# West Coast Algebraic Topology Summer School

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August 13, 2010

## 1 Sheaves on Categories

The last lecture we saw an equivalence between $\Omega^{\infty-1}MTO(d) \cong |D_d|$. Today, we're going to complete the proof from $|D_d|$ to $B\mathscr{C}_d$ I'll start with generalities about categories of sheaves, sheaves of categories, and then we'll approach the proof.

So again, $\mathscr{X}$ is the category of manifolds without boundary with smooth maps.

**Definition 1** *A sheaf of categories is a functor $\mathscr{X}^{op} \to Cat$, satisfying the sheaf condition. That is, we can look at ob $F(U)$ and mor $F(U)$, and we require them to be sheaves.*

For each sheaf of categories we will get a topological space in the following way

**Definition 2** *The realization of $F$ is a topological category $|F|$, where $ob|F| = |obF|$, the geometric realization, $[q] \mapsto obF(\Delta^q_e)$. Morphisms are the same: $mor|F| = |morF|, \cdots$*

Now we have a topological category, we can take its classifying space, $B|F| = |N.|F||$. The first time we use a simplicial set to evaluate a sheaf, and the second time we have a simplicial space given by the nerve of a topological category.

Now we have a trick, where we can do these in the opposite order. Given $F \in Sh(Cat)$, there is a sheaf of sets $\beta F$, so that $|\beta F| \cong B|F|$. This is in the Madsen-Weiss annals paper. I won't prove the equivalence but I'll describe $\beta F$ in some detail. Fix a big enough set $J$. An element of $\beta F(X)$ is $(U, \Phi)$, where $U = \{U_j\}$ is a locally finite open cover of $X$. Over each piece of $U$, we will have morphisms, for $U_S = \bigcap_{j\in S} U_j$, $\Phi = \{\varphi_{RS} \in morF(U_S) | R \subset S \subset J\}$ satisfying conditions.

1. $\varphi_{RR} = id_{C_R}$ for some $C_R$ in the objects of $F(U_R)$.
2. $\varphi_{RS}$ should be from $C_S \to C_R$, which is restricted to $U_S$.
3. a cocycle condition, for a triple $R \subset S \subset T$, we should have $\varphi_{RT} = varphi_{RS}|_{U_T} \circ \varphi_{ST}$

Here is an example. Let $F = Map(—, \mathscr{C})$, for $\mathscr{C}$ a topological category. Then the objects of $X_U$ are $\amalg_{R\subset J} U_R$ and the morphisms are $\amalg_{R\subset S} U_S$, the poset of $(R, X)$ for $X \in U_R$ an element $(U\Phi)$ in $\beta F(X)$ gives $\Phi : X_U \to \mathscr{C}$.

Now we're going to construct a map $X \to BX_U \to B\mathscr{C}$, with a partition of unity.

Look at the case where $\mathscr{C} = G$, then this data is exactly a principal $G$-bundle over $X$. The cocycle condition tells you how to patch together pieces.

So we've constructed $\beta F(X) \to Map(X, BG)$. Recall we have a notion of concordance, so when we go to concordance classes and homotopy classes of maps, we get $\beta F[X] \to [X, BG]$ which becomes $[X, |\beta F]$ and $[X, B|F|]$. So the equivalence of these two spaces implies exactly the classification of vector bundles.

Maybe I'll write the steps on the side of the board and leave them up for good.

$$\Omega^{\infty-1}MTO(d) \overset{\checkmark}{\cong} |D_d| \overset{A}{\leftarrow} |\beta D_d^{\pitchfork}| \overset{\checkmark}{\cong} B|D_d^{\pitchfork}| \overset{D}{\to} B|C_d^{\pitchfork}| \overset{C}{\leftarrow} B|C_d| \overset{B}{\cong} B\mathscr{C}_d.$$

We'll go alphabetically, proving things along the way.

The main tool will be the relative surjectivity condition (RSC). So $F_1 \to F_2$ of $Sh(Set)$ induces $|F_1| \overset{\cong}{\to} |F_2|$ if for all closed $A \subset X$ and all germs near $A$, that is, elements $S$ of $\mathrm{colim}_{U\supset A} F(U)$, the map

$$F_1[X, A, S] \to F_2[X, A, \tau(S)]$$

is surjective.

