DENNISFEST NOTES

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1. Kevin Costello, Supersymmetric, holomorphic, and topological field theories in dimensions two and four.

Thanks very much, and I think my title is misleading, I'm not sure, I'll be introducing other things before I talk about supersymmetric theories, that might be the last lecture.

Suppose we have a manifold X. A very familiar thing in geometry is to consider a moduli space $M(X)$ to some elliptic equations on X. This is something like an "elliptic moduli problem." For example, if X is a four-manifold, you might consider $M(X) = \{G-\text{bundles on } X \text{ with a connection } F(A)_+ = 0\}$.

Another example, if C is a curve (Riemann surface) and Y a complex manifold, then the space of holomorphic maps $C \to Y$. There the equation is that the map be holomorphic.

Take X, consider the solutions, perform an integral over that, get some invariants. For example, let's make a cartoon of Donaldson theory. Look at this first example, then the Donaldson invariants are some kind of integrals over $M(X)$ of some natural cohomology classes. This is a cartoon.

Similarly, in the second example you do the same thing and get Gromov-Witten invariants. Integrate over your moduli space of holomorphic maps some natural cohomology classes.

So the story I'd like to explain is the following.

(1) So the first part is, to any elliptic moduli problem like this, I'll give you a precise definition of an elliptic moduli problem in a little bit; I'll show you how to associate a classical field theory on X. There's this nice class of classical field theory any time you write down elliptical equations (I'll spend a lot of the lecture explaining this, but according to the Batalin-Vilkovisky formalism, this will be some space with a symplectic form.
(2) The second part, the quantization of this theory (I'll talk about quantization) gives a volume form on the moduli space.
(3) I'll explain how many theories in mathematics fit into this formalism.

I said in my abstract that I wanted to say how you can see the A and B models here, and also supersymmetric field theories, and I'll spend the third lecture talking about those connections.

Before going on to the first part, let me say a little bit about the volume form. If E is an elliptic curve (Riemann surface of genus one) and Y a complex manifold, we have $M(E)$, holomorphic maps $E \to Y$. If the second Chern character of Y vanishes, then there is a unique quantization, and we get a volume, and $Vol(M(E))$

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is called the Witten elliptic genus. This genus is a modular form, which depends on the elliptic curve, where it lives in the elliptic curve.

In the Gromov-Witten picture, if I change my elliptic equation a little bit, I get the same answer. Here the volume form reflects the geometry of the space you're working on.

If X is a four-manifold, and again $M(X)$ is anti-self dual G-bundles, then I think I can show that there is a unique quantization and that the volume of these moduli spaces depends on the conformal structure of X and is more closely related to physics and is some limit of Yang-Mills theory.

This was an advertisement that you do get something interesting in this story.

I want to spend some time talking about what it means to have an elliptic moduli problem. We should have some intuitive idea but it'll be helpful to formalize our definition. What is an elliptic moduli problem? If you look at these two examples, the moduli space of solutions is badly behaved, it's not of the expected dimension, it's very singular. It's often the case that the actual moduli space is badly behaved but there's this philosophy, well-known to many people, the "hidden smoothness" philosophy. I learned it from Kontsevich's paper.

[In Israel once, MacPherson gave a long, beautiful lecture about singularities and Kontsevich came up and said "there are no singularities."]

The philosophy is that in the derived world, everything is actually smooth. You might say, let's make a general theory of derived C_∞ stacks, you'll run into polyfolds, all these things. I'm lazy. Globally, this is very hard. Let me work locally and work this out at a local level. We'll give a definition of a formal derived elliptic moduli problem. Why is the formal story easier? In fact, it's because it's basically linear algebra.

There is what we could call the fundamental theorem of deformation theory over $\mathbb{Q}$. It says that formal pointed "derived spaces" (there is a theory where this is true, but I won't give a definition) are the same thing as differential graded Lie algebras (or if you prefer, L_∞ algebras) (this should include rational homotopy theory).

