific representations, come into focus as an object of study. I don't know the exact history, but I think this conceptual movement was actually extremely helpful in studying representations, because it facilitated the development of representation theory, which gave an abstract recipe for answering the kinds of questions that were being asked for individual groups.

I like to think of operadic algebra as occupying a similar sort of space. There are many kinds of algebra that are of interest to people, associative, commutative, Lie, Gerstenhaber or Poisson, BV, Hopf, Leibniz, coalgebras, bialgebras of various sorts, Frobenius, algebras coming from topological or geometric constructions, little disks (E_n), moduli spaces and compactified moduli spaces, racks, quandles, at this point I'm just riffing. There are algebras with invariant inner products, with traces, with units and counits, with differentials, with different compatibilities, different ground rings, quantum algebras, I could go on and on. This, to me, is like the situation with groups in the nineteenth century. There are permutation groups, alternating groups, braid groups, cyclic groups, groups of isometries of polytopes, classical matrix groups that are realized as the automorphisms of a vector space with some kind of bilinear scalar product, and then it turns out that all of these are representations of different abstract groups. In the same way, all of the kinds of algebra I have named are representations of different properads.

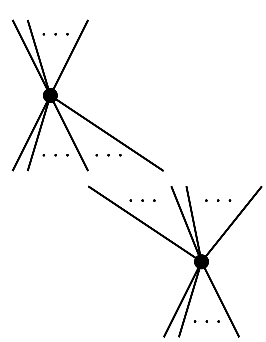
But this is enough of the big picture, let's look at my toy model.

Suppose I have a unital algebra, that is, a vector space with a bilinear associative multiplication μ and a unit 1. What is a (left) representation or module or action of this algebra on the vector space V? Well, one way of looking at it is that it's a map $\nu : A\otimes V \to V$ which satisfies compatibility with the product map: for a and b in A and v in V, you have $(ab)v = a(bv)$. You should also have unit compatibility: $1(v) = v$.

Another way to look at this, that I like a little better for what we are doing, is to use the duality between Hom and tensor to say that $Hom(A \otimes V, V)$ should be the same exact set as $Hom(A, Hom(V, V))$, so the data should be the same as a map φ from A to $Hom(V, V)$, that is, an endomorphism of V for each element of the algebra A. What does the compatibility become? Well, $(ab)v = \varphi(ab)(v)$ and $a(bv) = \varphi(a) \circ \varphi(b)(v)$ so this relation becomes the relation that $\varphi(ab) = \varphi(a) \circ \varphi(b)$. Similarly, we see that $\varphi(1)$ should be the identity map $V \to V$. The endomorphisms $Hom(V, V)$ form an algebra using composition of endomorphisms, and the identity map as the unit, and so we can encapsulate the whole structure by saying a representation of A on V is exactly the same thing as a map of unital algebras from A to $Hom(V, V)$.

So let's look at a really simple example. Let's look at the so-called "algebra of dual numbers" $A = k[\Delta]/\Delta^2$. What is a representation of this on the vector space V? Well, V is generated as an algebra by Δ, with $\Delta^2 = 0$. A representation should be a map to the algebra $End(V, V)$, so it should be specified by an endomorphism $\Delta_* : V \to V$ satisfying $\Delta_*^2 = 0$. So a representation of the algebra of dual numbers is the same thing as a vector space with a square zero operator, that is, a chain complex.

Now, let's say that we were interested not just in maps from V to itself, but among all of the tensor powers of V, so maps in $Hom(V^{\otimes k}, V^{\otimes \ell})$ for varying k and ℓ. Now this is not exactly an algebra, at least that's not the usual way to think about it, because how do you compose two maps with different ks and ℓs. But there are partial compositions where you compose some of the ℓ outputs into some of the k which we can picture like this:



So we partially compose two maps f and g by doing $(\mathrm{id}^{\otimes i} \otimes f) \circ (g \otimes \mathrm{id}^{\otimes j})$. We could imagine other ways of composing, but this is enough for now. If you write down what kinds of associativity, equivariance (because there is an action of the symmetric group on both the inputs and the outputs of the operation), and unit relations this structure satisfies, you get an algebraic structure.

