West Coast Algebraic Topology Summer School

Gabriel C. Drummond-Cole

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1 Introduction Søren Galatius

[I'm Dev Sinha, I've been doing some of the non-mathematical bits of putting this on. The mathematical organizers, most of them, are over there, Søren, Johannes, Oscar, and David. Nathalie and Paolo couldn't make it out. The one last bit of essential hosting that I plan to do is, today at 1:30, I'll give you a tour of Willamette Hall. We reserved a few different rooms whose locations might not be obvious from the numbers. We reserved a lot of rooms because the philosophy is that the most important part is not the talks but getting you guys to talk to each other and so on. So we're not sure where people are going to be. The organizers will have their offices in this building. Hopefully, magically, people will start doing good mathematics. There'll be, as is often the case, excursions on Wednesday afternoon, which could be as simple as taking a long walk on the Willamette river, but some of the locals will be willing to take people away, and that will be on the wiki. Some people may take a bus to a hike just south of town. I wanted to introduce Artema, who has done good work, like putting together the folders. We guessed that people with U.S. addresses need those kinds of reimbursement forms, and vice versa. If you have questions about reimbursement, you can talk to Artema. The books that we referenced will also be in Willamette hall. The articles should be on a website soon. That's about it, let me turn things over to Søren.]

Thanks, thanks to Dev for putting on everything. I'm not going to say that much. I wanted to say also a little bit of practical information. In particular the exercise sessions, maybe we should have called them discussion sessions. The point is to have time to talk about the talks, talk about related material. The plan is to improvise those. We tried to make some actual exercises as a way to help you think about what happened and what are the, but you should think of the exercises as a way, maybe, of starting discussion. They could also be used to, I know you come from slightly different backgrounds, we could use an exercise session to fill in some background. So I should say that, if there's something you'd like to hear more about, or hear again, well, there's also several different rooms, we can do several things in parallel.

[There is one room large enough to hold everybody. But the idea is not to do that the whole time.]

There are also evening sessions, maybe those we could have a beer at the same time. The point of thewhole thing is to learn something, and learning something here, the point, in the grant, was to have more than talks, also something interactive. We gave these exercises as sort of one thing you could try to figure out. For the lectures, all the lectures will be given by you, so they're all being given by nonspecialists, so as to have a relaxed atmosphere. Questions should be encouraged but the speaker can feel free to defer them to the exercise session, to say, "I'm not going to answer that right now."

Then I thought I would say a little about the recent history of what we're talking about. Are ther any questions so far? I'm happy to see that so many people showed up.

So the program is centered around a theorem of Madsen-Weiss, around 2002, but we tried to make it somewhat broad. The last day, Friday will be a lot about this theorem, but before that there will be a bunch of other stuff, background, and so on. I'll maybe talk for fifteen minutes, not say anything precise, just sort of outline a little bit. So Madsen and Weiss' theorem is about a space, $\mathscr{M}_g$. From one point of view it is the set of Riemann surfaces of genus g up to isomorphism of Riemann surfaces. Riemann, I think, realized that these come in families, and this is a space of [real] dimension $6g-6$. Skipping ahead, this has been studied from many different points of view. Mumford, in the mid 1980s, raised the question of the cohomology of this space, $H^*(\mathscr{M}_g,\mathbb{Q})$, and defined certain classes $\kappa_i \in H^{2i}(\mathscr{M}_g,\mathbb{Q})$, now called Miller-Morita-Mumford classes. It was known by Harer in the 1980s, he said, paraphrasing, that $H^k(\mathscr{M}_g,\mathbb{Q})$, if g is large enough is independent of g, if $g \gg k$. This is called the "stable range." Mumford made a conjecture that the Miller-Morita-Mumford classes are algebraically independent and generate the cohomology in the stable range. So $\mathbb{Q}[\kappa_1,\kappa_2,\ldots] \to H^*(\mathscr{M}_g,\mathbb{Q})$ is an isomorphism in the stable range.

Skipping ahead a little more, Madsen and Weiss proved this, actually a slightly stronger integral version, for integral homology. One of the goals of the program today is to give, formulate the integral homology version, let me say a few words about it. There is a space, which we will get back to, called $B\Gamma_g$, the classifying space of the mapping class group, which is classical, which has the same rational homology as $\mathscr{M}_g$, and so the same in cohomology. Harer proved that $H_k(B\Gamma_g,\mathbb{Z})$ is independent of g for $g \gg k$. That's the first step for proving something integral. In the mid 1990s, around 95, Ulrike Tillmann proved a theorem that started this way of thinking about it, that there is an "infinite loop space" $E_0 = \Omega E_1 = \Omega^2 E_2 = \cdots$ which has the same cohomology as $B\Gamma_g$ in a stable range, a map $B\Gamma_g \to E_0$ which induces an isomorphism in $H_*(\quad,\mathbb{Z})$.