The heart of this statement goes back to Quillen, Sullivan, and then was developed in algebraic geometry by Drinfel'd, Deligne, Kontsevich, Hinich, and recently Lurie (very clear and generalizing things even further).

What's the basic idea here? Functions on derived spaces are commutative dgas, so functions on formal derived spaces are some kind of completed pro-nilpotent commutative dgas. Then without loss of generality, we can choose a resolution, this is equivalent to something of the form $A = \widehat{Sym^*}(V)$ with a differential $V \to \widehat{Sym^{\geq 1}}V$ with V some graded vector space. I learned this picture from Dennis when I was quite young. The Lie algebra is V, dualized, and the Lie structure is given by the Taylor expansion of the differential.

The Lie algebra is $\mathfrak{g} = V^*[-1]$, the maps $d_n : V \to \widehat{Sym^n}V$ become maps $\ell_n : \mathfrak{g}^{\otimes n} \to \mathfrak{g}$, so ℓ_1 is a differential, ℓ_2 is a bracket, and so on. A classic example, of course, is that if I take A to be the minimal model of a space, then the Lie algebra is the homotopy groups of the space.

According to the hidden smoothness philosophy, every formal derived space is represented by an L_∞ or Lie algebra. Then if you want something a bit more concrete, if $\mathfrak{g}$ is a dg Lie algebra, and (R, m) is an Artinian ring with m a maximal ideal (over some field, $dim_{\mathbb{Q}} R < \infty$ and $m^N = 0$ for $N \gg 0$. The basic idea you should keep in mind is $\mathbb{Q}[t]/t^N$. If I have one of these guys, which I should think of a thickened point, I can ask, what are the maps from this thickened point to my formal moduli problem. So the idea is that $\mathfrak{g}$ defines a formal moduli problem by saying that $Hom_+(Spec\ R, M_g)$ (the + means preserving basepoint) are the Maurer Cartan elements of $(\mathfrak{g} \otimes m)$. There is an equivalence relation (that comes from saying this is a simplicial set) but we won't worry about it. Let me say what Maurer Cartan means, it's the $\{\alpha \in (g \otimes m)^1 | d\alpha + \frac{1}{2}[\alpha, \alpha] = 0\}$. If this were a dg Artinian ring, then this would be a simplicial set. In the simplicial formalism, these are the zero simplices. You can consider families over cochains on the n-simplices, I heard this from Dennis though he denies inventing it. So these are $MC(\mathfrak{g} \otimes m \otimes \Omega^*(\Delta^n))$. If you work out $\pi_0(MC(g \otimes m))$, then these are Maurer-Cartan elements up to gauge equivalence. Then M_g is a functor from dg Artin rings to simplicial sets. This is what a derived space is defined to be in the literature, a functor from dg Artin rings to simplicial sets.

I wanted to use this fundamental theorem of deformation theory to give a definition of an elliptic moduli problem.

The definition will be of an elliptic Lie algebra.

Definition 1. *An elliptic dg Lie algebra on a manifold X is a graded vector bundle L on X whose sections on $U \subset X$ I'll call $\mathcal{L}(U) = \Gamma(U, L)$. The structure should be that the global smooth sections should be given the structure of a dg Lie algebra, $\mathcal{L}(X)$ is a dg Lie algebra where:*

1. *The differential $d : \mathcal{L}(X) \to \mathcal{L}(X)$, this should be the elliptic condition, this is a differential making $\mathcal{L}(X)$ an* elliptic complex.
2. *The Lie bracket $\mathcal{L}(X) \times \mathcal{L}(X) \to \mathcal{L}(X)$ is something called a* bidifferential operator.

Let me say what this (and elliptic complex) mean. If I have two sections ϕ and ψ of $\mathcal{L}(X)$, then I can look at $[\phi, \psi](x) \in L_x$. To be a bidifferential operator, the axiom is that $[\phi, \psi](x)$ only depends on a finite subset the Taylor expansion of ϕ and ψ at x. We get it by differentiating a finite number of times and doing something linear.