All that said, I want to start with some more mundane things by laying down some groundwork so that we can get on the same page moving forward. Along the way, sort of before we get to properads, there are going to be a bunch of different types of algebraic structure that arise naturally in various settings. I'd rather not present them all as a laundry list, definition, definition, definition, but there is going to be some of that.

Here's a little outline of what I plan to talk about today or eventually:

(1) Chain complexes
(2) dg associative algebras and their representations (possibly commutative)
(3) dg coassociative coalgebras (possibly cocommutative)
(4) Frobenius algebras ("open" Frobenius algebras)
(5) dg Lie algebras
(6) dg Lie coalgebras
(7) dg Lie bialgebras
(8) dg Gerstenhaber algebras
(9) dg Batalin-Vilkovisky algebras
(10) dg properads

I'm hoping to discuss the first four today, five through seven tomorrow, eight and nine when I can, and the main goal of this part of the summer school is to describe ten, properads, and the beginnings of applications of them. Properads constitute an algebraic structure at a different level than these other structures, and you should think of two through nine as all constituting different representations of different properads.

2. Chain complexes, Associative and Lie algebras

This lecture is (hopefully) going to contain a lot of review. I want to spend an hour on this more accessible material in order to draw connections to help motivate the properadic framework that I'm going to be talking about later. Let's get started.

2.1. **Chain complexes.** I want to work over the rational numbers, and so everything I say for the rest of the week will be over the rationals or possibly the reals or complex numbers. A chain complex is a set of vector spaces C_n indexed by the integers (we call this indexing number the degree) along with a degree -1 differential or boundary map $\partial : C_n \to C_{n-1}$. A *chain map*, $C \to C'$ is a set of maps $C_i \to C'_i$ that commute with the boundary map: $f\partial = \partial f$ although we'll look at a different notion of maps between complexes with the Hom complex right away. Chain complexes are one of the basic places to work in algebraic topology, and essentially everything I describe will be some kind of extra structure that sits on top of an underlying chain complex. Because of this, I want to establish some properties of chain complexes.

The most basic thing that you can do with a chain complex is take its homology. This is a graded vector space $H(C)$ which is $\ker \partial / \mathrm{im} \partial$, and it inherits its grading from C. This is functorial, so maps between chain complexes induce maps at the level of homology, and we will call a map which induces an isomorphism at the level of homology a *quasiisomorphism* or *weak equivalence*. There is also a notion of *chain homotopy equivalence* which is very different over a ring, but since we are working over the rationals, we will not need to focus on the minor differences in the data involved in these two notions, and will focus on quasiisomorphism.

Let's do some constructions with chain complexes.

Definition 1. *If C and C' are chain complexes, then we can take the tensor product $C \otimes C'$, where we define $(C \otimes C')_n$ to be*

$$\bigoplus_{i+j=n} C_i \otimes C'_j$$

The differential on $C \otimes C'$ will be $\partial_C \otimes \mathrm{id} + \mathrm{id} \otimes \partial_{C'}$, which doesn't mean what you might think at first glance. Rather, if I'm applying this to $x \otimes y$ in $C_i \otimes C'_j$, then instead of $\partial_C(x) \otimes y + x \otimes \partial_{C'}(y)$ you get $\partial_C(x) \otimes y + (-1)^i x \otimes \partial_{C'}(y)$, so there's an extra sign.

