# West Coast Algebraic Topology Summer School 

Gabriel C. Drummond-Cole

August 14, 2010

## 1 Wrap-up

[Dev: The organizers will tell us various things. I wanted to thank them very deeply. They did a lot of work, preparing lectures, helping with the material, attending lectures, answering questions. I'm very grateful]

Great, so, after five days of all this stuff, I'm going to talk about a summary of what happened, what the key ingredients were. Oscar's figuring out unstable cohomology of moduli spaces, and connections to other things.

What happened? We started out by studying surfaces. The cobordism category was for general manifolds, but we started out studying surfaces using Teichmüller theory. We defined $\mathscr{M}_{g}$ in various ways, and realized it as $\mathscr{M}_{g}=\mathscr{T}_{g} / \Gamma_{g}$ or $\mathcal{J}_{g} /$ Diff $^{o r} S_{g, 0}$.

There was also a realization of $\mathscr{M}_{g}$, its rational homology, as $\operatorname{Emb}\left(S_{g, 0}, \mathbb{R}^{\infty}\right) \times_{\Gamma_{g, 0}} \mathcal{J}_{g}$, which is equivalent for standard reasons to $B \Gamma_{g}$. The second part is that the map BDiffor $S_{g, 0} \rightarrow$ $B \Gamma_{g}$ is a homotopy equivalence, this is also from Teichmüller theory. We saw the same equivalence happening with boundary: $B D i f f^{o r}\left(S_{g}, r\right) \rightarrow B \Gamma_{g, r}$. Then we wanted to say that these various homologies were independent of the coefficients.


We have maps among these which induce isomorphisms on homology in a range. So one thing that would be nice would be to construct a space that has the homotopy type of this stably. So this was

$$
B \Gamma_{g_{0}, 1}{\xlongequal{H_{g \gg}}}^{\operatorname{hocolim}( }\left(\rightarrow B \Gamma_{g, 1} \rightarrow B \Gamma_{g+1,1} \rightarrow \cdots\right)
$$

Then we had the four-author paper, which defined the cobordism category $\mathscr{C}_{d}$, and we saw that $B \mathscr{C}_{d} \cong\left|D_{d}\right|$, which was formal, using sheaves, and then compared this to the spectrum $\Omega^{\infty-1} M T O(d)$, and here's where we used Pontrjagin-Thom theory, and the $h$-principle for submersions.

I guess I'll come back to surfaces, we showed there was the positive boundary subcategory $B \mathscr{C}_{d, \delta} \rightarrow B \mathscr{C}_{d}$ which is an equivalence in $d \geq 2$. So group completion comes in here for the $d=2$ oriented version, I don't know exactly what said, but there was a particular functor,


This is a homology fibration, and so we get that $\mathbb{Z} \times B \Gamma_{\infty, 1}$ is homologically equivalent to $\Omega B \mathbb{C}_{2, \delta}^{o r}$. After a calculation of the right hand side, this gives the Mumford conjecture. So hopefully, this at least gives an idea of what all these equivalences are for the Mumford conjecture.

Let us move to part two.
So I'm going to talk about the purpose of this week, to talk about the Mumford conjecture, ttrying to prove something about the stable cohomology of $\mathscr{M}_{g}$. What happens outside the stable range? There was this nice answer to the stable situation, but for the situation unstably, fixing the genus, the situation is rather complicated.

1. So, let me remind you of what we know about this cohomology first. We have the Mumford-Morita-Miller classes, the $\kappa_{i}$, which we defined whenever you have a surface bundle. So $\Sigma_{g} \rightarrow E \rightarrow B$ is a surface bundle, then we defined $K_{i} \in H^{2 i}(B, \mathbb{Z})$, defined to be $\pi_{!} e\left(T^{v}\right)^{i+1}$. Let me remind you, $T^{v}$, the vertical tangent bundle, are the tangent vectors of $E$ which are in the kernel of $D_{\pi}$. Take the Euler class of that, take the $i+1$ power, that has degree $2 i+2$, and when you fiber integrate, you get $2 i$. In particular, you have $\kappa_{i} \in H^{2}\left(\mathscr{M}_{g}, \mathbb{Q}\right)$, which, we have to be careful because there's no surface bundle, but rationally we have an equivalence to $B \Gamma_{g}$ which does have a surface bundle. Let's consider the subring of the cohomology generated by these classes. This is $R$, the image of the map that sends $\mathbb{Q}\left[\kappa_{1}, \kappa_{2}, \cdots\right] \rightarrow H^{*}\left(\mathscr{M}_{g}, \mathbb{Q}\right)$ People call this the tautological ring. Stably, this is all you have. There are no relations among these $\kappa$-classes. There is no genus where this is true but it becomes more and more true as the genus gets bigger.
2. There is a conjecture about what the structure of this ring is. It's surprising, but he calculated it in a great many cases. This is something that's true unstably. Before stating the conjecture, let me recall, we took $\mathscr{T}_{g}$, which turns out to be a vector space, and quotient out by the mapping class group, which has the same rational homology as $B \Gamma$ but isn't a free action, so is not homotopy equivalent. In particular, though, it's dimension $6 g-6$ so it has no comology above this degree.

Something that is surprising is

Theorem 1 (Looijenga)
$R^{j}\left(\mathscr{M}_{g}\right)=0$ for $j>2(g-2)$, and $\mathbb{R}^{2(g-2)}$ is at most one dimensional.