If you know that a space is an infinite loop space, there's a good chance that it is equal to something you've seen before. Tillmann's theorem was kind of abstract. She didn't say which infinite loop space it is. Madsen made a conjecture about which infinite loop space it would be, and Madsen and Weiss proved that. They called it $\Omega^\infty \mathbb{CP}^\infty_{-1}$ and later renamed it $\Omega^\infty MTSO(2)$. This is a space with rational cohomology $\mathbb{Q}[\kappa_i,\kappa_2,\ldots]$, and Madsen and Weiss proved that this infinite loop space is in fact this particular one. We'll talk about this space later. This rational cohomology is in some sense "easy." So the theorem of Madsen and Weiss is that there is a map $B\Gamma_g \to \Omega^\infty MTSO(2)$ which induces an isomorphism in homology in the stable range (integrally). Harer proved that $g \geq 3k$ was a good range, and now it means $g > 1.5k + c$. That settled Mumford's conjecture, but it's a much more precise statement.

We'll hopefully get to that by the end of Friday. We tried to partition the talks into different topics each today. Today there will a talk about $MTSO(2)$ and about $\mathscr{M}_g$. Tomorrow will be about, one main ingredient in their proof is h-principles. I think it's an important technique. Thursday will be about Harer's theorem, that $H_k(B\Gamma_g)$ is independent of g when g is large, and Friday will be about Madsen and Weiss' result.

Saturday we said wrap up. One thing we thought of doing was talking a bit about what are the directions this is moving in, what are the relations to other topics. Questions?

[What does the theorem say about integral homology of $\mathscr{M}_g$?]

The first thing is that it's very complicated. You might think there's a nice formula, but it's much uglier than you thought. You can work quite hard and calculate the mod p homology. The answer is not something I can just state now. The mod 2 homology is a polynomial ring on infinitely many generators, but saying what they are is kind of complicated. I think there's no reason for a half hour break, so now we'll start the next lecture a bit early, how about twenty five minutes past.

2 Teichmüller theory and moduli spaces

The main object of study will be F_g, a Riemann surface of genus g, smooth, oriented, connected, compact, two dimensional manifold without boundary. Consider compact structures on F_g, that is, smooth endomorphisms $J : F_g : TF_g \to TF_g$ so that $J^2 = -I$ and $\langle Jv, v \rangle > 0$ for $v \in TF_g$.

The space of complex structures on F_g can be identified with, well, there is a bundle with fiber $C = GL^+(\mathbb{R}^2)/GL_1(\mathbb{C}) \cong GL^+(\mathbb{R}^2)/GL_1(\mathbb{R}) \times SO_2(\mathbb{R})$ where M goes to MJ_0M^{-1}, where J_0 is the standard endomorphism $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then C is homeomorphic to the unit disk D, via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \mu = \frac{1 + i\tau}{1 - i\tau}$$

where $\tau = \frac{a+ib}{c+id}$.

Theorem 1 *For Riemann surfaces, complex structures and almost complex structures are the same.*

Theorem 2 *the space of complex structures on F_g is contractible.*

Now I want to introduce $Diff(F_g)$, the group of orientation preserving diffeomorphisms of F_g, and $Diff_0(F_g)$, those which are homotopic to the identity. So $Diff(F_g)$ acts on the complex structures $H(F_g)$ by sending (J, f) to $df^{-1} \circ J \circ df$.

So here are some interesting things about the complex structures.

1. $H(F_g)$ is a connected complex manifold.
2. $Diff(F_g)$ acts continuously, effectively, and properly on $H(F_g)$.

Moreover, $Diff_0(F_g)$ acts freely on $H(F_g)$, and the quotient T_g is $H(F_g)/Diff_0(F_g)$, the Teichmüller space. This quotient is also a complex manifold.