This is as good of a time as any to introduce the sign convention that indicates that this is the appropriate thing to do, the so-called Koszul sign convention. Algebraic topology is full of signs and this convention does a really good job of coming up with the "right" answers that make everything work coherently. The convention says that, you need to write things in a line, that's how we write things, but if you need to move symbols x and y that have signs past each other then you pick up a sign, $(-1)^{|x||y|}$. So for instance, when you apply maps to a tensor product $(f \otimes g)(x \otimes y)$ we need to move the symbols g and x past each other so you get $(-1)^{|g||x|} f(x) \otimes g(y)$. In the case that we were just doing, we're applying $(\mathrm{id} \otimes \partial_{C'})$ and $\partial_{C'}$ has degree -1 so we get the sign $(-1)^{|c|} = (-1)^i$, as promised.

Specifically, sometimes we'll act on the tensor product $C^{\otimes n}$ on the left by the symmetric group S_n by permuting factors, so that if $\sigma \in S_n$, then

$$\sigma(x_1 \otimes \cdots \otimes x_n) = \epsilon(x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)})$$

where ϵ is the sign induced using this rule.

So we just described a tensor product of chain complexes, now let's describe an "internal Hom."

Definition 2. *Let C and C' be chain complexes. The Hom complex $Hom(C, C')$ has as its n-degree space the vector space of degree n maps between C and C' (collections of maps $C_m \to C_{m+n}$); these may or may not be equivariant with respect to the differentials of C and C'. Then the differential D in the Hom complex is given by "bracketing with ∂" so*

$$D(\psi) = [\partial, \psi] = \partial_{C'} \circ \psi - (-1)^{|\psi|} \psi \circ \partial_C.$$

What does this mean? Well, let's look at the degree zero homology of the Hom complex. The kernel of the hom D are degree zero maps ψ so that $D(\psi) = 0$, that is (since the degree of ψ is 0), maps satisfying $\partial_{C'}\psi = \psi\partial_C$. These are exactly the chain maps we described before. If we have $D(\phi) = \psi - \psi'$, then that means exactly that ϕ is a chain homotopy between ψ and ψ'. So the homology is spanned by chain maps, and two are equivalent if they induce the same morphism on homology.

These are basic concepts and you may have seen them before; I'm emphasizing them because in the subsequent lectures we'll be dealing with the endomorphism properad of a chain complex, which will be made up of the Hom complexes

$Hom(C^{\otimes m}, C^{\otimes n})$, and I want to be clear on the convention so that this particular ingredient is not a surprise.

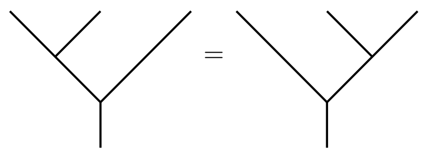
So let's turn now to differential graded algebras.

Definition 3. *A* differential graded algebra*, or* dga*, is a chain complex A along with a product map $A \otimes A \to A$ which is an associative chain map. We'll often write ab to mean $\mu(a \otimes b)$.*

Associative, let me write this in two ways, can be written in terms of the commutativity of a diagram:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

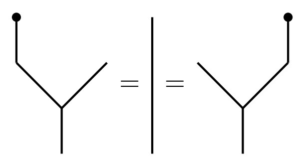
This is not my preferred way to write this, I'd rather encapsulate the same thing with the picture:



We read the diagram from top to bottom; the left hand tree corresponds to moving along the top and right of the commutative diagram and the right hand tree corresponds to moving along the left and bottom.

The fact that I have asked for the map to be a chain map means that $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$, the Leibniz property for multiplication. We also call this property ∂ being a derivation of the product.

Definition 4. *A unital dga is a dga A equipped with a chain map $k \to A$ that is a unit for the product:*



Definition 5. *A commutative dga is a dga A so that μ is commutative. I'll write that by saying that μ composed with the permutation map $\sigma \in S_2$ is again μ: $\mu\sigma = \mu$ or $\sigma(\mu) = \mu$.*

So a good example of a dga is the cochains of a space, with the boundary map as differential and the cup product as the product. A good example of a commutative dga is the differential forms of a smooth manifold, with de Rham d and the wedge product.