The first thing he showed was that this vanishes above a far lower dimension than where you a priori expect to see, the geometric dimension of the space, and also that in the top place it's at most one-dimensional. If it was one-dimensional, it starts to look like the cohomology of a manifold, has one in top degree and so on.

So then, work of various people: Witten, Kontsevich, Faber, [unintelligible], implies that it is in fact one dimensional, so that $K_{g-2}\left(\mathscr{M}_{g}\right) \neq 0$. Witten stated conjectures, Kontsevich proved them, and [unintelligible]translated this from $\tau$-classes to $\kappa$-classes. These moduli spaces have compactifications which are the topic of the Witten conjecture, and this is just one tiny piece of a very complicated story, this is just one thing that you get out of it.

Based on this, and on computer calculations over a period of many years. He made a conjecture about what the tautological ring looks like:

Conjecture $1 \mathbb{R}^{*}\left(\mathscr{M}_{g}\right)$ is (rationally) a Poincaré duality algebra of dimension $2(g-2)$. The group being a rational vector space of dimension 1 is a fundamental class. [For $B \Gamma_{g}$, this can be stated rationally but it's wrong, the higher $\kappa$-classes give torsion.]

Further, $\kappa_{1}, \ldots, \kappa_{\left\lfloor\frac{g}{3}\right\rfloor}$ generate with no relations on dimension les than $2\left\lfloor\frac{2 g}{3}\right\rfloor$. We proved this part in this course with stability. The first part is also known to be true, but it is far more difficult, due to Morita.

The third part of the conjecture is certain explicit proportionalities in dimension $2(g-2)$. That also has been proved, and follows from the Witten conjectures as well.

The first part has not been proven. The ring doesn't have the right to be a Poincaré duality algebra. There should be a $g-1$ dimensional subvariety of $\mathscr{M}_{g}$ that carries the tautological ring.

Theorem 2 (Faber)
This holds for $g \leq 23$
This was computational. There's an algebro-geometric process that tells you when certain products are zero. You may not have this in dimension 24, they haven't found this yet.

Let me write down some examples, here's genus 9:
$\kappa_{1}^{7}=26011238400 \kappa_{7}, 7 \kappa_{1}^{5} \kappa_{2}=6195753664 \kappa_{7}$, and so on, you see the pattern [laughter]
You can say $\kappa_{1}^{g-2}$ and $\kappa_{g-2}$ are related by

$$
\kappa_{1}^{g-2}=\frac{1}{g-1} 2^{2 g-5}((g-2)!)^{2} \kappa_{g-2}
$$

This was a kind of silly example. Let me do $g=3$. So $\kappa_{1} \neq 0$ and $\kappa_{1}^{k}=0$. So you have a $\mathbb{Q}$ in dimensions 0 and 2.

For $g=4$ you get $\kappa_{1} \neq 0$ and $\kappa_{1}^{2}=\frac{32}{3} \kappa_{2}$.
There's another conjecture, that $R^{*}\left(\mathscr{M}_{g}\right)$ "is like" the cohomology of a nonsingular projective variety of dimension $(g-2)$. It has the structure and properties that these things should satisfy. This is also true in the same range or at least up to his original calculation of 15 . The bigger the genus gets there's kind of more that we don't know about because we only know about the bottom third, and then by duality we know about the top third, but we don't know much about the middle third.
[Why is it natural to do this?]
Every geometrically meaningful class should be in this ring, that's the slogan. Every geometrically meaningful concept should respect this subring. It's the smallest subring containing something like, that is closed under pullbacks and pushforwards and contains 1.
[What about the torsion?]
This $\mathscr{M}_{g}$ is the coarse moduli space, so we don't know anything about the integral homology.
The cohomological degree is roughly $4 g$. He can show a subcomplex that contracts, which is $4 g$-dimensional. Another thing which is shocking, the orbifold Euler number of the mapping class group, not quite the Euler number of $\mathscr{M}_{g}$, it is the values of the Riemann $\zeta$ function at negative integers. That's the first part of the paper, and then in the last part of the paper, they used that to calculate the actual Euler number. The formulas are really long, you get numbers with twenty or forty digits. They grow exponentially, but the Euler number is $(-1)^{g}$ times some fast growing function of $g$. So there has to be a tremendous amount of cohomology in odd degrees for the negative ones. It grows exponentially.
[There's a stage of the proof in the last week where you pass to sheaves and sheaves of categories. It's not clear to me what's happening there? Can you tell me what lesson to take there?]

So, I don't think sheaves are, the lesson should not be that proving the Mumford conjecture is about sheaves. It's like simplicial sets, like topological spaces but a little more convenient. We wrote the paper that way because that's what Madsen and Weiss did, but I could imagine that you could do it with simplicial sets, or with spaces.

The category, maybe doesn't have things you can understand because there are isomorphism classes of manifolds in $\pi_{0}$, but when you take the classifying space it becomes something you can understand. You take some complicated spaces and glue them together in a complicated way and get something less complicated. It's just as much about taking $B$ of a category, it's not really about sheaves.
[I'm trying to remember if anyone mentioned what $B \mathbb{C}_{d}$ classifies?]
Everyone knows that $B G$ classifies principal $G$-bundles. So maybe that's one lesson, that
this $\beta$ construction gives you an answer of what $B$ of a category classifies.

