DENNISFEST NOTES

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1. John Morgan III

Until this point I've been assuming everything is simply connected. Now we'll consider what happens for a general connected algebra, meaning the cohomology in negative degrees is 0 and the cohomology in degree 0 is $\mathbb{Q}$. Now I want to mimic the cohomology of A with a differential algebra starting in degree 1. So I start with the exterior algebra on the first cohomology $\wedge^*(H^1)$, and by the usual construction I map this into A^* by taking a basis and choosing representatives. This map is an isomorphism on the first cohomology. The inductive step you need, an isomorphism in one dimension but you should have an injection in the next dimension if you want to move up. If the cup product is injective into H^2 we're okay, but otherwise we'll need to put more generators in degree 2 to annihilate. Call this $M_{1,1}$, and $H^2(M_{1,1})$ is $\wedge^2 H^1$. So we look at the kernel $V_{1,2}$ of $\wedge^2 H^1 \to H^2(A)$. So now we take $M_{1,1} \otimes \wedge^*(\underbrace{V_{1,2}}_{\text{degree one}})$ with a differential $d : V_{1,2} \to \wedge^2 H^1$ by the inclusion map. This extends to A^*. We call this stage $M_{1,2}$, and it includes $M_{1,1}$ as a sub dga. We know the kernel of $H^2(M_{1,1}) \to H^2 A^*$ is the same as the image of $H^2(M_{1,2})$. Unfortunately, or interestingly (I can't keep track of which is better, optimist or pessimist), this might create new problems.

If I were working on the complement of the Borromean rings from yesterday, $M_{1,1}$ would be $\wedge^*(a, b, c)$, and $V_{1,2}$ would be $\wedge^2 H^1$ since the cup product is 0. So we1d have to add $\eta_{ab}, \eta_{bc}, \eta_{ac}$, which annihilates $\wedge^2 H^1$, but we've created new cohomology classes. So for instance $\eta_{ab} \wedge c + a \wedge \eta_{bc}$ is closed. What does it represent in A^*? We don't know until we do some calculation. In some topological spaces with the same cohomology ring these classes would be trivial. This is something captured in the differential forms and the minimal model but not in the cohomology ring. The kernel at the next stage, you have to continue the construction if there is a kernel. You continue this construction, and haev $M_{1,1} \subset M_{1,2} \subset \cdots$ and at each stage we're adding more things in degree one to kill the kernel of the previous stage. So $M_{1,n} = M_{1,n-1} \otimes \wedge^*(V_{1,n})$ with $d : V_{1,n} \cong Ker\ H^2(M_{1,n-1}) \to H^2(A^*)$. Then M_1 will be the infinite union (this may not terminate at a finite stage) and M_1 maps to A^* with an isomorphism on H^1 and injective on H^2.

This could be dualized, submanifolds, codimension one and two and their intersection patterns, so on. I want to say what this is about the homotopy theory of the space. Now I'm imagining that A is $\Omega_{\mathbb{Q}}(X)$. This is related to a very nice part of the fundamental group, which I'll derive before telling you what it is. Let's think about these differential algebras. At each stage we have an exterior algebra on a

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finite dimensional vector space and a d. There's a well-known construction that will give us a Lie algebra.

So we take the maximal ideal modulo their square I/I^2, and these are the indecomposables. Let me shift the indecomposables L_n is $[I(M_{1,n})/I^2(M_{1,n})]^*$ in degree 0. So $L_1 = V_{1,1}^*$, L_2 is $(V_{1,1} \oplus V_{1,2})^*$, and so on. Dualizing reverses my arrows and I'll get surjections $\cdots \to L_2 \to L_1$. Now, d is decomposable, so I get $d : I/I^2 \to I^2/I^3$ which is $\wedge^2(I/I^2)$. Take its dual, and that will be a map $d^* : L_n \wedge L_n \to L_n$, a skew-symmetric map. The relation $d^2 = 0$, when I dualize I get Jacobi, so this is a Lie bracket. We get a projective system of Lie algebras.