If we dualize everything, we get the idea of a dg (coassociative) coalgebra, which is a chain complex C along with a chain map called the coproduct $\Delta : C \to C \otimes C$ which is coassociative. A counit for this is a map $C \to k$ that is a counit for the coproduct. We can make pictures for this in the same way as before by just turning the pictures for a dga upside down. The diagonal map $X \to X \times X$ induces a coassociative coalgebra structure on the singular chains of a space, using the identification of the singular chains on $X \times X$ with the tensor product of the singular chains on X with itself. We would call this cocommutative if $\sigma\Delta = \Delta$.

Whenever you have a structure that is defined by chain maps, like a dga or a dg coalgebra, it descends to the homology, since homology is functorial. This is *not* always the best thing to do with a structure like this because it often forgets homotopical information, but it's a first approximation to the chain level data. So if you have a dga, you get an associative algebra structure on the homology, and if you have a dg coalgebra, you get a coassociative algebra on the homology of the chain complex. In both of the examples I gave, the cohomology and homology of a space, when you pass to the homology, the structure becomes commutative. So the cohomology of a space is a commutative algebra (we can view it as a commutative dga with zero differential) and the homology of a space is a cocommutative dg coalgebra with zero differential.

2.2. **Lie algebras.** Let's turn to the next most common type of algebra, a nonassociative type of algebra, namely a Lie algebra.

Definition 6. *A dg Lie algebra is a pair* $(\mathfrak{g}, [\ ,\ ])$ *where* $\mathfrak{g}$ *is a chain complex (with internal differential* ∂*) and* the bracket $[\ ,\ ]$ *is a bilinear map* $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ *which:*

(1) *is* graded *skew-symmetric:* $[x, y] = -(-1)^{|x||y|}[y, x]$*, and*
(2) *satisfies the graded Jacobi relation, which we could describe in a few ways:*

$$[[x, y], z] + (-1)^{|x|(|y|+|z|)}[[y, z], x] + (-1)^{(|x|+|y|)|z|}[[z, x], y] = 0$$

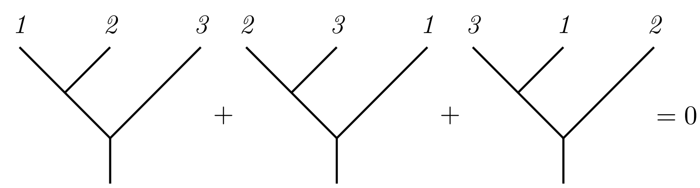
This has some signs, which we can get rid of with our convention by saying, if τ *is a cyclic permutation of order* 3*, that*

$$[\ ,\ ] \circ ([\ ,\ ] \otimes \mathrm{id}) \circ (\mathrm{id} + \tau + \tau^2) = 0.$$

We could describe it conceptually by saying that the bracket is a graded derivation of itself, in which case we'd write it a little differently:

$$[[x, y], z] = (-1)^{|y||z|}[[x, z], y] + [x, [y, z]],$$

or we could write it using the same kind of pictures as before:



So I want to make a point about this. If you are used to ungraded Lie algebras, then you're used to the idea that skew-symmetry means that $[x, x] = 0$. But in a graded Lie algebra, we have $[x, x] = -(-1)^{|x||x|}[x, x]$ so that if $|x|$ is odd then we don't necessarily get $[x, x] = 0$.

I'm describing Lie algebras as a purely algebraic structure, but if you have seen any Lie theory, you have probably seen Lie algebras as arising from Lie groups, where you start with a Lie group G and then you find you have a Lie algebra structure on the tangent space to G at the identity. These are the most familiar Lie algebras, I guess. The first nontrivial Lie algebra we see is usually $\mathfrak{so}(3)$, in multivariable calculus, in the guise of the cross product.