Now an important theorem tells us

Theorem 3 *$\Phi : H(F_g) \to T_g$ is a $Diff_0(F_g)$-principal bundle, and moreover a topologically trivial bundle. We can show that Φ is holomorphic, but not holomorphically trivial.*

Theorem 4 *(Teichmüller) T_g is homeomorphic to $\mathbb{R}^{6g-6}$*

Theorem 5 *$H(F_g) \to T_g$ is topologically trivial.*

*Proof.*There is a map $g; T_g \times I \to T_g$ so that $g(T, 0) = \tau_0$ and $g(\tau, 1) = \tau$ and then by lifting, $f : T_g \times I \to H(F_g)$ and $f(\tau,1) = \sigma(\tau)$ is a section for Φ. Then $H(F_g)$ is homeomorphic to $T_g \times Diff_0(F_g)$. I didn't introduce the topology, but the idea is that $T_g \times Diff_0(F_g) \to H(F_g)$ by $(\tau, f) \mapsto \sigma(\tau)f$ and this is a homeomorphism. □

What is a Riemann surface with boundary? This is obtained by removing n disjoint open disks from a Riemann surface. We denote this by $F_{g,n}$, and we can imagine this like this picture.

What is the difference between boundary and punctures? A puncture is just a removed point. A boundary component is fixed by an automorphism and punctures can be permuted by automorphism.

Now I would like to show, we've seen $Diff_0(F_g)$ is contractible, and the same holds for Riemann surfaces with boundary. So $Diff(F_g, D)$ is the group of diffeomorphisms $F_g \to F_g$ which fix D pointwise. This is the same as the group $Diff(F_g - D, \delta(F_g - D))$, that is, maps that are the identity on the boundary. The components of this group are also contractible.

Consider the space of embeddings of D into F_g. So $Diff(F_g)$ acts transitively on $Emb(D, F_g)$, so I can suggest with this picture, so if you embed the disk, you can find a diffeomorphism between these disks. So the map $Diff(F_g) \to Emb(D, F_g)$ is in fact a fiber bundle with fiber $Diff(F_g, D)$. So we get

$$Diff(F_g, D) \to Diff(F_g) \to Emb(D, F_g)$$

and so we get that $Emb(D, F_g)$ is a $K(\pi, 1)$.

For a proof, consider immersions $Imm(D, F_g)$. Let $S(f) = \max\{r | f \text{ is injective on } |Z| < r\}$. If $S > 0$ then $f \mapsto f((n-t)S(f)z)$ is a retraction.

The differential $d : Imm(D, F_g) \to C^\infty(D, Fr(F_g))$ is continuous but not surjective. For this map, by an h-principle, this is a weak homotopy equivalence. We deduce that $Fr(F_g)$ is aspherical, D is contractible.

We will need surface bundles to define some classes.

Definition 1 *Let F be a smooth, oriented manifold. An F-bundle is a bundle $\pi : E \to B$ with fiber F and structure group $Diff(F)$ (orientation preserving). If $F = F_g$ or $F_{g,n}$, then we would call this a surface bundle.*

One motivation for this, we know the topological classification of F bundles, we know that isomorphism classes of F-bundles are the same as homotopy classes of maps $[B, BDiff(F)]$. If $B = S^n$ and $F = F_g$, then $[S^n, BDiff(F_g)] \cong \pi_n(BDiff(F_g))/\pi_1(BDiff(F_g)) = \pi_{n-1}(Diff(F_g))/\pi_0(Diff(F_g))$.

So now we want to study the object by which we're quotienting.

Definition 2 $\Gamma_g = Diff(F_g)/Diff_0(F_g)$.

Remark 1 *Γ_g is discrete.*

So Γ_g acts properly discontinuously on T_g, but Γ_g does not act freely. So the quotient space will not have similar properties.

Definition 3 *$\mathscr{M}_g = T_g/\Gamma_g$, the moduli space of genus g.*

So we should get an exact sequence

$$Diff_0(F_g) \to Diff(F_g) \to \Gamma_g$$

and we've shown that the first of theses is contractible, so that $BDiff(F_g) \to B\Gamma_g$ is a homotopy equivalence.

This was already said in the introduction, that we have a rational homology equivalence $\varphi : B\Gamma_g \to \mathscr{M}_g$. So consider the universal bundle $EDiff(F_g) \to BDiff(F_g)$. Look at the associated bundle with fiber F_g, mapping to $BDiff(F_g)$. Then $\pi^{-1}(b)$ is a Riemann surface and $BDiff(F_g)$ maps to $\mathscr{M}_g$ by taking $b \mapsto [\pi^{-1}(b)]$. [To get this map, you need to choose complex structures on $\pi^{-1}(b)$. So because this is contractible, you can pick such complex structures coherently.]