Before I say what an elliptic complex is, you can generalize this easily to the L_∞ world, to define elliptic L_∞ algebras.

What is an elliptic complex? The basic example is the de Rham complex $(\Omega^*(X), d_{dR})$ or the Dolbeaut complex of a complex manifold (with values in a vector bundle) $(\Omega^{0,*}(X), \bar{\partial})$. The characterizing property is that on a compact manifold, an elliptic complex has finite dimensional cohomology. The formal statement is technical so bear these examples in mind, but it's a graded vector bundle E on X, a differential operator $d : \Gamma(X, E^i) \to \Gamma(X, E^{i+1})$, $d^2 = 0$, and any time you have a differential operator you can take its symbol. If you look at the complex of vector bundles on the cotangent bundle minus the zero section given by $T^*E^i \to T^*E^{i+1}$, where I use the symbol $\sigma(d)$ of this operator, this complex has zero cohomology, is acyclic or exact. This may be unilluminating. Another example is that a two-term elliptic complex is the same as an elliptic operator. $E \xrightarrow{d} F$, d is an elliptic operator, so that $\sigma(d)$ is an isomorphism away from the zero section.

[The key property is that if X is compact, then the cohomology is finite dimensional.]

Example 1. *If X is a manifold, and say I have a vector bundle $V \to X$ with flat connection ∇. I can look at the formal moduli problem of deformations of V and ∇ (as a flat connection, V can't deform). I can ask, what is the corresponding elliptic*

Lie algebra. It's $\mathcal{L} = \Omega^*(X, End(V))$, and the differential arises from the flat connection coupled to the de Rham differential, and the Lie bracket is a combination of wedging of forms and bracketing of matrices. Why does this correspond to these deformations of V and ∇? I need to tell you about Maurer Cartan elements. If I take $R = \mathbb{R}[t]/t^3$ and the maximal ideal $= \{t, t^2\}$, then $MC(\mathcal{L} \otimes m)$ is something of the form $t\alpha + t^2\beta$, with $\alpha, \beta \in \Omega^1(X, End\ V)$, such that $[\nabla, \alpha] = 0$, α is closed and $[\nabla, \beta]$ (the failure of β to be closed) satisfies $[\nabla, \beta] + \frac{1}{2}[\alpha, \alpha] = 0$. This is the same thing as that $\nabla + t\alpha + t^2\beta$ is flat mod t^3.

[Is there an analog to the universal property of moduli spaces?]

If R has things in negative degrees, you'd see forms in positive degrees and you'd get something like a superconnection. Ω^1 is deformations, Ω^2 gives the flatness. Ω^0 gives the gauge group, and Ω^n for $n > 2$ are higher versions of flatness. If you consider the whole moduli space you get something very close to a superconnection.

I'm almost out of time, so maybe I'll give a couple more examples and next time I can explain what classical field theory is.

Example 2. *If X is a complex manifold, $E \to X$ is a holomorphic vector bundle, then $\Omega^{0,*}(X, End\ E), \bar{\partial}_E$ is the elliptic Lie algebra that controls deformations of this complex vector bundle E. The only thing that can move is $\bar{\partial}_E$, and this can move by a $0-1$ form in the endomorphism group, and the same argument applies.*

One more:

Example 3. *The self-duality equation: if X is a four manifold, V a vector bundle with connection ∇, and the anti-self dual part of the connection $F(\nabla)_-$ vanishes, then the elliptic Lie algebra looks like the following:*

$$\Omega^0(X, End(E)) \xrightarrow{\nabla} \Omega^1(X, End(E)) \xrightarrow{\nabla_-} \Omega^2_-(End(E))$$

You can work out how this degree two part imposes the anti-self dual part.

