# DENNISFEST NOTES

GABRIEL C. DRUMMOND-COLE

## 1. Kevin Costello III

So I know I've been barraging you guys with definitions and I can't expect you to have absorbed everything. What I've done so far is introduced language to describe moduli spaces of solutions to elliptic equations at a formal level.

So $M$ is the derived moduli space of solutions to an elliptic equation. We were thinking we should define this in terms of a Lie algebra, $\mathcal{L}_M$ is a sheaf of $L_\infty$ algebras on $X$. Oh, sorry, Dennis. Can you see?

[You should write a bit larger.] [Dennis moves.] [No, that's good.] [Laughter.]

$\mathcal{L}_M(X), d$ is an elliptic complex and there are some conditions, unintelligble, so maybe an example, if $X$ is a four-manifold, I can look at the instanton equations, solutions, self-dual connections, and the Lie algebra $\mathcal{L}_{M_{\text{inst}}}$, in degree 0 is the gauge group, in degree one is the ways of making connections, and in degree 2 I want the anti-self dual part to vanish.

$$\Omega^0(M, \mathfrak{g}) \to \Omega^1(M, \mathfrak{g}) \to \Omega^2_-(X, \mathfrak{g})$$

The reason this is a good space is because this is an elliptic complex. $H_1$ of the complex is finite dimensional if $M$ is compact.

We were also interested in geometric structures on these guys, like symplectic forms. So a symplectic form of degree $k$ on $M$ corresponds to an $\mathcal{L}$-invariant pairing with a shift of degrees (degree $k-2$). We're interested in symplectic forms because classical field theories are critical points of a function, which correspond in this derived world to symplectic forms of degree $-1$. One example to bear in mind, if I take any elliptic equation, I can take the cotangent bundle, that has a symplectic form, so I can take the shifted cotangent bundle. If $M$ is any moduli problem then $T^*[-1]M$ is a classical field theory.

Suppose I take the instanton equations and take its cotangent bundle. The Lie algebra $\mathcal{L}_{T^*[-1]M_{\text{inst}}}$ has two pieces, the original one and then the duals of these ones:

$$\begin{tikzcd} & \Omega^2_- \arrow[r] & \Omega^3 \arrow[r] & \Omega^4 \\ \Omega^0 \arrow[r] & \Omega^1 \arrow[r] & \Omega^2_- & \end{tikzcd}$$

We shifted it because we wanted a pairing of degree $-3$. In physics this is the self-dual Yang Mills, actual Yang Mills is a deformation of this.

[Does actual Yang Mills fit in your framework?]

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You can do it for $\mathbb{R}^4$, I don't know how to do it in an arbitrary 4-manifold. You add a differential $\Omega^2_-$ to itself by $c\mathrm{Id}$. When $c = 0$ then you get the self-dual Yang Mills.

Many theories have a large volume limit that looks like a cotangent theory, and then you add small perturbations.

The aim is to say something about quantization. If $U$ is a subset of $X$ then I can think of my moduli space $M(U)$, and functions on that, $\mathcal{O}(M(U))$ I'll define to be the Chevalley cochains of the Lie algebra $C^*\mathcal{L}_M(U)$.

If my moduli problem has a symplelctic form, then functions on it have a Poisson bracket of degree 1. We'd like to talk about a quantization of this structure. What is quantization of a degree 1 bracket? That has a nice, uniform theory, but the most unintelligblecase, let me remind you, is ordinary deformation quantization.

$A$ is a commutative algebra, and $\{\ \}$ is a degree 0 Poisson bracket, then a quantization of $A$ is a product $a \times_\hbar b$, $\hbar$ dependent, on $A[[\hbar]]$, which is $\hbar$-linear, such that $a \times_\hbar - b \times_\hbar a = \hbar\{a, b\} \mod \hbar^2$.

We'd like a similar description when we have a degree 1 Poisson bracket. There's a very natural degree 1 Poisson bracket. [Is it the BV bracket?] Yes. If $A$ has a degree 1 Poisson bracket, and again is a commutative dga, then

**Definition 1.** *A quantization of $A$ is an algebraic structure on $A[[\hbar]]$, for deformation quantization it was an associative structure, here it will be something else, a commutative product and Poisson bracket of degree 1 and a differential on $A[[\hbar]]$, lifting the structure we already had, such that $d(ab) = d(a)b + (-1)^{|a|}ad(b) + \hbar\{a, b\}$. When we deform, d is not a derivation, and the failure is the bracket.*

*The failure of $A[[\hbar]]$ ot be an dga at all (that is, to have a product compatible with d), is measured by $\{\}$. You know as a topologist that commutativity is a lot more information than associativity.*

*[Is that a BV algebra with an $\hbar$?] I call this a BD algebra, for Beilinson-Drinfeld. In the mid-90s, this was shifted and given degree $-1$, it's kind of annoying.*

I can see from the audience reaction that this is confusing. Let $M$ be a manifold. Then $\mathcal{O}(T^*[-1]M)$ has a Poisson bracket of degree 1.

[What is equivalence?] It's homotopy equivalence. This should be a family where the base is unintelligbleon the $n$-simplex, like in John's talk.

