West Coast Algebraic Topology Summer School

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August 13, 2010

1 Introduction to homological stability III

[Please turn in forms before you leave, if yossible. Receipts can be emailed. You can give them to Dev or bring them to Susan Campbell Hall, south across 13th from here, next to the museum on the east side.]

[There will be a wiki to coordinate departures. There is a shuttle from campus to PDX.]

[Exercises etc. this morning in Deady 209, 210, etc., and here.]

I'm going to go around several messy bits but I'll give you the idea. Yesterday we saw a fairly complicated sequence of arguments building off the ordered arc complex. The goal today is to analyze this complex and see that it is $g-2$-connected. Stop me if you have questions. It's very technical, I'll draw as many pictures as I can.

A reminder. If X is a simplicial complex and σ is a simplex, then $STAR(\sigma)$ is the subcomplex consisting of all simplices containing σ and their faces. The link of σ is the subcomplex of the star of σ that doesn't intersect σ. I need the simplicial approximation theorem:

Theorem 1 *If K and X are finite simplicial complexes with $L \subset K$ and $f : |K| \to |X|$, if $f|_L$ is a simplicial map $L \to X$ then there is a relative subdivision (K', L) and a simplicial approximation $g : K' \to X$ so that $g|_L = f|_L$ and g is homotopic to f relative to L*

The two cases I care about are $(S^k, \emptyset)$ and (D^{k+1}, S^k). Lastly I want to say -1-connected means nonempty, and -2-connected, et cetera, are vacuous.

Recall that our setting, we have a surface $S_{g,r}$ which I'll call S, of genus g with r boundary components. By an arc I mean an oriented embedding of $I \to S$ and we have these things, $O(S, b_0, b_1)$, the ordered arc complex, which is the simplicial complex whose vertices are given by isotopy classes of nonseparating arcs in S with boundary $-\{b_0\}, \{b_1\}$, with b_0 and b_1 in boundary components.

The p-simplices are collections of $p+1$ distinct isotopy classes of collectively nonseparating arcs representable by pairwise disjoint arcs (except at b_0 and b_1) so that the ordering induced by the orientation at b_0 in the counterclockwise direction is the same as that at b_1 in the clockwise direction.

That's the ordered arc complex. What we want to know is that its connectivity has a particular bound. But this thing is complicated. So we're going to look at some simpler complexes and sort of work our way toward this thing. We're going to define four total complexes, this is one of them.

So fix Δ, a finite nonempty collection of points in the boundary of S, and I'll say an arc a in S with boundary in Δ is trivial if it separates S into two components, one of which only intersects Δ in a.

Now I'll define $A(\Delta, S)$, the full arc complex, to be the simplicial complex whose vertices are isotopy classes of nontrivial arcs in S with boundary in Δ. The p-simplices are collections of $p+1$ non-trivial isotopy classes of arcs with disjoint interior. This is so big it has to be contractible.

Let Δ_0 and Δ_1 be finite disjoint non-empty subsets of δS, and let $B(S, \Delta_0, \Delta_1)$ be the subcomplex of $A(\Delta_0 \cup \Delta_1, S)$ so that if $\delta a = -\{b_0\} \cup \{b_1\}$, then b_0 is in Δ_0 and b_1 is in Δ_1.

Then there's another one, $B_0(\Delta_0, \Delta_1, S)$, which is the subcomplex of $B(\Delta_0, \Delta_1, S)$ of non-separating collections of arcs.

I want to line these up. I have

$$A(\Delta, S) \hookleftarrow B(\Delta_0, \Delta_1, S) \hookleftarrow B_0(\Delta_0, \Delta_1, S) \hookleftarrow O(S, b_0, b_1).$$

Now I'm going to state four theorems about the connectivity of these complexes. The two that I'm going to prove should demonstrate all the techniques.