The last thing I wanted to say was to justify the terminology. Suppose that $\mathcal{L}$ is an elliptic Lie algebra. If R is $\mathbb{R}[t]/t^3$, I can look for Maurer-Cartan elements, and these are of the form $\{t\alpha + t^2\beta\}$ where α and β are in $\mathcal{L}^1(X)$ and these satisfy equations, $d\alpha = 0$ and $d\beta + \frac{1}{2}[\alpha, \alpha] = 0$. These are literally elliptic equations. So $d\alpha = 0$ is a linear elliptic equation on α, and because I required the bracket to be a differential operator, the equation $d\beta + \frac{1}{2}[\alpha, \alpha] = [$ is a nonlinear system of PDEs, and the key point is that its linearization is elliptic.

I'll stop here, and next time I'll talk about what a classical field theory is and how to construct these objects.

[What does it mean that the linearization is elliptic?]

You're looking at the tangent space, which is a cochain complex, and it's an elliptic complex.

[Suppose you write down a PDE and look at nearby solutions, is it described by this picture, and the answer is yes. This is just a way of writing the equations in an algebraic way.]

2. Moira Chas, string topology and three manifolds

I do not take lecture notes at slide talks.

3. Jim Simons, Dennis saves the day

It's nice to be here, to give a talk by chalk, I used to be able to do it well. Modern technology gives a new definition of well. I met Dennis in 1968 when I came here to chair the Stony Brook department and hire a bunch of people. Dennis was at the top of the list, but I failed, he made a big mistake. He subsequently learned the error of his ways and came, which is terrific. No fault of his, ten years later I stopped doing mathematics, but it was always in the back of my head. In 2004 I started thinking about a problem that I had thought about before, no progress for a long time, and I more or less went back to doing mathematics, and it turned out that I collaborated with Dennis. I'll give you a little background on that. So, I have some chalk, in the early 70s, I worked with Chern, is this visible? I suppose it must be. We did things with principle G-bundles E over M with θ a connection and Ω the curvature, P an invariant polynomial on the Lie algebra, you'll just have to remember that. If I plug the curvature into the polynomial $P(\Omega^\ell)$ is a closed 2ℓ-form downstairs on M whose cohomology is independent of θ. This is the famous Chern-Weil homomorphism, these represent real characteristic classes, et cetera, et cetera. If you lift this form. Call the map π. This is the softest chalk with which I've ever worked, so either I've gotten stronger or...

You lift to something, the bundle is trivial so when I lift, it will become exact, and $\pi^*(P(\Omega^\ell))$ is $dTP(\theta)$. So $TP(\theta)$ is well-defined up to something exact. There are other formulas you could write down, to do this job functorially it's only defined up to being exact. With Chern we found many uses (anyway, some uses) for these forms.

[Are these TP what they call the Chern Simons forms?] Yes, I suppose they are, T for Chern and P for Simons, Chern didn't know the alphabet very well, he wasn't born in this country. [Laughter]

The forms are up there and the manifold down here. It would be nice to get something downstairs. There was something down there. You can do the following. I want to have, suppose $P(\Omega^\ell)$ represents a universal real image of an integral class in the classifying space BG. Suppose it's a Chern class or Pontrjagn class or whatever, so then, if I pick one, such a class, then we can define what I call $SP_\mu(\theta)$ which is a homomorphism from $2\ell-1$-cycles into $\mathbb{R}/\mathbb{Z}$. Moreover, this map from cycles into the circle has the property that $SP_\mu(\partial Y) = \int_Y P(\Omega^\ell) \mod \mathbb{Z}$. These are, uh, things. They're natural, it turns out, you can do all that, they're functions to the circle, but if it's on the boundary, it's nothing but this integral, it's well defined, that's well defined, all the rest.

That's interesting. I'll give you one example. Suppose the group is $SO(2)$, and E is an $SO(2)$ bundle with connection, and χ is the integral Euler class, and P_χ is $\frac{1}{2\pi}\Omega$, the normalized curvature.

Now what is SP? If I have a closed curve (one dimensional cycle) downstairs, then $SP_\chi(\gamma)$ is $\frac{1}{2\pi}$ times the angle of holonomy. This can be extended to all cycles. So holonomy around the cycle turned into something between 0 and 1 is the most elementary of these SP things.