Lemma 1 *Γ_g has a torsion free normal subgroup H of finite index.*

Now consider Γ_g/H and you get the diagram

$$\begin{tikzcd} BH \arrow[r, "\text{h.e. by LES}"] \arrow[d, "\text{finite } K\text{-covering}"'] & T_g/H \arrow[d, "\text{quotient}"] \\ B\Gamma_g \arrow[r, "\varphi"'] & M \end{tikzcd}$$

Lemma 2 *If X is a CW complex and G is a finite group acting cellularly on X. If g fixes a cell then it fixes it pointwise.*

Then $H_*(X,\mathbb{Q})\otimes_{\mathbb{Q}G}\mathbb{Q} \cong H(X/G,\mathbb{Q})$. We can follow and see that $H*(B\Gamma_g,\mathbb{Q}) \to H_*(\mathscr{M}_g,\mathbb{Q})$ is an isomorphism.

Now we construct the MMM-classes. We look at F_g-bundles π and consider the associated $Diff(F_g)$ bundle $E_g \to B$. The fibers are diffeomorphsm preserving maps $F_g \to \pi^{-1}(b)$.

Let ξ be the bundle $E_g \times_{Diff(F_g)} TF_g \to E_g \times Diff(F_g)F_g \cong E$, and $\xi|_{E_b} = \pi^{-1}(b) \cong T\pi^{-1}(b)$. There is an Euler class $e(\xi) \in H^2(E,\mathbb{Z})$ and $e^{i+1} \in H^{2(i+1)}(E,\mathbb{Z})$ and $\pi(e^{i+1}) = e_i \in H^{2i}(B,\mathbb{Z})$.

Now taking $E = EDiff(F_g)$ and $B = BDiff(F_g)$, you get exactly $e_i \in H^{2i}(BDiff(F_g),\mathbb{Z}) \cong H^{2i}(B\Gamma_g,\mathbb{Z})$, and by the isomorphism we get classes $\kappa_i \in H^{2i}(\mathscr{M}_g,\mathbb{Q})$. The κ_i are the classes from the introduction.

We can once again formulate the Mumford conjecture, that there is an isomorphism $\mathbb{Q}[\kappa_1,\kappa_2,\cdots] \to H^*(B\Gamma_\infty,\mathbb{Q})$.

3 Pontryagin-Thom Theory I
Hiro Tanaka

[I want to thank the organizers for assigning me this talk. I apologize that you have to learn it from someone who just learned it. There are two parts. The first part will cover the Pontrjagin Thom theorem, which gives an isomorphism of two groups. I think it's actually rings, but we'll keep it simple. So I'll write down $\pi_1(MB)$, which looks like the fundamental group of a space, but it's actually a spectrum, and I'll define that. The other side will be Ω_n^{Bf}. n is a dimension, and this is the equivalence classes of n dimensional manifolds up to cobordism. This is easier than diffeomorphism. I'll be using smooth conditions.

I've explained almost completely what this is between. I didn't explain what this is an isomorphism between. B and f will be what I care about. In general it'll be oriented or unoriented cobordisms for these talks.

Part two will be about characteristic classes and characteristic numbers. The upshot is going to be, I'll have Stiefel-Whitney classes and numbers. These are unoriented. In the case of unoriented cobordisms, these are the best you can do. I gave a big picture, any questions?

All right, Part I A, we'll talk about Thom spaces. This is a pointed topological space associated to vector bundles over B. Let me give you the general context. Assume you have a vector bundle $\begin{array}{c} E \\ \downarrow \\ B \end{array}$. I'm going to define $Th(E)$ as the bundle of disks of unit length $Disk(E)$ mod out by the spheres $Sph(E)$, and it comes with a basepoint which is the collapse of the sphere bundle.

Here are some examples.

1. What if you have $\begin{array}{c} \mathbb{R}^n \\ \downarrow \\ pt \end{array}$? Then $Th(\mathbb{R}^n) = \mathbb{S}^n$.
2. If you have $\begin{array}{c} V \\ \downarrow \\ B \end{array}$ and $\begin{array}{c} \mathbb{R}^n \\ \downarrow \\ B \end{array}$, what is $Th(\mathbb{R}^n \oplus V)$? It's actually $\Sigma^n Th(V)$.
3. This also is an important example. I hope people have seen this before. So we can look at the Grassmannian $Gr_k(\mathbb{R}^{n+k})$, and we can ask of the colimit of these over the embeddings into $Gr_k(\mathbb{R}^{n+k+1})$. So we'll call this $BO(k)$ or $Gr_k(\mathbb{R}^\infty)$. There's a tautological $\mathbb{R}^k$ bundle over each one of these spaces that passes to the colimit and gives the bundle $\begin{array}{c} \gamma^k \\ \downarrow \\ BO(k) \end{array}$. The fiber over any point in the Grassmannian is the plane that that point defines.