Theorem 2 *$A(\Delta, S)$ is contractible unless S is a disk or an annulus with Δ in a single boundary component in which case $A(S, \Delta)$ is $(|\Delta| + 2r - 7)$-connected, where r is the number of boundary components*

Theorem 3 *$B(S, \Delta_0, \Delta)$ is $4g + r + r' + \ell + m - 6$-connected, where g is genus, r is the number of boundary components, r' are boundary components intersecting Δ_i, ℓ are pure edges, and m are impure edges. Note that Δ_0 and Δ_1 partition δS into edges and copies of S^1. Call an edge pure if the endpoints are in the same Δ_i and impure if they are in different Δ_i.*

I'm not going to prove that. The proof takes about four pages of Nathalie's paper, and I don't have that stamina.

Theorem 4 *$B_0(S, \Delta_0, \Delta_1)$ is $2g + r' - 3$-connected*

Theorem 5 *$O(S, b_0, b_1)$ is $g-2$-connected.*

So we start with this big complex, we collapse it, and use that data to give us information. I'll map a sphere in, push it over to the other complex, and then lift that information back. That the general process for this thing, outside of the first one, where I actually have to think, as opposed to the next ones, which you don't have to think.

Everybody okay?

I'll need this lemma before we get going

Lemma 1 *Suppose $A(S, \Delta) \neq \emptyset$ and Δ' is obtained by adding a point to Δ in a boundary component that already intersects Δ. So $\Delta' = \Delta \cup \{q\}$. Then if $A(S, \Delta)$ is d-connected, $A(S, \Delta')$ is $d+1$-connected.*

You can actually prove that this is a suspension up to homeomorphism.

The proof, here's my boundary component, with a point p and now I've added the point q. I know there are at least two points. I can draw points on the other sides of them, which may or may not coincide:

$$\neq p \text{ ------ } p \text{ ------ } q \text{ ------ } \neq q$$

I have two arcs here, one from $\neq p$ to q and one from $\neq q$ to p The arc from p to $\neq q$ was trivial before adding q, so it was not in $A(S, \Delta)$.

Let $X(I)$ be the subcomplex of $A(S, \Delta')$ consisting of all simplices not containing I. In particular, it lives in $A(S, \Delta)$ as long as q is not involved. We can decompose $A(S, \Delta')$ in the following way: it is

$$STAR(I) \cup_{LINK(I)} X(I)$$

We'll assume for now that I is nontrivial and figure out what happens if it's trivial in a minute.

So $STAR(I)$ is contractible, and the best possible case, what we want to show is that $X(I)$ deformation retracts onto $STAR(I')$, which will be contractible by the same argument. Then I'll apply Van Kampen and Mayer-Vietoris, which will show that we are 1-connected and that the homology $0 \to H_{*+1}(LINK(I)) \to H_*(A(S, \Delta'))$, and those are things that don't come out of q (which would make it intersect I) and I. Then I'm left with $A(\Delta, S)$. This is a pretty picture, and then I'll have to take the time to explain. So $STAR(I')$ are the complex with no arcs containing endpointns at p. Hore's what we're going to do. Suppose we have a k-simplex in $X(I)$ containing an arc ending at p. So I have I' between $\neq p$ and q, and suppose I have an arc coming in and ending at p. This is a p simplex, and I'll construct a $p+1$ simplex. I have vertices corresponding to these arcs at p, a_0 and a_1. I know there's no line from a_i to I'. So what I want is to replace a_i with a_i' which have a connection to I' and then connect the a_i to a_i' so that I can contract to I'. So I copy a_i and then follow along I' where they would intersect to attach to q. These can be connected to each other: a_0, a_1, a_1' in a simplex, then a_0, a_0', and a_1', and finally a_0', a_1', and I'.

So we can retract any simplex in $X(I)$ to one in $STAR(I')$. How do we do it all compatibly? We write it down, I'm not going to do that. So this tells me if I tack on a point, I've suspended my complex. Then I can prove this for one point in every boundary component and it will be good enough. So first you deal with the special case, look at disks, when are you going to have a non-empty complex? you need at least four points, in which case you count. If $|\Delta| = 4$, and $r = 1$, you get $4 + 2(1) - 7 = -1$, and this is nonempty. For an annulus, if I have a points only in a single boundary component, then the other boundary, well, you play around with this, and this is the form that falls out. I claim that if you discard these bad cases, that should fix the probelms from earlier.