It's even better than that, because, we have the equation that $d(V_{1,n})$ is contained in $M_{1,n-1}$ and when you dualize this, the kernel of $L_n \to L_{n-1}$ which is $V_{1,n}^*$, is central in L_n. So we have a short exact sequence $0 \to V_{1,n}^* \to L_n \to L_{n-1} \to 0$ with $V_{1,n}$ central. The first space here L_1 is $H_1(X)$ with 0 bracket. We get a tower of central extensions, higher and higher order nilpotent Lie algebras. So L_n is a nilpotent Lie algebra of length of nilpotency $\leq n$ over $\mathbb{Q}$. This is the same thing as a group, because you can use the exponential. The multiplication is given by the Baker Campbell Hausdorff formula. This is a power series in general but here it truncates and becomes a polynomial with rational coefficients. So we have a tower of $\mathbb{Q}$-nilpotent groups $\mathcal{L}_n \to \mathcal{L}_{n-1} \to \cdots \to \mathcal{L}_1$. This is dual to the construction putting more generators in degree 1 and taking d. So every center is a rational vector space.

This is the rational nilpotent completion of the fundamental group. Associated to any group is a series of nilpotent groups: G, $[G, G] = G_2$, $[G, G_2] = G_3$, and so on. Then G/G_n is nilpotent of length n and maps to G/G_{n-1} with kernel G_{n-1}/G_n which is a central subgroup, and this is exact. This is universal with respect to maps from $\pi_1(X)$ into nilpotent towers.

Any time you have a nilpotent group, you can tensor it with $\mathbb{Q}$, one stage at a time. You can tensor with $\mathbb{Q}$ and get a tower of rational nilpotent groups, and this is what the 1-minimal model gives.

That's all I'll say about the fundemental group. Let me just make one remark about mixing the fundamental group with the higher degree construction. If you start with a connected algebra, you can build the 1-minimal model, which produces something isomorphic on H^1 and injective in H^2.

[What goes wrong if it's not nilpotent?]

It's not "wrong," this is just what you see. For a knot complement, H^1 is just rank one and then you're done and you haven't seen the knotting, this is deeper in the fundemental group than the lower central series.

[Discussion]

I want to give you three kinds of consequences. First of all, I want to look at the automorphisms of homotopy type. X is a finite complex, simlply connected. I can look at $Aut_{\mathbb{Q}}(M)$, which is an affine algebraic group over $\mathbb{Q}$. This means it's a subgroup of GL_n defined by polynomial equations. This minimal model for X may be an infinite construction, which may go on above the dimension of X. You can truncate and forget a couple of dimensions above the space. This automorphism should commute with multiplication, d, and this is a quadratic and linear formula, so a rational algebraic group. One problem is that some of these may be homotopic to the identity. The thing that Dennis wrote down was the following. Think about the maps $i : M \to M$ that are derivations of degree -1 and consider their commutator with d: $id + di$. The collection of maps like that, commutators, make a Lie algebra of elements commuting with d. Then exponentiate, it's easy to see that $exp(id + di) = id + di + \frac{1}{2}(id + di)^2 + \cdots$ terminates and gives isomorphisms of M, and all are homotopic to the identity. It's not hard to show that all automorphisms of M homotopic to the identity arise this way.

What we might call the rational outer automorphisms of the minimal model are this algebraic group divided by, well, we'll get a unipotent algebraic group in the image.

It now follows that the automorphisms (actual homotopy automorphisms) is commensurate with the integral points of this algebraic group. The two groups have a group of finite index in common. In particular, the automorphism group was finitely generated.

[That was the whole motivation, I wanted to study that automorphism group. Surgery theory gave a description of invariants, and this group acts on it. What's the nature of it? There's some algebraic structure.]

Surgery is mainly for connected manifolds of dimension five and greater. These are true for PL, topological, and smooth manifolds. Such a (simply connected) manifold is determined up to finitely many possibilites by $H^*(M, \mathbb{Z})$, the ring, the rational Potrjagin classes, and the minimal model. Now we begin to understand how the automorphisms act.

In the last ten minutes I want to talk about formality. A differential algebra or space is formal if the minimal model for the algebra is also the minimal model for the cohomology; there's a map $M \to H^*(A)$ which induces the identity on cohomology.