**Definition 3** *Let $D_d(X)$ be the set of sets $W \subset X \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}$ equipped with $\pi : W \to X$ is a submersion with d-dimensional fibers and $(\pi, f) : W \to X \times \mathbb{R}$ is proper.*

*Now let $D_d^{\pitchfork}(X)$ be pairs $(W, a)$ where $W \in D_d(X)$ and $a : X \to \mathbb{R}$ so that $f : W \to \mathbb{R}$ is fiberwise transverse to $a$. So $a(X)$ is a regular value for $f_X : W_X = \pi^{-1}X \to \mathbb{R}$.*

*The morphisms are a poset $(W, a) \leq (W', a')$ when $W = W'$ and $a \leq a'$. As a poset it becomes a category with a single map between comparable objects from the lesser to the greater.*

This defines our first couple of sheaves of sets and categories, so now let's show the equivalence labelled $A$ above.

Let's first think about what this map is, $|\beta D_d^{\pitchfork}| \to |D_d|$. An element $(U, \Phi) \in \beta D_d^{\pitchfork}(X)$ is a cover $U$ of $X$ and $\Phi = \{\varphi_{RS} : (W_S, a_S) \to (W_R, a_R)\}$ where $W_S = W_R|_{U_S}$ and $a_S \leq a_R$. We get $(U, \Phi) \mapsto W = \cup W_j$.

Let's first do global surjectivity $\beta D_d^{\pitchfork}[X] \to D_d[X]$. Pick a $W \in D_d(X)$. We have $W \subset X \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}$. So $\pi$ will always project to $X$ and $f$ to $\mathbb{R}$. For each $x \in X$ we can find $a_X \in \mathbb{R}$ which is a regular value for $f_X : W_X = \pi^{-1}(X) \to \mathbb{R}$.

So there is an open neighborhood of $x$ so that $a_x$ is a regular value for $b_y : W_y \to \mathbb{R}$, for $y \in U_x$. We're picking an element of $D_d[X]$, and we want to lift it, we need to find an open cover and then find these objects over our open cover. So we can find a nice covering $U$ and numbers $a_j : U_j \to \mathbb{R}$ such that $b_j : W_j = \pi^{-1}(U_j) \to \mathbb{R}$ is fiberwise transverse to $a_j$. Thus we have constructed an object $(W_j, a_j) \in obD_d^{\pitchfork}(U_j)$. Let $a_R$ be the minimum of $\{a_j\}$ for $j \in R$. Then if $R \subset S$ it will follow that $a_S \leq a_R$.

In other words we have a morphisms $\varphi_{RS} : (W_S, a_S) \to (W_R, a_R)$ so $(U, \Phi)$ is a lift of $W$.

Now we have to do surjectivity for a closed subset of $A$. We're given an element on all of $X$ with a lift in a neighborhood of $A$. The hard part is figuring out how to align the elements of the neighborhood of $A$ with elements nearby, and there's some fiddling one needs to do near the boundary of $A$.

Now let's go to the other side and look at $B$. I'll have to define $C_d^{\pitchfork}(X, \epsilon : X \to (0,\infty)) = \{(W, a_0 \leq a_1 : X \to \mathbb{R}) | W \subset X \times (a_0 - \epsilon, a_1 + \epsilon) \times \mathbb{R}^{d-1+\infty}\}$ $W$ is given by $(x,t)$ so that $|a_0 - \epsilon)(x) < t < (a_1 + \epsilon)(x)$, so that $\pi : W \to X$ is a submersion with $d$-dimensional fibers, $(\pi, f)$ is proper, and $(\pi, f) : K_i = W|_{X \times a_i - \epsilon, a_i + \epsilon)} \to X \times (a_i - \epsilon, a_i + \epsilon)$ is proper. Then we want to let $\epsilon$ not be part of the data so we let $morC_d^{\pitchfork}(X)$ to be the colimit as $\epsilon \to 0$ of $morC_d^{\pitchfork}(X, \epsilon)$.

Let's do $B$ next, the classifying space is the realization of the nerve of $C_d$, that is, the simplicial space $N_\cdot|C_d|$, which itself is the realization of the simplicial set which at level $[q]$ is $N_k C_d(\Delta_e^q)$, and now we have to use the fact that this is a represented sheaf, so that this space is $C^\infty(\Delta_e^q, N_k \mathscr{C}_d)$. We didn't define a manifold structure on either of these spaces. The intuitive idea was discussed yesterday. I want to say that this is equivalent to $Map(\Delta^q, N_k \mathscr{C}_d)$, which is $S_q N_k \mathscr{C}_d$, and so this is equivalent, under realization, to $N_k \mathscr{C}_d$. So when you take classifying spaces you get equivalences, since you have equivalences beforehand. That does B.