Let me tell you about how this next bit is proved. Assume that you each boundary component contains at most one point of Δ (not in these cases). If I put more points in the boundary, I'll just raise the connectivity. You can define a retract of $A(S, \Delta)$ onto the star of a nontrivial arc. Call my nontrivial arc I, and I have an arc that is incompatible. I'll build a larger simplex using the same trick. Because there's only one point p, it turns out that one of the two things you can do is nontrivial.

That's the proof, that picture.

I have negative two minutes left. Now you stick spheres of appropriate dimensions into the complexes further down the line, pull them back to here, and fill them and push back out.

What you can do is show that you have enough homotopies to get connectivity up to a certain level. In the range that Nathalie proves, you can then show that this one sphere, you can replace the things that you don't think you have with some other chunk that is nicer.

[You have the map of this disk, and you can cook up a better way to improve your bad simplices, and you show that that process will end. There's a measure of badness, and you can produce a process that strictly decreases the badness.]

2 The cobordism category

So, thank you. It's a new subject, and my talk, I planned not to get too technical, so please relax. [Laughter] So I'll define the cobordism category C_d—can everyone hear me? And I'll extend the map $BDiff(W^d) \to \Omega^\infty MTO$, where W is a closed manifold, to a map $C_d \to \Omega^{\infty-1} MTO(d)$. Then the main theorem is that this will be a weak equivalence.

I'll start by defining the cobordism category. First I'll describe this set-theoretically. This consists of objects and morphisms. An object will be a pail (M, a) where a is a real number and $M \subset \mathbb{R}^\infty \times \{a\} \subset \mathbb{R}^\infty \times \mathbb{R}$. This is a $(d-1)$-dimensional compact manifold without boundary. The space of nonidentity morphisms C_d will be cobordisms (W, a, b) where a and b are reals, $a < b$. Then $W \subset \mathbb{R}^\infty \times [a, b] \subset \mathbb{R}^\infty \times \mathbb{R}$. W should be a compact d-dimensional manifold and I require it to have a collar. I need δW to be $W \cap \mathbb{R}^\infty \times \{a, b\}$ and also, let me call these parts of the boundary δ_{in} and δ_{out}, respectively. There should exist a nonzero ϵ so that $W \cap \mathbb{R}^\infty \times [a, a+\epsilon) = \delta_{in} W \times [a, a+\epsilon)$, and the same for the outgoing boundary.

I described this as a set, but I am interested in the homotopy type. One way to do this is to topologize this set. Let me start with some $(d-1)$-dimensional closed manifold. If I fix the diffeomorphism type of M, this will be topologized as the disjoint union

$$\amalg_M Emb(M, \mathbb{R}^\infty / Diff(M)) \times \mathbb{R}$$

where the disjoint union runs over diffeomorphism types of closed $(d-1)$ dimensional manifolds M. The space of embeddings has the C^∞ topology. The embeddings $Emb(M, R^\infty)$ sits inside $Map(M, R^\infty)$ and I give this the subspace topology. I can basically give any topology on $\mathbb{R}$. Let me give the single $\mathbb{R}$ factor the discrete topology.

An important thing is that the object space of C_d will be a disjoint union of $BDiff(M)$, which we multiply by $\mathbb{R}$. For the morphisms, I have similar spaces, and I let ϵ go to zero. Then the morphism space of C_d will be $ob\ C_d \sqcup \amalg_W BDiff(W, collar) \times \mathbb{R}^2_+$. Here $\mathbb{R}^2_+$ are the pairs (a, b) where $a < b$. This W should be given a collar, and the diffeomorphisms should respect the collars.