So now let me give

Definition 4 *$MO(k)$ is the Thom space of the vector bundle γ^k.*

I want to point something out that will lead me to spectra. Inside each of these guys $Gr_k(\mathbb{R}^{n+k})$ into $Gr_{k+1}(\mathbb{R}^{n+k+1})$ so that the pullback of the canonical bundle is the canonical bundle summed with the trivial bundle. So we get

$$\begin{tikzcd} \gamma^k \oplus \mathbb{R} \arrow[r] \arrow[d] & \gamma^{k+1} \arrow[d] \\ BO(k) \arrow[r] & BO(k+1) \end{tikzcd}$$

So I get a map $Th(\gamma_k \oplus \mathbb{R}) \to Th(\gamma_{k+1})$. But the left hand side is the same as $\Sigma Th(\gamma^k)$. So we get a sequence of spaces which are Thom spaces of the tautological bundles, with maps $\sigma_k : \Sigma Th(\gamma_k) \to Th(\gamma_{k+1})$. This is a spectrum (or a prespectrum).

Definition 5 *A* spectrum *is a sequence of pointed spaces X_k and structure maps $\Sigma X_k \to X_{k+1}$ indexed over $k \geq 0$.*

Let me give you enough information for you to understand the right hand side of what I wrote down. Let me define the homotopy group of a spectrum.

Definition 6 *If $X = \{X_k\}$ is a spectrum then $\pi_n(X) = \mathrm{colim}\pi_{n+i}X_i$.*

What diagram is this a colimit over? Say you have a map $f \in Hom(S^{n+i}, X_i)$, we can apply suspension to get a map in $Hom(S^{n+i+1}, \Sigma X_i)$ and by composing with the structure map of X, I get a map in $Hom(S^{n+i+1}, X_{i+1})$. So then MO is this spectrum, and the right hand side is a homotopy group of this spectrum.

Let's talk about the unoriented cobordism group. I'm just going to fix a dimension to make things easier. The empty manifold has whatever dimension you want. The empty manifold is a manifold of dimension n. So as, well,

Definition 7 *There are several ways I can say this. Two n-manifolds M_0 and M_1 are cobordant if there exists a compact manifold N so that the boundary of N is $M_0 \sqcup M_1$, the disjoint union of the two manifolds.*

I'm assuming people have seen this but examples help. As an example, this is reflexive, since the boundary of $M_0 \times I$ is $M_0 \sqcup M_0$.

As a set, Ω_n is the set of closed n-manifolds modulo cobordism. The unit of the group I'm putting on this is the empty manifold, and the addition is disjoint union.

[Do you want to say a few words about this being a set?]

[Whitney embedding theorem.]

I couldn't find a definition of this B, f thing that didn't seem a little hairy, so I apologize. The spirit is that after defining a bunch of trivial line bundles, things are the same. That's the spirit.

So let me say something about cobordisms with orientations. What I want to note is that there is a natural space that looks like a cover of the Grassmannian of k-planes, that's the Grassmannian of oriented k-planes $Gr_k^{or}(\mathbb{R}^\infty)$, and I can denote the sequence of spaces as B_k.

Say that we have a manifold M with a bundle on it of dimension k, $\begin{array}{c} E \\ \downarrow \\ M \end{array}$, then by general classifying space theory, this gives rise to a homotopy class of maps $M \to Gr_k(\mathbb{R}^\infty)$. What we can ask is, we have the space Gr_k^{or}, and we can ask if there's a lift:

$$\begin{tikzcd} & Gr_k^{or} \arrow[d] \\ M \arrow[ur, dotted, "\tilde{\xi}"] \arrow[r, "\xi"'] & Gr_k \end{tikzcd}$$

Such a lift is called a Gr_k^{or}, f_k structure. This would generalize to a B_k, f_k structure.

An observation: Since Gr_k^{or} embeds in Gr_{k+1}^{or}, a Gr_k, f_k structure induces a Gr_{k+i}, f_i-structure on $E \oplus \mathbb{R}_i$.

[Discussion]

To be continued, maybe. You can forget the last three pages of your notes. A cobordism of oriented manifolds M_0, M_1 is a compact manifold N, oriented, such that the induced orientations on the boundary are the assumed and reversed orientations of M_0 and M_1, respectively.

This still allows the cylinder to be a cobordism from M to itself. Now we have an idea about the construction of the map. I'll just give you two maps.

$$\begin{tikzcd} \Omega_n^{unor} \arrow[r, shift left, "\alpha"] & \pi_n(MO) \arrow[l, shift left, "\beta"] \end{tikzcd}$$

So to construct α, given $i : M^n \to \mathbb{R}^{n+q}$, the normal bundle $\begin{matrix} \nu \\ \downarrow \\ M \end{matrix}$

1. gives a classifying map $M \to BO(q)$ and
2. gives a Thom space $Th(\nu)$.