Anyway, because of this relationship, Lie algebras are intimately related to deformations of many different kinds of structure, for the following reason: if you have a structure, any kind of structure, then you can look at the automorphisms of that structure, which form a group. We can think of deformations as being automorphisms that are somehow "near the identity" automorphism. If we pretend this is a Lie group, then every automorphism "near the identity" comes from an infinitesimal tangent vector at the identity automorphism, and these fit together into a Lie algebra. So up to first order, at least, deformations of a structure "should" be related to this Lie algebra. We'll explore this relationship more tomorrow or the next day.

3. Topological background I

I'm actually not going to talk about the Goldman bracket or the Turaev cobracket, but thinking about these questions is what led Chas and Sullivan to basically stumble upon string topology. So I want to, this afternoon, talk about some of the topological tools we'll need for this, intersections, loop spaces, and equivariant homology.

Let me remind you of a couple of things that Gabriel talked about this morning. If X is a topological space, then you have the cup product on singular cochains of X

$$C^*X \otimes C^*X \to C^*X$$

and this descends to cohomology,

$$H^*X \otimes H^*X \to H^*X$$

One structure he didn't mention is the cap product, which goes from chains tensor cochains to chains:

$$C_i(X) \otimes C^j(X) \to C_{i-j}(X)$$

so we think of the cochain cutting down the chain by the degree. This mixes homology and cohomology:

$$H_*(X) \otimes H^*(X) \to H_*(X)$$

I should have put up a big motivating question, what is the algebraic topology of a smooth manifold? You can think of this qustion as motivating a lot of what we do. Part of one answer is by talking about Poincaré duality. Here I'll let M be an orientable d-dimensional manifold, closed, and $H_d(M, \mathbb{Z})$. This is an infinite cyclic group, and so I can choose one of two generators for it. A generator $[M]$ of $H_d(M, \mathbb{Z})$ is called an orientation class or fundamental class.

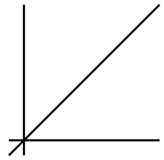
You can choose your favorite definition of orientable or orientation, my favorite one is that it is a choice of this generator. I will pick a representative and give it the same name. Let a cycle $[M]$ represent $[M]$, and there is a map $C^*(M) \to C_{d-*}(M)$ where I'm capping with this representative of the fundamental class, the duality map $[M]\cap$. This descends to homology and the theorem is

Theorem 7. *The induced map on homology which goes from $H^*M \to H_{d-*}M$ is an isomorphism.*

This is a special thing we can say because we have a fundamental class because we've got a closed orientable manifold. This is Poincaré duality.

Poincaré duality is great for two reasons, it says something about the algebraic topology of manifolds, but we're going to use it to do some intersecting.

For any set we have the diagonal map $X \xrightarrow{\Delta} X \times X$ where $x \mapsto (x, x)$. I always have the same picture that Gabriel drew this morning, this is the diagonal:



Then the Kunneth theorem tells us that

$$H_*(X \times X) \to H_*(X) \otimes H_*(X)$$

is an isomorphism (over $\mathbb{Q}$), so Δ induces a map $H_*(X) \to H_*(X) \otimes H_*(X)$ and a map $H^*(X) \leftarrow H^*(X) \otimes H^*(X)$. When people say that the cup product comes from the diagonal, they mean that the cup product is the induced map Δ^*. The map of the Kunneth formula, let me give you some names to google, Eilenberg-Zilber or Alexander Whitney.

So what we're going to want on a manifold is a product on homology. So we can take two homoloogy classes, $H_i(M) \otimes H_j(M)$. By Poincaré duality, each factor is isomorphic to the appropriate cohomology $H_i(M) \otimes H_j(M) \to H^{d-i}(M) \otimes H^{d-j}(M) \to H^{2d-i-j} \to H_{i+j-d}(M)$. This is Poincaré duality again.

If I take this composition, I've got a two to one map on homology that is degree $-d$. We're dealing with a manifold. You might have a more general space where you can do this, certainly on a monifold. This we'll call the intersection product, and you might notice that this has degree $-d$ (like Bryan said).