Now let me move over to this board here. So is Cheeger here today? He promised me he wouldn't attend this talk, and he kept this promise. He and I started talking about this stuff and we decided to make a definition, differential characters. We're in the smooth category, we have manifolds. Then

Definition 2. *Differential characters are:*

$$\hat{H}^k(M) = \{f \in Hom(Z_{k-1}(M), \mathbb{R}/\mathbb{Z}) | f(\partial Y) = \int_Y \Omega_f \mod \mathbb{Z}\}$$

This generalizes what we were talking about.

It's easy to show the following:

(1) Ω_f is closed, has integral periods, and is unique.
(2) Associated to f is $\mu_f \in H^k(M, \mathbb{Z})$. This has the porperty that the de Rham image of Ω_f is the real image of μ_f. This is a fairly simple construction that I won't go into. You have a differential character. To it is associated a closed form and an integral class.

This is the only one that moves?

[That board moves.]

Well I'll be a son of a gun, how many boards are there?

Of course, as the gentleman said... $\wedge^{k-1}_{\mathbb{Z}}$ are the closed forms with integral periods. Then let $\{\theta\}$ be $\wedge^{k-1}/\wedge^{k-1}_{\mathbb{Z}}$ and $\{\theta\}(a) = \int_a \theta \mod \mathbb{Z}$. For this particular guy, Ω sub this guy is just $d\theta$.

So the $\mathbb{R}/\mathbb{Z}$ cohomology also sits in this. So $H^{k-1}(M, \mathbb{R}/\mathbb{Z})$ is homomorphisms from $Z_{k-1}, \mathbb{R}/\mathbb{Z}$ that vanish on the boundaries.

I want to put u something that I want to leave up. I want to fit this gadget, this thing, into some exact sequences. Let's see where this fits into the galaxy of all previously known functors. We'll put it in the middle [Chalk breaks] Jesus Christ.

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{k-1}(M,\mathbb{R}/\mathbb{Z}) \arrow[rr, "-\mathrm{Bock}"] \arrow[dr, "i_1"] & & H^k(M,\mathbb{Z}) \arrow[ur] \arrow[dr, "\mathrm{Real}"] & \\
H^{k-1}(M,\mathbb{R}) \arrow[ur] \arrow[dr] & & \hat{H}^k(M) \arrow[ur, "\delta^2"] \arrow[dr, "\delta_1"] & & H^k(M,\mathbb{R}) \\
& \wedge^{k-1}/\wedge k-1_{\mathbb{Z}} \arrow[ur, hook, "i_2"] \arrow[rr, "d"] & & \wedge^k_{\mathbb{Z}}(M) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

[When did you say that map was there?] [Number two is that map in a sentence.] That's exactly right. What's your name? [Nate.] Nate's exactly right.

Some of this isn't so obvious. This functor, which is kind of nice, fits into this diagram of all these familiar functors. It's natural if you take C^∞ map, everything in sight commutes, we'll call this thing the character diagram, or just the diagram.

Now, Jeff Cheeger and I made this stuff up. It looks awful that I wrote diagram, can you remember that I wrote that? This gave some simple proofs about things we had done with these TP forms. The Weil homomorphism factors through this thing, so if you have a pair, an invariant polynomial and an integral class who represents, then you get a differential character, and when you get down to here you recover it. Even if both δ_1 and δ_2 are zero, then there is a simultaneous kernel, which is $H^{k-1}(M,\mathbb{R})/H^{k-1}(M,\mathbb{R})$. It's more than either the images or the kernels.

There it is, and it's good for some things.

In around 2003, is Blaine Lawson here? Another guy who promised not to come. Is Zweck here? I never met Zweck. They came up with a lot of other constructions that turned out to be naturally, what do you call it, equivalent, some were graded, some were similar, some were different. In each case they proved that their functors were equivalent to differential characters. It occurred to me that maybe the diagram itself characterizes the functor. Maybe nothing else could do it. That's not such a crazy thought. It's sort of trapped in here. How many guys could you come up with, up with which you could come, I like to keep my grammar good, and it turns out that it's true. This diagram characterizes the functor, and that's what we'll sketch, and that's where Dennis comes in.