The first thing means we have a map $Th(\nu) \to Th(\gamma^q)$ and the tubular neighborhood theorem gives a map $\mathbb{R}^{n+q} \to Th(\nu)$, which we can think of as $\mathbb{S}^{n+q} \to Th(\nu)$ because everything near infinity is mapped to the basepoint. The composition gives an element of $\pi_{n+q}(MO(q))$.

Once you have this, you can collapse $\mathbb{R}^{n+q}$ in two stages, collapsing everying off near infinity, and then collapsing the points at ∞.

The resulting element of the group does not depend on the choices involved, the representative of the cobordism class and the embedding. You can show independence of the embedding by embedding in a big enough $\mathbb{R}^n$. You also need to show independence of cobordance class.

So now let's give β. Given $f : S^{n+q} \to Th(\gamma^q)$, let's look at $f|_{\mathbb{R}^{n+q} \subset S^{n+q}}$ and let's also look at the the zero section $Gr_q(\mathbb{R}^\infty) \subset Th(\gamma_q)$. I can find something in the homotopy class transversal to this section, so find a homotopic f transversal to this. This is what they teach you in your first differential topology class. Then I can look at the preimage of the intersection and the preimage is again a smooth manifold sitting inside $\mathbb{R}^{n+q}$. What is the codimension? The codimension is q so we get a manifold of dimension n. This one it's easy to answer, whether two homtopic maps give rise to a cobordism. You'll get a cobordism in the preimage assuming transversality.

I claim this is an isomorphism of groups. This should respect the group map in both directions. The theorem is that these are inverse group homomorphisms.

I have ten minutes for part II, which is what I thought would be the fun part.

Let me tell you what Stiefel-Whitney classes are. I apologize for thinking I had a ninety minute talk.

It would be nice if we had cobordism invariants. So let me give a definition.

Definition 8 *A* characteristic class *is a way to associate to a vector bundle* $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$*, elements in $H^*(M)$. The coefficients depend on your interest. We will use $\mathbb{Z}_2$ coefficients. Let $w_i(E)$ be in $H^i(M,\mathbb{Z})$. These should satisfy (at least) the properties:*

1. $w_0(\mathbb{R}_n) = 1$ *and* $w_i(\mathbb{R}_n) = 0$ *otherwise.*
2. $w_*(V \oplus W) = w_*(V) \oplus w_*(W)$
3. *Naturality:* $w_*(f^*(E))) = f^* w_*(E)$
4. *If* $|E| = n$ *then* $w_k(E) = 0$ *for* $k > n$
5. $w_*(\begin{matrix} \gamma^1 \\ \downarrow \\ \mathbb{RP}^1 \end{matrix})$ *is not* 1 *[0]*

We have cohomology classes. You can get numbers by integrating over a fundamental class. We can take products that end up in the top dimension and integrate over $[M]$. So such products for TM integrated against $[M]$ are the Stiefel-Whitney numbers of M.

Proposition 1 *If $M = \delta W$ then all Stiefel-Whitney numbers are zero.*

Proof. If $M = \delta W$ then $TW|_i = TM \oplus \mathbb{R}$ over M. Using the properties I wrote down, when we pull back Stiefel-Whitney classes associated to TW we get those associated with TM. Then $[M] \in H_n(M, \mathbb{Z}_2) = \delta z$ for $z \in H_{n+1}(W, M, \mathbb{Z}_2)$. So given a polynomial f of Stiefel-Whitney classes, $\langle f, \delta z\rangle = \langle i^* f_W, \delta z\rangle = \langle f_W i_* \delta z\rangle$, but $i_*\delta$ is part of the long exact sequence of a pair and thus gives zero.

I've gone five minutes over so I have twenty-five minutes. Just kidding. There's just one more observation I want to make. When I did the Thom space construction we were looking at normal bundles. If we have an embedding $M^n \subset \mathbb{R}^{n+r}$, we can look at two vector bundles associated to this, the tangent or normal bundles. The Stiefel-Whitney [unintelligible]vanish for the tangent bundle if and only if the same is true for the normal bundle. You can see this from axiom number two. Then this is a cobordism invariant. The really big theorem is that these form a complete invariant, two manifolds are cobordant if and only if their Stiefel-Whitney numbers are the same.