So the only difference between $C_d$ and $C_d^{\pitchfork}$ is that one of them has collars. We'll prove that at each level of the nerve we have a weak equivalence which will show that their realizations are equivalent. I'll use relative surjectivity and just draw a picture for $k = 1$. [picture]

We haven't yet defined the map $D_d^{\pitchfork} \to C_d^{\pitchfork}$. Recall that we have $W \subset X \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}, a : X \to \mathbb{R}) \in D_d^{\pitchfork}(X)$. Choose $\epsilon : X \to (0, \infty)$ such that $(\pi, f) : W_\epsilon = (pi, f)^{-1}(X \times (a - \epsilon, a + \epsilon)) \to X \times (a - \epsilon, a + \epsilon))$ is a proper submersion. [Missed some.] We'll start instead with $|N_k D_d^{\pitchfork}| \to |N_k C_d^{\pitchfork}|$, and we'll show this is a weak equivalence with RSC. Then $W \in N_k C_d^{\pitchfork}(X)$. So pick $W, a_0 \leq \cdots \leq a_k)$ so that $W \subset X \times (a_0 - \epsilon, a_1 + \epsilon) \times \mathbb{R}^{d-1+\infty}$. Now choose a diffeomorphism which is the identity near the collars to $X \times \mathbb{R}$.

So that shows $D$ and I guess I'm done.

# 2 Group completion

[Before the next talk, I want to say a little bit about what's going on tomorrow. It would be nice to talk about something that you're interested in. Tell one of the organizers if you have an idea about what would be interesting to hear.]

Here's my outline: we're going to cover a lot of background.

I. Toplogical categories acting on spaces
II. Group comp. theorem
III. Applying GCT to prove the generalized Mumford conjecture

I'm always going to let $\mathcal{E}$ denote "spaces" which can be spaces or simplicial sets. We'll start with a definition. Let $h_*$ be a homology or homotopy theory on $\mathcal{E}$. An $h_*$-equivalence is a map that induces isomorphisms on the homology. The examples we should keep in mind are $h_*$ as integral homology or homotopy groups. This works for $h_*$ a generalized homology theory that commutes with filtered colimits and takes inclusions which are $h_*$-equivalences to $h_*$-equivalences [?]

So $\mathscr{C}$ is going to be a *category* in spaces $\mathcal{E}$ (also called a topological category). Recall that $\mathscr{C}$ is determined by what I'm just going to call domain and codomain maps that go from the space of morphisms $mor(\mathscr{C})$ (remember this is a space) to the space of objects $ob(\mathscr{C})$

$$\begin{tikzcd} mor(\mathscr{C}) \arrow[r, shift left, "d_0=\text{domain}"] \arrow[r, shift right, "d_1=\text{codomain}"'] & ob(\mathscr{C}) \end{tikzcd}$$

Let me define $X$ as a *$\mathscr{C}$-diagram.* $X$ is determined by maps

$$X \xrightarrow{\pi=\text{proj}} ob(\mathscr{C})$$

$$mor(\mathscr{C}) \times_{ob\ C} X \xrightarrow{a=\text{action}} X$$

So these satisfy standard relations you can find in [unintelligible]-Moerdijk.

From a $\mathscr{C}$-diagram $X$ we form a new topological category $(X \wr \mathscr{C})$ the *category of elements*. This is a topological category, and it has object space $X$ and the morphism space is $mor(\mathscr{C}) \times_{ob\ C} X$, the pullback over $d_0$.

The domain map of $X \wr C$ is

$$mor\ \mathscr{C} \times_{ob\mathscr{C}} X \quad \overset{\pi=\text{proj}}{X}$$

and the codomain is

$$mor\ \mathscr{C} \times_{ob\mathscr{C}} X \quad \overset{\alpha=\text{action}}{X}$$

We'll make an observation that $X \to ob\ \mathscr{C}$ induces a functor of topological categories $(X \wr \mathscr{C}) \to \mathscr{C}$. We can take the nerve of this map, the projection map, and we get a map on spaces $\mathcal{N}(X \wr \mathscr{C}) \to \mathcal{N}(\mathscr{C})$ in $\mathcal{E}^{\Delta^{op}}$, and now we can work on the group completion theorem.