So spheres and projective spaces are formal. S^3 minus the Borromean rings is not formal because in a formal space all Massey products vanish. You can compute those on both sides, but if you have a dga isomorphism, you can compute the Massey products, and if it's zero in cohomology, since $d = 0$, the way to solve the equation is to take $0 \to 0$ so they vanish. Formality is the vanishing of all Massey products in a uniform way. That will all be captured somewhere in the minimal model.

What does formality say about the fundamental group? It says that rational nilpotent completion of the fundamental group is as free as possible given the cohomology ring.

The theorem that I want to end with, due to Deligne, Griffiths, Morgan, and Sullivan, was that any compact Kähler manifold is formal. I'd hoped to talk about how the rational homotopy theory is related to interactions with current stuff today with A_∞ structures. The minimal model in dgas, you can find a quasiequivalent object has an A_∞ structure, but you need the higher structure. Formality means you don't need any higher structure.

The basic ingredient is the $d\bar{d}$ Lemma. I'll work with the complex differential forms $\Omega^{p,q}(X, \mathbb{C})$, into $\bar{\partial}$ and ∂, and the $d\bar{d}$ lemma says that if α is closed under ∂ and $\bar{\partial}$ (and d) and exact under one of them, then it's exact under all three consistently, so there is a β with $\alpha = \partial\bar{\partial}\beta$.

It's proved using the basic Kähler identities, and then this allows us to make a diagram of differential graded algebras

$$\begin{tikzcd} (Ker_{\bar\partial}, \partial = d) \arrow[r] \arrow[d] & \Omega^*(X,\mathbb{C}), d \\ H_{\bar\partial,\partial=0} & \end{tikzcd}$$

The $d\bar{d}$ lemma implies that the maps are equivalences and the bottom differential is zero, so these are formal, at least over $\mathbb{C}$. Then you can see rational formal.

[What about derivations of other degrees?] I don't know. [If you make a complex Lie algebra, this makes a model for the space of automorphisms.]

[What is the geography of formal and nonformal spaces?] I think that formality is a tight restriction. Some spaces don't have any choice. They're formally formal. If the cup product is an embedding, then they're formal. If you have a possibility of non-formality, then generically you have non-formality.

[Just about any elliptic space is non-formal.]

2. Graeme Segal

It's an honor to talk here at Dennis' birthday. I met him more than forty years ago and almost at once he changed my view of mathemetics. He made me feel that in areas I had never felt the tiniest interest in, there were many questions of interest. I won't talk about my experiences at his seminar. I thought it was one of those examples of structures like the eye involving independently in different regimes. There's something about this unlike other things. Dennis can be the best listener and extract the real idea that is sometimes inchoately expressed.

I'm going to talk about field theory, just defining it. This was explained well by Smirnov, there were problems in the lattice model, once you saw there was a two dimensional field theory, that many things could come easily. Seeing it was hard.

Field theories are supposed to be crystalline, rare. Above two-dimensions, they form finite dimensional families, above 4 there aren't any at all. There's supposed to be one in dimension six, all alone, like a six leaf clover.

It's hard to say what one is. Let me say something now about the traditional formulation of a quantum field theory. I'll only consider a toy model. We'll talk about a d-dimensional space-time, which I'll think of as being Riemannian. For each point $x \in X$ we'll assigno a vector space $\mathcal{O}_x$ of local fields, observables, whatever, not an algebra but a mysterious thing that you can observe at a point. When you pick a certain number of disjoint points, there's an expectation map $\mathcal{O}_{x_1} \otimes \cdots \otimes \mathcal{O}_{x_k} \to \mathbb{C}$ ofter written like $\psi_1 \otimes \cdots \otimes \psi_k \mapsto \langle \psi_1 \cdots \psi_k \rangle_X$. This should depend on the set, and vary smoothly as we move the points around.