This is kind of really great. We knew that the homology formed a coalgebra, but it also forms an algebra. So:

Theorem 8. *$H_*(M)$ is an associative algebra.*

Later Gabriel will talk about what you have here with a coproduct and product that interact on the same space, you get a richer structure. I want to make a big deal, you should have one to two, but we made a map that goes the other way on homology. We've created a "wrong-way" map on homology.

I want to discuss this in terms of immersions, which I can do because I'm working over $\mathbb{Q}$. Let a be an i-dimensional homology class, and b a j-dimensional homology class, and assume that a and b are represented by immersed submanifolds (closed, oriented). Let me assume that these immersions $A \to M$ and $B \to M$ intersect transversally. Consider, I have maps

$$\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & M \end{tikzcd}$$

and I can fill this diagram in with what is called the fiber product

$$\begin{tikzcd} A \times_M B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & M \end{tikzcd}$$

or I can say that $A \times B$ maps to $M \times M$, which contains the diagonal, and $A \times B$ intersects Δ transversally. I'll say then that equivalently, $A \times_M B$ is the (transversal) preimage of the diagonal.

How do these things get to have the same name? Let me write this as a set. What is it? This fiber product is some subset of $A \times B$. It's points (x, y) where x is in A, y is in B, and such that $a(x) = b(y)$.

Because of transversality, $A \times_M B$ is an oriented $i + j - d$-dimensional manifold, and $A \times_M B$ sits by the restriction of $a \times b$ to $A \times_M B$ to $\Delta(M) \subset M \times M$, which is isomorphic to M.

So define $a \cdot b = (a \times b|_{A \times_M B})_*[A \times_M B] \in H_{i+j-d}(M)$. This is exactly the same thing as perturbing a and b slightly so that they intersect transversally and then intersecting them and getting a representative of the homology class which is literally the intersection product.

It would be nice to draw the picture right now that shows why this is the same as the cup product. I could draw that picture later, unofficially, maybe. Any picture that you draw for transversality is the right picture. If A and B are submanifolds of N, they intersect transversally if, well, for all points in the intersection, I've got tangent spaces to A, to B, and to N, then for all p in the intersection of A and B, $T_pA + T_pB = T_pN$. So the picture we drew earlier with two curves on a surface, you get two independent vectors and so they span the whole space. You write this as $A \pitchfork B$

More generally, if A is a submanifold and f is a smooth map $B \to N$. You write $f \pitchfork B$ if $T_{f(b)}(A) + df_b(T_bB) = T_{f(b)}N$ for all b so that $f(b)$ is in A.

This intersection product has us choosing two representative cycles, moving them slightly, and making them transverse. Here's a picture of two curves that don't intersect transversally. There is something non-transverse here, if you move one or the other of these curves in the homotopy class, you'll either get a picture with no intersection or a picture where there are two points. As soon as I move a map just a little bit, it's okay. If f is not transversal to A then it is homotopic to one that it is. It's really close to a transversal map.

If we apply this to this setting where a and b are homology classes represented by A and B, then $A \times B \to M \times M$ is transversal to the diagonal.

We said that we formed the pullback of the diagonal under this transversality assumption, let me draw a couple of pictures. If $A \times B$ goes to $M \times M$, here's the diagonal, here's the image, and you get something that intersects transversally, I can pull this back and get a submanifold of $A \times B$. Maybe I have something that looks like this degenerate picture; when I pull this back I get something that looks a little sad, not quite a good submanifold.

So transversality is an important property to have to get submanifolds of the proper dimension. We can move something just a little bit and get the things we want, get something generic.