I'm going to state the theorem, then [unintelligible]

**Theorem 1** *(The group completion theorem)*
*Let $M$ be a topological monoid. The canonical map of $M$-spaces*

$$M \to \Omega BM$$

*induces an isomorphism $H_*(M)[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*\Omega(BM)$ if $\pi_0(M)$ is in the center of $H_*(M)$*

Before I try to prove it, I'm going to make a remark from algebra. If you take $R$ a ring and $S$ a countable multaplicative subset, and $A$ a right $R$-module, then we can construct a universal $R$-module $A[S^{-1}]$ where all of the elements of $S$ act by the invertibly. This is the colimit of the sequence $A \xrightarrow{p_{s_0}} A \xrightarrow{p_{s_1}} \cdots$ where each $s$ appears infinitely often. If $R$ is commutative, then you don't need countable.

Sketches for the proof, starting with a few notes:

1. $H_*(M)[\pi_0(M)^{-1}]$ and $H_*(\Omega BM)$ both commute with filtered colimits, and if $\pi_0$ is in the center of $H_*(M)$, then we can write $M$ as a union of submonoids, [unintelligible]you can enumerate elements of the components, [some handwaving], assume $\pi_0(M)$ is countable.

Pick an $m \in M$ on each component. Let $\{m_i | i \in M\}$ is a sequence where each vertex $m_i$ appears countable many times. I'm going to denote by $\rho_{m_i} : M \to M$ the multiplication by $m_i$ on the right.

Now let $\overline{M}$ be the homotopy colimit of the sequence $M \xrightarrow{\rho_{m_0}} M \xrightarrow{\rho_{m_1}} \cdots$, and $M$ acts on this homotopy colimit from the left. Then $\bar{M} \wr M$ is the category of elements associated to the $M$-action.

This is a homotopy colimit of many copies of $M \wr M$. Also, $\mathcal{N}(M \wr M)$ is contractible, and together, this means $\mathcal{N}(\overline{M} \wr M)$ is contractible.

The homotopy fiber of $|\mathcal{N}(\overline{M} \wr M)| \to |N(M)| = BM$ is $\Omega BM$ is to show that the homology $H_*(\overline{M})$ is $H_*(M)[\pi_0(M)^{-1}]$, and we want to show that this has the same homology as the homotopy fiber $H_*(hofiber|\mathcal{N}(\overline{M} \wr M)| \to |N(M)|)$ by Quillen theorem B.

Let's use $\mathscr{C}_{d,\delta}$, the positive boundary category. The objects in this topological category is the same as the objects of $\mathscr{C}_d$, but we're taking the space of morphisms to be a subset of $mor(\mathscr{C}_d)$ where we onl/y take the disjoint union over $W$ where each nonempty component has nonempty outgoing boundary, so that $\pi_0(M_1)$ surjects on $\pi_0 W$. The hard theorem for the next talk

**Theorem 2** *(Galatius-Madsen-Tillman-Weiss)*
*For $d \geq 2$, the inclusion $B\mathscr{C}_{d,\delta} \to B\mathscr{C}_d$ is a weak equivalence.*

The proof is the next talk. Using this, we consider a topological category $\mathscr{C} \subset \mathscr{C}_{d,\delta}$, a special one, just so you know, which has objects pairs $(M, a)$ with $a < 0$ and for which we have a stability theorem.

Okay, we have a diagram $X : \mathscr{C} \to \mathcal{E}$, so if you look in the paper, you'll see something ($\mathscr{C}^{op} \to \mathcal{E}$), and you construct a diagram in $\mathscr{C}$, right? And, I think I might have time to explain roughly how you do it. It's in the following way:

1. Fix $S^1 \subset \mathbb{R}^{2-1+\infty}$. Choose $b_i = \{i\} \times S^1$ in $C_{d,\delta}$. Then choose $\beta_i \subset [i, i+1] \times \mathbb{R}^{2-1+\infty}$. These are morphisms from $b_i \to b_{i+1}$. The $\beta_i$ are connected surfaces with $g = 1$. I'll just say all the orientations are compatible. So we're going to make a sequence of objects, it'll be a homotopy colimit of these little diagrams. Each $X_i(c)$ is $\mathscr{C}_{d,\delta}(c, b_i)$. Then I construct $X$ to be the hocolimit of the maps

$$X_0(C) \overset{\circ\beta_0}{\to} X_1(C) \overset{\circ\beta_1}{\to} \cdots$$

So due to our general setup, skipping quite a bit, we have $X(C) \cong \mathbb{Z} \times B\Gamma_{\infty,n+1}$ Then we can apply group completion, or a similar idea, to imply the theorem