[Let me give a quick proof. You can get $M \to BO(n)$. If it's oriented, you get $[M] \mapsto H_n BO(n)$. All characteristic numbers are doing is telling you about where this lands. This is like evaluating the monomials in $H^*(BO(n))$. If all of the numbers vanish it's like getting a null homology. They're just encoding which homology class you get.]

[I think Thom's theorem, a surprising thing is that you can calculate these explicitly, the cobordism groups. You can get a product with the product. The cobordism ring is $\mathbb{F}_2[y_i, i \neq 2^k - 1]$.]

4 Pontryagin Thom Theory II

Let's define $MTSO(n)$. So now $G(n,k)$ is the Grassmannian of oriented n planes in $\mathbb{R}^{n+k}$, and these sit inside each other, $G(n,k) \hookrightarrow G(n,k+1)$, where the pullback of γ is $\mathbb{R} \oplus \gamma$.

We can apply the Thom space construction to get a spectrum with spaces $MTSO(n)_{n+k} = Th(\gamma_n^\perp)$ with structure maps. Given a fiber bundle $M \overset{i}{\hookrightarrow} E \to B$ of closed manifolds, Embedding $E \hookrightarrow B \times \mathbb{R}^{n+k}$ for large enough k, over B, this has a normal bundle, let's call it $-\tau^i$.

Now over each point of the base we have a map $S^{n+k} \times \{b\} \to Th(-\tau^i)|_{E_b}$. Taking the maps together gives a parameterized $S^{n+k} \wedge B_+ \to Th(-\tau^i)$.

Let τ be the tangent bundle along the fibers with classifying map $E \to G(n,k)$, the fibers are n-dimensional. This has two structures on it, the maps $\tau \to \gamma_n$ and $-\tau \to \gamma_k^\perp$. So we get $Th(-\tau) \to Th(\gamma_n^\perp)$ so you get $S^{n+k} \wedge B_+ \to Th(-\tau) \to Th(\gamma_n^\perp$ by composition.

So $B \to \Omega^{n+k} Th(\gamma_n^\perp) \hookrightarrow \Omega^\infty MTSO(n)$. You can always associate to a spectrum an infinite loop space, by colimit of spaces. So it's the colimit of $\Omega^{s+n} Th(\gamma_s^\perp)$.

This map I want to call α_B. These are in one to one correspondence with maps $\Sigma^\infty B_+ \to MTSO(n)$ by adjunction.

First, homotopy class of α does not depend on i. Second, $H^*(E) \cong H^{*+k} Th(-\tau)$ and we can use β to get from there to $H^{*+k}(S^{n+k} \wedge B_+) \cong H^{*-n} B$ which coincides with p. I won't prove this. I can use the Serre spectral sequence or take this as a definition [laughter].

We have an embedding from $-\tau \xrightarrow{j} E \times \mathbb{R}^{n+k}$ over E and we have an Euler class $e = e(\tau)$ and now, let $x \in H^*(E)$, then the following formula is true: I can take $p_!(e \cup x) = \sigma \beta^* j^* \sigma(x)$

All of these remarks in the construction, they are due to a paper in the [sixties? 74? Names given that I missed]

Now we consider the universal case. Fix an oriented manifold M^n and let $G = Diff_+(M)$, the orientation-preserving diffeomorphisms. Then we equip the space of embeddings of M into $\mathbb{R}^{n+k}$ with the Whitney C^∞ topology and define $Emb(M, \mathbb{R}^\infty)$ as the colimit of $Emb(M, \mathbb{R}^{n+k})$. I want particular embeddings, which are actually embeddings restricted to the unit disk bundle. You can show that the inclusion of such into all embeddings is a weak homotopy equivalence.

Now $Emb(M, \mathbb{R}^\infty)$ is weakly contractible. This follows from Whitney embedding theorems. Furthermore, it carries a free G-action via precomposing with diffeomorphisms. The projection is locally trivial so we get the model for the universal G-bundle

$$G \to Emb(M, \mathbb{R}^\infty) \to Emb(M, \mathbb{R}^\infty)/G$$

So this is a model for $G \to EG \to BG$.

[Do you need fat embeddings?]

[They are preserved by the diffeomorphism group.]