Let me move to a different topic and go to loop spaces and equivariant homology. String topology is concerned with the homology of a loop space of a manifold. That's where the word string comes from in this context. I might rush this a little bit. The most important thing is the definition. So loop spaces. I'll use S^1 to be the unit interval $[0, 1]$ quotiented by the relation $0 \sim 1$. I have a distinguished point on the circle. The free loop space of M is the maps from S^1 into M. You could be talking about different kinds of maps. Let me say once that I'll be using piecewise

C^∞ maps. There's a related definition, the based loop space of M at $p \in M$, and I'll call this $\Omega_p M$, and thats maps from $(S^1, *)$ to (M, p). Both of these are infinite dimensional manifolds. If that freaks you out, I'll share a confession with you, I pretend they're not. There are lots of analytical considerations here that I don't worry about. Don't tell anyone.

So let me make a couple of remarks.

(1) If γ is a point in LM, it has a basepoint. It doesn't have to be a particular one, it has $\gamma(0)$. This can be any point of M.
(2) If I'm talking about the based loop space $\Omega_p(M)$, you can do something from the first fundamental group, where I can concatenate loops, $\Omega_p M \times \Omega_p M \to \Omega_p M$. In your exercises, you'll show that this product is not associative on strict loops, although the map it induces on homology is associative.
(3) Also, if I have two different points in M, I can consider $\Omega_p M$ and $\Omega_q M$, these are actually homotopy equivalent. You can choose a path going from one point to the other, and I'll write ΩM.
(4) I have an evaluation map $ev : LM \to M$ which takes γ to $\gamma(0)$, and I can write this as a fibration with fiber ΩM:

$$\begin{tikzcd} \Omega M \arrow[r] & LM \arrow[d, "ev"] \\ & M \end{tikzcd}$$

(5) The last thing I want to say is that S^1 acts on the loop space LM by rotation. If I take an element of S^1, I can precompose and everything is going to shift around, and I get a loop tracing the same path at the same speed but a different parameterization.

    This is what equivariant homology is about. You want something that captures not just the structure of the space, the group action.

    This action is not free. What are the fixed point of this action? Constant loops.

We'll be looking at the homology of the loop space, sometimes on its own and sometimes with this action, meaning equivariant homology. Let me say what that means. Let G be a topological group that acts on a space X, the action is continuous, and X/G is the orbit space. The naive thing to do is to have the G-equivariant homology of X, $H_*^G(X)$ should be something like $H_*(X/G)$, the regular homology of the orbit space.

That's the thing to keep in the back of your head. That's what we would like for it to be. I'd be happier if it actually were. The orbit space in general might be badly behaved, it might be non-Hausdorff, those things don't exist in my world, I don't know what to do, or the orbit space might be contractible, that's not quite what I want either. This definition is okay if the action is free. Your naive thing is exactly right. What are we going to do. There's a really nice trick if the action is not free. Even if the action is free, the trick won't give you anything new.

The trick is the Borel construction: fix G, there is a unique (up to homotopy) contractible space EG on which G acts freely. This is going to solve all of our problems. Why? One is, if I have X, then X is homotopy equivalent to $X \times EG$. Then $X \times EG$ has a free action by G, G acts diagonally on each factor, since the action on EG is free. So we can study the orbit space of that action, which will give us the G-equivariant homology of the space X.

Definition 7. *– $(X \times EG)/G$ we'll write as $X \times_G EG$, the "homotopy quotient" of X by G, and then*
– $H_*^G(X)$ is $H_*(X \times_G EG)$

If you've seen the notation EG before in your past life, this agrees with what you've seen before. Moreover, if you look at EG/G, this is BG, the classifying space for the group G, and moreover EG is the total space of a principal G-bundle over BG:

$$\begin{tikzcd} G \arrow[r] & EG \arrow[d] \\ & BG \end{tikzcd}$$

As a last remark, let me go back to the loop space. For an S^1 action, ES^1 is S^∞ and BS^1 is $\mathbb{CP}^\infty$.

The action is, you can think of S^∞ as the colimit of S^{2n} in $\mathbb{C}^n$, and S^1 acts on each of these. We do have tea and coffee, or let's check.