**Theorem 3** *(Madsen-Weiss, generalized Mumford conjecture)*

$$\alpha : \mathbb{Z} \times B\Gamma_{\infty,n+1} \to \Omega^\infty MT(2)^+$$

*is a homotopy equivalence.*

Let's just say $X_i(C) \cong \amalg_{g\geq 0} BDiff(W_{g,n+1}, \delta) \times_{Diff} Bun^\delta(—)$, that will be enough of a description for now. Fundamentally, $n$ depends on $c$.

Somehow my attempts to cut out everything possibel still didn't work. I guess we can talk about some of the middle stuff later.

# 3 The positive boundary category

So this talk was billed as a proof of the fact that the classifying space $B\mathscr{C}^+_{\delta,d} \to B\mathscr{C}^+_\delta$ is a weak equivalence. I'll prove this indirectly. Recall that the main theorem of Galatius-Madsen-Tillman-Weiss is that $B\mathscr{C}^+_\delta \to \Omega^{\infty-1}MTSO(2)$ is a weak equivalence.

So I'll start by showing that we have $BC^+_{d,\delta} \to \Omega^{\infty-1}MTSO(2)$, and then that gives the weak equivalence. So let me remind you about the positive boundary category. The objects are basically manifolds, and the morphisms are basically cobordisms, of a special kind. [Pictures]. Every component intersects the outgoing boundary. So how am I going to prove this second equivalence? We can take the proof from earlier today, and modify everything in obvious ways, and everything goes through, except we have to show $|\beta D^\pitchfork_d| \to |D_d|$ is a weak equivalence. If you do the program that I just described, you need a positive boundary version of that sheaf $|\beta D^\pitchfork_{d,\delta}|$. This is the non-easy part, and this is what I want to talk about today.

Let me define the three sheaves. For $X$ a smooth manifold, let $D_d(X)$ be the set of submanifolds $W$ in $X \times \mathbb{R}^{d-1} + \infty \times \mathbb{R}$ with projections $\pi$ and $f$ to $X$ and $\mathbb{R}^1$, which satisfy three conditions: that $\pi$ is a submersion with $d$-dimensional fibers, that $(\pi, f) : W \to X \times \mathbb{R}^1$ is proper, and that for all compact sets $K$, $\pi^{-1}(K) \subset X \times \mathbb{R}^{d-1+n} \times \mathbb{R}^1 \subset X \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^1$ for some $n$.

$$\begin{tikzcd}
 & W \subset X \times \mathbb{R}^{d-1+\infty} \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\
X & & \mathbb{R}^1
\end{tikzcd}$$

Let's start with the example where $d = 1$ and $X$ is a point. Then my space is $\mathbb{R}^1 \times \mathbb{R}^\infty$. This is a one-manifold that extends to $\infty$. If $X$ is not a point, the fiber looks like this. If we have a path between two points, you can vary the fibers smoothly. To extend the path, you may move compact components off into $\infty$.

**Definition 4** *Let me define $\beta D_d$. This is a functor from [unintelligible], but I'm just going to describe the set-valued sheaf. Let $J$ be a fixed uncountable set, and $X$ a smooth manifold. Let $BD_d^{\pitchfork}(X)$ be a set of triples $(W, \{U_j\}, \{a_R\})$, where $W \in D_d(X)$, for all $j \in J$, $U_j \subset X$ is open, and for all finite $R \subset J$, $a_R$ is a smooth function from $\bigcap_{j \in R} U_j \to \mathbb{R}$. This should satisfy the conditions that $\{U_j\}$ is a locally finite open cover of $X$, that if $x \in U_R$ then $a_R(x)$ is a regular value for $f|_{\pi^{-1}(x)}$ ("$f$ is fiberwise transverse to $a_R$), and for finite $R \subset S \subset J$, $a_S \le a_R|_{U_s}$ and this inequality is strict except on an open set.*

What does this look like? [Picture]

**Definition 5** *$\beta D^{\pitchfork}_{d,\delta}(X)$ is a subset, the "positive boundary version," which is the subset of triplets so that $\pi^{-1}(x) \cap f^{-1}(a_R(x)) \hookrightarrow \pi^{-1}(x) \cap f^{-1}[a_S(x), a_R(x)]$ induces a surjection in $\pi_0$. That is saying that the cobordism from $a_S$ to $a_R$ intersects the positive boundary in each component.*

There's an obvious inclusion and forgetful maps $\beta D^{\pitchfork}_{d,\delta} \to \beta D^{\pitchfork}_d \to D_d$ The forgetful map takes $(W, \{U_j\}, \{a_R\})$ to $W$. The composition of these two is $\alpha$.