In this situation, what does it mean to quantize? In degree 0 it's smooth functions, in degree $-1$ it's vector fields and so on.

$$\Gamma(M, \wedge^2 TM) \quad \Gamma(M, TM) \quad C^\infty(M)$$

A volume form on $M$ gives one of these. This is an operadic definition of a volume form. Let me explain why this is true.

If I choose an $n$ form on $M$, $\omega$, then we get an isomorphism like $C^\infty(M)$ I can associate with top forms, vector fields with $n-1$-forms, and so on. I'll write down the quantization from this. I can choose to only deform the differential. How can I do this? I have the de Rham differential on $\Omega$. We can do this using our isomorphism. That's the divergence for the volume form $\hbar Div_\omega$

The point is that polyvector fields with this differential, $\mathcal{O}(T^*[-1]M), \hbar Div_\omega$ is a quantization, so that $Div_\omega(\alpha, \beta) = Div_\omega(\alpha)\beta + \alpha Div_\omega(\beta) + \{\alpha, \beta\}$.

There is a natural set of deformations equivalent to volume forms. Observe that $\mathcal{O}(T^*[-1]M)$ has an $\mathbb{R}^*$ action by scaling the cotangent fiber $\{\ \} \to \lambda\{\ \}$ where we scale by $\lambda$. Now, $\hbar$ and the Poisson bracket are next to each other. Let $\mathbb{R}^*$ act on functions on $\hbar$ by sending $\hbar$ to $\lambda^{-1}\hbar$, and then our axioms are $\mathbb{R}^*$-invariant and we can ask for an $\mathbb{R}^*$-invariant quantization.

The lemma is, that here, volume forms give quantizations, invariant quantizations, and these are all you find. There is a bijection between volume forms on $M$ up to multiplication by a scalar and $\mathbb{R}^*$-invariant quantizations.

In particular, we can talk about volume forms for these infinite dimensional things.

[Can you think of this as the unintelligble?] Yes, exactly. I am doing quantum field theory, so there's a measure, formally $e^{S/\hbar}d\mu$.

Let's talk about the infinite dimensional situation. So $M$ is an elliptic moduli problem. $T^*[-1]M$ has a degree $-1$ symplectic form which correpsonds to $\mathcal{L}_M \oplus \mathcal{L}'_M[-3] = \mathcal{L}_{T^*M[1-]}$.

If $U \subset X$, I'll define, again, $\mathcal{O}(T^*[-1](M(U)))$ to be $C^*(\mathcal{L}_{T^*[-1]M}(U))$, the Chevalley cochains on the Lie algebra. This has a Poisson bracket of degree 1 (it's nonessential that this be a cotangent).

**Definition 2.** *A quantization is a quantization of each of these $\mathcal{O}(T^*[-1]M(U))$, $\mathbb{R}^*$-invariant, compatible with the natural maps (let me explain these.*

There are maps $\mathcal{O}(T^*[-1]M(U_1))\otimes\mathcal{O}(T^*[-1]M(U_2))\otimes\cdots$ which maps to $\mathcal{O}(T^*[-1]MV)$ where $U_i$ is in $V$, all disjoint, maps of Poisson algebras. If $U_i$ are not disjoint, this is not a Poisson map.

So we want quantization to be compatible with this structure. This leads to what we call a "factorization" algebra, where $Obs(U)$ is the cochain complex $\mathcal{O}(T^*[-1]MU[[\hbar]]$, with maps as above.

If $X$ is compact, then this moduli space $M(X)$ is finite dimensional. The upshot is that a quantization of the field theory leads to, on the whole space, a volume form on the finite dimensional dg manifold $M(X)$. In good situations this might be the germ of a smooth manifold. This is a volume form given by local data on $X$.

Let me explain how you might construct these and indicate why the volume form is unique.

**Theorem 1.** *(Gwilliam)*
*There is a deformation complex, a sheaf you associate by a simple procedure, obstructions on $X$ associated to $M$, a sheaf of cochain complexes such that $H^1(X, Obstr^n(M))$ are the obstructions to quantization, and if the obstructions vanish, then the space is described by $H^0$, but this describes the deformations of a given quantization. In many examples, these cohomology groups vanish.*

Here is an example. Let $E$ be an elliptic curve, $Y$ a complex manifold, and $M$ the holomorphic maps $E \to Y$ near constant maps. What you get from the cotangent theory then, $T^*[-1]M$ is what Witten calls the "$N = (0, 2)$ susy twisted $\sigma$-model. Maybe I'll tell you, this global moduli space is $T[-1]Y$, and $H^{\leq 0}(Obstr^n) = 0$, $H^1(Obstr) = \mathbb{C}$, and the obstruction to quantization is $ch_2(Y)$, so if that vanishes there is a unique quantization (this gets to $Y$ by pushforward). Then $M(E)$ has a volume form, $Vol(Y^E) = Wit(Y, E)$.

Grady and Gwilliam have this result: $Y$ is copmlex, lecally constant maps $S^1 \to Y$, then there is a unique quantization $Vol(Y^{S^1}) = \int_Y Td(Y)$. A last one, if $M$ is the instanton moduli space, $X$ a 4-manifold, $G$ a simple group, thene there is a unique quantization which gives a canonical, conformally invariant measure on the instanton moduli space.

I think I'll stop there.

[unintelligble] [Are these obstructions related to what physicists call anomalies?] Yes, they're the same.