What properties ought these to have? There's mythology overlying this situation. If you imagine there is a mythological space Φ_X of "classical fields," locally defined on X, like functions with values in a vector space or curved manifold, or connections with values in whatever. There's meant to be an action functional $S : \Phi_X \to \mathbb{R}$, which enables one to define a measure $d\mu$ on Φ_X which formally is meant to be written $d\mu(y) = e^{-\frac{1}{\hbar}S(y)}dy$ (in the Riemannian setting). For Kevin, a classical field theory is just the germ of the infinitessimal structure of this space of fields around the critical set of the function S. The relation is that if we have $d\mu$, then what we would do is define $\mathcal{O}_x$ as contained in the C^∞ functions on Φ_x so that $\psi(\varphi)$ depends only on the jet of φ at x. Then $\langle\psi_1 \cdots \psi_k\rangle$ is meant to be $\int_\Phi \psi_1(\varphi)\cdots\psi_k(\varphi)d\mu(\varphi)$.

About twenty-five years ago, rather as a joke, I proposed a different way to come at this thing, and it's been taken in many directions having nothing to do with me. Let me say it quickly to take it somewhere. The idea I had in mind has not been pursued very much. My idea is one that I hardly need to mention. I suggested that one could define a field theory in dimension d as a functor from the cobordism category to vector spaces.

To a closed manifold Y^{d-1} it should give a vector space $\mathcal{H}_y$ and to a cobordism $X^d : Y_0 \to Y_1$ a ctrace class operator U_X.

[How did you get this idea?]

There was a meeting near Munich and Witten was giving a talk on how he looked at this, and the essential property was what happened when you divided into two regions. You're saying we have a functor on a cobordism category and everyone laughed. I kept saying, maybe it's not such a stupid idea. Atiyah wrote it down in the topological case. People have talked about it since.

This is the data, and the axioms that this is meant to satisfy is two or three. The output should be smoothly dependent on the data. There should be functoriality, secondarily, which I'll write down as concatenation, and thirdly, the tensoring property meaning that if you take a disjoint union of two Ys, then this should be the tensor on the range, so for the empty manifold you get the complex numbers, and similarly for morphisms. So it's a tensor functor.

There are lots of refinements you need to put in. For this talk, the only important one we need immediately is that $\mathcal{H}_Y$ is defined not for Y but for the germ of a d-manifold along Y. Think of Y as being a little bi-collared thing from which another cobordism might step off. You need to keep track of the germ or else you wouldn't have a smooth structure on the gluing.

That's a simple definition, how does it give back the kind of data? Imagine X is closed, and we cut out a piece around each point, $\hat{X}$ which is X minus some little disks. This is a cobordism from $\amalg \partial D_i \to \emptyset$ so you should get a map $\bigotimes \mathcal{H}_{\partial D_i} \to \mathcal{H}_\emptyset = \mathbb{C}$. So we can define $\mathcal{O}_x$ as the inverse limit of $\mathcal{H}_{\partial D_i}$. This is an ordered system by inclusion.

The kind of question I was interested in at the time was the question of when two field theories were the same. I was familiar with the fermion-boson correspondence. It was an art to show they were equivalent. Describing a theory in the first way is disingenuous. You pick a vector, you have actual fields, and construct things in cunning ways using normal ordering and other ingredients and get local fields. You never know when there might be some new trick. A theory was a murky piece of data, so it was difficult to say what it would mean to speak of the moduli space of two dimensional field theories. People think above two dimensions they are rare, but there are the conformal field theories in dimension two, and string theory begins with the observation that a certain class of two dimensional field theories are conformal, they don't depend on a Riemannian structure, these fall into families of finite dimension. [Discussion of σ-model.]

There's a map, one hopes, and it's not really defined, which to target manifolds give theories, and Ricci flat manifolds to formal theories. If you take a general manifold as target, you have the Ricci flow, and on theories of this kind, you have a one-parameter family by rescaling, and you can hope that two-dimensional theories have to do with manifolds and scaling flow to Ricci flow.

I was fascinated at the time with a paper doing perturbation theory near unintelligble, and this still exists in what mathematicians call a "physical level of rigor." This person started with a conformal theory and parameterized a little neighborhood in the space of theories, not necessarily conformal. He defined a Riemannian structure and function, and argued something about gradient trajectories.

What is the structure of these things? One thing you see from the second definition is that you have an algebra structure on these things, but only of the up-to-homotopy type that we have been hearing so much about.