**Theorem 4** *For $d \ge 2$, $\alpha$ is a weak equivalence.*

The forgetful map is a weak equivalence, so this will imply the weak equivalence we want and hence the main theorem. As a remark, this can be modified to handle tangential structures.

It suffices to sho that $BD^{\pitchfork}_{d,\delta}[X, A, s] \xrightarrow{\alpha} D_d[X, A, \alpha(s)]$ is surjective, where $X$ is a manifold, $A$ is closed, and $s$ is a germ over $A$. Assume $A = \emptyset$; the other case is "similar." We want to show that up to concordance $W \in D_d(X)$ can be pulled back.

We want $W' \in D_d(X \times \mathbb{R})$ such that $W'|_{X \times 0} = W$ and $\{U_j\}, \{a_R\}$ such that $(W'|_{X \times I}, \{U_j\}, \{a_R) \in \beta D^{\perp}_{d,\delta}(X)$.

First I'll choose promising choices of $U_j$ and $a_R$, and then choose my concordance, and finally show that the surgeries in each fiber fit together.

For $x \in X$, choose a regular value $a_x \in \mathbb{R}$ of $f|_{\pi^{-1}(x)}$. I claim that $f$ is fiberwise transverse to $a_x$ over a neighborhood $U_x$ containing $x$. I can extract a finite subcover $E_j \subset \{U_x\}$. Since each one had a regular value $a_x$, we have a regular value $a_j$ for each $E_j$. Let's define $E_{jk} = E_j \cap E_k$.

For each finite $R \subset J$, define $a_R = min_{j \in R} a_j$. So I have a triple, $W$, $\{E_j\}, X$. This doesn't quite lie in the positive boundary component. Now I'll introduce my first attempt at surgery. A slight modification will make it work and I'll get to that at the end.

So our two components, take a little disk, and draw a move where we pull that litle disk up. [Picture]

I desire a conccordance $W'$ in $D_d(X \times \mathbb{R})$ such that the original inadmissible configuration is the fiber over $(x \times 0)$ and the fixed version satisfies the positive boundary condition. But we're missing coherence.

First let's make sure that we can do the surgery in a little neighborhood. I'll claim that $\pi^{-1}(E_{jk}) \cap f^{-1}[a_j, a_k]$ is a fiber bundle over $E_{jk}$. We can demand that $E_{jk} \cong *$ or 0. So the fiber bundle is trivial, and

$$\begin{tikzcd}
\pi^{-1}(E_{jk}) \cap f^{-1}[a_j, a_k] \arrow[dr] & & E_{jk} \times M \arrow[dl] \\
& E_{jk} &
\end{tikzcd}$$

I want to specify surgery sites, that is, embedded disks $D^d \hookrightarrow M$ in each offending component. I want to cut out $D^d$, cut them out, and then glue in some prescribed manifold.

$$\begin{tikzcd}
\square \arrow[r] \arrow[d] & \pi^{-1}E_{jk} - \amalg D^2 \arrow[d] \\
E_{jk \times \mathbb{R}} \arrow[r] & E_{j,r,p}
\end{tikzcd}$$

We'll need bump functions $\lambda_j : E_j \to [0, 1]$ and so that the support of $\lambda_j$ is in $E_j$. Let $\tilde{E}_j$ denote the interior of $\lambda_j^{-1}(1)$. We demand that $U\tilde{E}_j = X$, that $\tilde{E}_{jk} = \tilde{E}_j \cap \tilde{E}_k$ and $\lambda$.

[Picture]

If you've done this rigorously, we hope that $(W'|_{X \times 1}, \{\tilde{E}_j\}, \{a_R\}\}$. There is one catch. [picture]

Now let me revise the surgery so that it does work and we'll be done. Okay, ready? [Picture.]

Our new surgery involves pulling down, not up, going out through infinity, and continuing to pull down to get to negative infinity again from the top, and then bringing up the two different components with the pulled down part attached, and then smooth everything out.