This is ordinary differential cohomology, called ordinary, because it's associated with ordinary cohomology. There are similar diagrams you could draw for any of the extraordinary cohomologies. Singer and unintelligbleshowed something more or less like this. This was the first example, and completely independently of Jeff and I concocting this, Deligne was inventing Deligne cohomology, I think it was holomorphic, it wasn't smooth manifolds particularly, but in the smooth category, it's a functor isomorphic to this. There was something in the air that made one want to come up with things like this. It turns out in the ordinary case there's only one. That's what we'll talk about today.

[In even K-theory it is unique, Andrew, sitting up there had a little help along with a couple of Germans, Bunke and Schick. In the odd case it's not enough, you need something else. It's maybe pretty much unique if it has pushforward, that's Andrew's thesis. An exotic cohomology can be very exotic. I didn't know that K-theory was an extraordinary cohomology. I learned a lot coming back, not enough to make up for 27 years. One thing that's weird is that Hom of homology into $\mathbb{R}/\mathbb{Z}$ is not $\mathbb{R}/\mathbb{Z}$ cohomology. That's true for K-theory, but Dennis tells me that's a miracle. I think this should always be true but what I think doesn't matter a heck of a lot.]

I went first to Jeff with my idea. I have a fake functor, an imposter, I want to map it to $\hat{H}^k(M)$. I need a function on cycles. I'll take a cycle and represent it with a lower dimensional manifold that I'll push in, its fundamental class will be homologous to the cycle, well, anyway, it was an approach. I told Jeff I had this idea, you evaluate it, you get an $\mathbb{R}/\mathbb{Z}$ number, and everything was Jake. Jeff said, "I don't know, you'd better talk to Dennis about that."

I was sitting around quietly in my office, and Dennis came to ask me a question. It wasn't math, I said, can every homology class be represented by a manifold stuck in? He said no, good try, Poincaré asked that question, it's not stupid to ask that question, he came up with the same answer, no. Some multiple can always be represented with a manifold stuck in there. If we were only working over $\mathbb{Q}$, you can get every rational class by a manifold. I was bound to get the integral classes.

That was the end of my theory of how to prove this theorem, but Dennis got kind of interested in the problem. He said, maybe there is something else we can do. We did, mostly he did, and I sort of hung around.

I'm going to make a couple of definitions and then outline the proof, the right homomorphism, and the three facts about topology that Dennis either recalled or invented or both. They're very pretty facts and I'd never heard of any of them.

[The other side of the eraser works better.] Nate, why didn't you tell me that? [Laughter.]

Definition 3. *A character functor into Abelian groups is a* 5*-tuple* $\hat{G}^*, i_i, i_2, \delta_1, \delta_2$ *which satisfies the diagram.*

Theorem 1. *Any character functor is equivalent to* $\hat{H}^*$ *via a unique natural transformation which commutes with the identity map on the other four functors.*

This is as good as it gets, and it's unique. I'm going to sketch how this gets proved, and so we need a Dennis definition.

Definition 4. *An open* U *in* M *is called* k*-good if* $H^\ell(U, \mathbb{Z}) = 0$ *for* $\ell > k$*.*

Did you ever hear that? No one has heard of that, so he even invented the definition.

[What is important about M] Well, usually open sets are in a topological space, you don't ust walk around with an open set.