Suppose you are near x and take the disk D. Then this will inject into the space $\mathcal{H}_{\partial D}$, it'll be a dense subspace, so think $\mathcal{H}_{\partial D} = \hat{\mathcal{O}}_x$. If you pick a disk, you can take out tiny disks around them. What you read off is always maps, if you pick k points, always have maps $\mathcal{O}_{x_1} \otimes \cdots \otimes \mathcal{O}_{x_k} \to \hat{\mathcal{O}}_x$. If you're willing to work with completions, you have a product, but you have things defined only up to homotopy because it's parameterized by the many points in the small disk.

Suppose one is interested in making a moduli space of theories. What is the tangent space? What are the infinitesimal deformations? From the physicists' point of view, it's supposed to be the space on fields.

Suppose we're deforming a theory and we get a map $\mathcal{H}_{Y_0} \to \mathcal{H}_{Y_1}$, and we can chop the cobordism into pieces. If we could chop it up into little bits that moved by a tiny little piece, then U_X would be a composite of a whole lot of little pieces. Since U_X is meant to be the composite, if you make a small change, then the first order change will come from one of those things, which should be completely described by a tiny piece of X. This should occur on the boundary of those little disks. In a deformation, you should always have a field ψ in $\mathcal{O}_x$. Assume that the manifold looks the same near every point, so $\mathcal{O}_x$ is independent of x. Then you'd like to say that the only deformations are of the form $\delta_\psi U_X$ which is $\int_X U_{X,\psi_x} dx$.

[Picture.]

To make sense out of this, you need more structure. I can't prove a theorem of this kind. I need to know something more about the system $\mathcal{O}_x$ which was the inverse limit of the $\mathcal{H}_{\partial D}$. The case we started off with was one we knew well. In the space of two dimensional conformal theories, any two disks are the same. All of these spaces are the same, let's call them $\mathcal{H}$. Also, the formal embeddings of D on $\circ D$ acts on $\mathcal{H}$ by trace class operators. These should preserve the center. Then you can say that these fields aren't all the same, so by differentiating at the identity, you can get to $\mathbb{C}^\times$.

We can pick a cofinal system around every point, so we actually have a rather obvious action just of $\mathbb{C}^\times$. In particular, you could look for eigenvalues so that when you've contracted it, well, $\theta'(0) = \lambda$. You might look for ψ so that $\theta_*\psi = \lambda^k\psi$ or $\lambda^p\bar{\lambda}^q\psi$.

The resulting field, we can identify these things, but to get this to a point on the manifold, we have to choose a coordinate patch tells us that the way theat ψ behaves is like a differential form on the manifold. We'll say that the fields have properties that let us do this. WE could have something that wasn't an eigenvolue. That wuold be a derived field.

You immediately see that to deform a theroy you look for a field like a volume element.

That was conformal so what's more general?

I don't know how to go further without an axiom. This is related to asymptotic freedom. At small distances, a nicely behaved theory will become something simple, where a think behaves as it would in Euclidean space.

I don't see a way of going anywhere without adding an axiom that says the inverse limit is asymptotic to something conformally invariant on Euclidean space. It's naturally attached to the metric tangent space, so the conformal group formally acts on it, and so far. In all the theories that one can think about, that's true. There aren't many theories, but they all satisfy this.

We still need more properites. We wanted to cut things up into pieces that are small. My axioms aren't restricted enough because they don't tell you in what senes the space associated to Y depends locally on Y.

[Missing some.]

It would be nice to say that $\mathcal{H}_Y$ were a tensor product $\mathcal{H}_{Y_1} \otimes \mathcal{H}_{Y_2}$ If that were true, physics would be trivia. So the next best hope is that, well. . .

The simplest case is Φ which is maps $X \to \mathbb{R}$ with action $\int_X = \{d\varphi * dp + unintelligble* = \varphi$ and $\mathcal{H}_Y = L_2(\Omega^0(Y, \mathbb{R}))^X$

What is the quadratic form that we have to use? You can compare and think of a quadratic form as a map like $\Omega^0(Y, \mathbb{R}) \to \Omega^{d-1}(Y, \mathbb{R})$.

[Missed.]

If we were working in deformation quantization, the spillover will only involve an infinitesimal deformation.

[Missed, had to stop.]