Let me sketch what this means. You want to have something like what follows. If $P \to B$ is a principal G-bundle, and P is weakly contractible, then we have a one to one correspondence between classes of maps from X to B and principal G-bundles over X. Let me sketch that this is onto. Given some bundle over X, I can form the associated bundle over X with fiber P and this admits a section. The fiber is weakly contractible. You can show that a section of this bundle corresponds to a G-equivariant map $Q \to P$, and you can take quotients, and get

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] & P \arrow[d] \\
Q/G \arrow[r] & P/G
\end{tikzcd}$$

Now we have an embedding from $Emb(M, \mathbb{R}^\infty) \times_G M \to BG \times \mathbb{R}^\infty$ over BG:

$$\begin{tikzcd}
Emb(M, \mathbb{R}^\infty) \times_G M \arrow[rr] \arrow[dr, "p"'] & & BG \times \mathbb{R}^\infty \arrow[dl] \\
& BG &
\end{tikzcd}$$

So we can take $I(\varphi, M) = ([\varphi], \varphi(M))$. Now we want to proceed, consider the normal bundle of M.

So let BG_k be the space $\{x \in BG | I(p^{-1}(x)) \subset BG \times \mathbb{R}^{n+k}\}$. Then define $EG_k = \pi^{-1}(BG_k)$. Now we can define the normal map $-\tau_k$ as $(EG_k \times_G TM)^\perp \hookrightarrow (EG_k \times_G M) \times \mathbb{R}^{n+k}$. Note the following. If we pull back $-\tau_{k+1}$ over $EG_{k+1} \times_G M$ this again splits as $-\tau_k$ and a trivial line bundle. Now we can parameterize TP. We get

$$S^{n+k}BG_{k+} \to Th(-\tau_k) \to Th(\gamma_k^\perp).$$

This gives us a map $BG_k \to \Omega^{n+k}Th(\gamma_k^\perp)$ which in the colimit gives $BG \to \Omega^\infty MTSO(n)$. For this construction it was not necessary for B and E to be manifolds. We started when B was a manifold, and then we did it on BG_k. The last step is to extend this all to BG.

Now let me compute the cohomology of this loop space with $\mathbb{Q}$ coefficients. This is done in three steps, to $H^*\Omega^\infty MTSO(n)$. First, we calculate

$$H^*BSO(n) = \begin{cases} \mathbb{Q}[p_1, \ldots, p_m] & , \quad n = 2m+1 \\ \mathbb{Q}[p_1\, ldots, p_m, e]/(e^2 - p_m) & , \quad n = 2m \end{cases}$$

Then we show the stable Thom isomorphism $H^*BSO(n) \cong H^{*-n}MTSO(n)$, and then we define a sectrum $B_m = \Sigma^m G(n, m-n)_+$, and we can show that $B \cong \Sigma_+^\infty BSO(n)$.

Now we consider from these spectra, the $n+k$ space and consider cohomology

$$H^*(B_{n+k}) = H^{*-n-k}G(n,k) \cong H^{*-n}Th(\gamma_k^\perp)$$

The isomorphisms on the spaces induces an isomorphism on the inverse limit in $\mathbb{Q}$ coefficients. In general it is not true that isomorphisms of all the spaces are isomorphisms of spectra.

The last step is to take the infinite loop space of $MTSO(n)$. Let's start with the identity, which gives us a spectrum map $\Sigma^\infty\Omega^\infty MTSO(n)$ to $MTSO(n)$, which induces a map in cohomology, w^*. The next statement is that this induces an isomorphism if we take the universal graded commutative algebra $U(H^{*>0}(MTSQ(n)))$ which maps via w^* to one connected component $\Omega_0^\infty MTSO(n)$. The right side is a Hopf algebra and you can ask, you can say it's generated by the primitive elements, but I'm not totally sure about the proof.

The last part will be about characteristic classes. I want to define the universal MMM-classes. Let $c \in H^*BSO(n)$ and e the Euler class of τ and f be the classifying map of τ. Then an MMM-class with respect to c is $p_!(e \cup f * c) \in H^*BG$. This is a class in BG, the natural place for characteristic classes of G-bundles.

This definition works for all manifolds M You can construct characteristic classes in the cohomology of BG. Here we define them for all diffeomorphism groups G and all manifolds M. Here I can take c_1^i and pull back along f^*, getting e^i and then get $p_!(e^{i+1}) = \kappa_i$.

I want to close with one diagram:

$$\begin{tikzcd}
\Sigma^\infty(EG \times_G M)_+ \arrow[r] & \Sigma^\infty BSO(n)+ \\
M(-\tau) \arrow[u, hook] \arrow[r] & MTSO(n) \arrow[u, hook, "j"'] \\
EG \times_G M \arrow[r] \arrow[d] & BSO(n) \\
BG \arrow[uu, bend left] &
\end{tikzcd}$$

Proposition 2 $MMM_c = \alpha^* \circ j^*(c)$

[Some discussion of the fat embeddings]