Fact 1 (Fact 1)**.** *Any neighborhood of the image of a smooth, singular* k*-chain contains a* k*-good sub-neighborhood. This isn't such a shocking statement, but it's not so easy to prove. Why can't you just thicken it a little bit? It is not as simple as what I just said; in any event, is Chris Bishop here? Another guy, I think he helped Dennis with this.*

Now I'll define the right map to $\hat{H}$. Here's what we do. I want to map $\Phi : \hat{G}^k \to \hat{H}^k$. I'll take a cycle $a \in Z_{k-1}(M)$ and take U a $(k-1)$-good neighborhood of the support of a. Now we'll call the inclusion map $\lambda : U \to M$. Since everything is natural, I can look at, if I have $g \in \hat{G}^k$, I can look at $\lambda^*(g)$, I can pull it by restriction, it's in $\hat{G}(U)$. Then $H^k(U, \mathbb{Z})$ is 0 since it's $k-1$-good. That means, think of G as sitting in the middle of the diagram, it's one of Nate's favorite characters, just a form, so $\lambda^*(g) \in \wedge^{k-1} / \wedge^{k-1}_{\mathbb{Z}} (U)$, call it $\{\theta\}$, and I can say, okay, $\Phi(g)(a) = \int_a \theta \mod \mathbb{Z}$. If I picked a different θ, it'll differ by something integral. Suppose I'd picked a different U. Their intersection, in that I could find something $k-1$-good inside that. Naturality would ensure that this was independent of the choice of U, so this is a good definition.

Now, it's also, it's pretty easy to show that this is going to be a homomorphism. We've sent this to $Hom(Z_k(M), \mathbb{R}/\mathbb{Z})$, but we need to show whta it does on boundaries. We need to show that if $a = \partial e$ then $\Phi(g)(a) = \int_e \delta_1(g) \mod \mathbb{Z}$. That's what we need to show, see? It's not just a character, we have this right form. I have to show that. This, I'll go to this. I want to keep one picture up here, and I'll try to finish quickly, I was told Hershel's was going to be half an hour. So here's a, it's a boundary, here's e.

Here's fact 2, a wonderful fact.

Fact 2 (Fact 2)**.** *Assume the dimension of* M *is at least* k*. Then a smooth singular* k*-cycle is homologous in every neighborhood of its image to the fundamental cycle of an embedded pseudomanifold.*

These are allowed to have singularities. The top, every second to top level is the side of two top levels, but they have singularities, this is the level of things that *do* represent homology. Not only that, it's, you only have to move it a teeny bit. No matter what the neighborhood is, inside is a homologous embedded pseudomanifold. That's a terrific fact.

So in my picture here's a pseudomanifold P and a homology between these two. The name of that homology is b. This b and P and e and a are all in U.

Here's fact three, I'll just write it down and then talk through to the end. This is a beaut.

Fact 3 (Fact 3)**.** *If the fundamental cycle of embedded P, a k-pseudomanifold, is a boundary, then it is also a boundary in some $k-1$-good neighborhood of P.*

Now it's saying, if this is a boundary somewhere, then you can get it to be a boundary in a $k-1$-good neighborhood. You inch your way along in two steps. It's a boundary in two ways, and I'll just write all this down, you conclude that $\Phi(g)(a) = \int_e \lambda_1(g) + \int_{b+y-e} \lambda_1(g) \mod \mathbb{Z}$. You chase around this thing, you use naturality. We'll see that $b+y-e$, this is a cycle, this is a cycle, so I'm integrating over a cycle and it goes away. This last fact three, I presume, you really need that it's a pseudomanifold to get the $k-1$ good neighborhood of its whole homology to 0. You inch your way along, you use two different k-good neighborhoods and have this, so this Φ is a natural transformation from G to H. It's mostly algebra, I don't have time, and at the end the famous five lemma.

Now, we worked, we kept getting stuck and Dennis came up with fact one, and got stuck and Dennis came up with fact two, and we got stuck again and he came up with fact three, so he really saved the day three times. That's the end of my talk.

[What's the problem for exotic cohomology.] The model we've been working with as vector bundles with connections, it's starting from a very different point of view. Isomorphism is not as easy.

[The diagram starts with $H^{k-1}(M,R)$ and ends with $H^k(M,R)$, is there a map $\hat{H}^k(M) \to \hat{H}^{k+1}(M)$.]

No. If you stay on the train too long on a long exact sequence you get killed. [Like the LIRR!]

[What is the codimension of the singular set on a pseudomanifold?] At least 2.