So to be precise, let W be a compact d-manifold, with $\delta W = \delta_{in} W \sqcup \delta_{out} W$, and $h_{in} : \delta_{in} W \times [a, \epsilon) \to W$ and $h_{out} : \delta_{out} W \times (\epsilon, b] \to W$. I should consider embeddings of W in $\mathbb{R}^\infty \times \mathbb{R}$ which respect collars. So you take the limit (colimit?) as $\epsilon \to 0$ of $Emb(W, \mathbb{R}^\infty \times [a, b]; \epsilon)/Diff(W, \epsilon)$

There is another way to give this a homotopy type. I want to give another description. Let $\mathscr{X}$ be a category of finite dimensional possibly open manifolds with smooth maps. Let $X \in \mathscr{X}$ and then $C_d(X)$ will be a small category with objects $Map(X, obC_d)$ and morphisms $Map(X, morC_d)$. I want to describe these functors directly without using the descriptions of the topology I gave. I want to choose smooth maps, and there is a way to do this coherently. I will give a different description. The original thing C_d as a set theoretical category is the same thing as C_d of a point. The reason that this functor gives this homotopy type will be the subject of the next talk. The reason that these two homotopy types are equivalent will be the subject of the next talk too, I think.

I want to give the description of $C_d(X)$. So I want to assume that X is connected. If it's not, you are taking a direct product. I want to know what the maps are from $X \to ob\ C_d$. One has to know, what are the maps from X to $Emb(M, R^\infty)/Diff(M)$. Let me call this $BDiff(M)$. So over $BDiff(M)$ I have the space of embeddings of M into $\mathbb{R}^\infty \times_{Diff\ M} M$. This is a fiber bundle with fiber M and a canonical embedding to the trivial bundle with fiber $\mathbb{R}^\infty$. So a map $X \to BDiff(M)$ induces a pullback $f^*(E)$ which sits inside $X \times \mathbb{R}^\infty$. So each time I give a map to $BDiff(M)$ that gives me a bundle which is embedded in $X \times \mathbb{R}^\infty$. If I am given an embedded fiber bundle with fiber M, then each fiber gives me a point in $BDiff(M)$.

So the conclusion is that $C_d(X)$ is a fiber bundle paired with a real number $\left(\begin{array}{c} E \\ \downarrow \\ X \end{array}, a\right)$. Before, I took the discrete topology, so I can take a to be constant. Otherwise I should take a as a function defined on X. Here $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ is a smooth fiber subbundle in $X \times \mathbb{R}^\infty$ with fiber a

$d-1$-dimensional closed manifold. The morphisms are done similarly. It can be guessed, I want to leave it as an exercise.

What I want to do is this. I want to define the classifying space. This can be done with either description [?]. The classifying space of a topological category is as follows. If C is a topological category, then BC is the realization $|N_{.}C|$. So $N_n(C)$ is just a composable series of n morphisms. To give this a topology, consider it in $mor\ C \times_{ob\ C} mor\ C \times_{ob\ C} \cdots mor\ C$. You can consider this as $Fun([n], C)$.

Should I say why this is the same thing as BG in the case of groups?

Earlier in the third lecture, we saw the map $BDiff(W) \to \Omega^\infty MTO(d)$. We actually saw this for $MTSO$ but it's the same map. Here W is a closed d-dimensional manifold. So consider W, let me say that, consider, okay, so, what I want to say is that so, $BDiff(W)$, this can be, if I fix a and b, $a < b$, this can be considered as embeddings $W' \subset \mathbb{R}^\infty \times [a,b]$, where $W' \cong W$. This is contained in the morphism space of C_d or $N_1(C_d)$. Then $N_1(C_d) \times \Delta^1$ gives me a morphism by the construction of the geometric realization to BG so I get a map $BDiff(W) \times \Delta^1 \hookrightarrow N_1 C_d \times \Delta^1$. This gives me, oh, I wanted a map to $Map(\Delta^1/\delta\Delta^1, BC_d)$.

[Some confusion about the difference between relative paths and loops.] I have paths $(\emptyset, a) \to (\emptyset, b)$ and I can use these to extend the paths to loops.

So we eventually get maps

$$\begin{tikzcd} BDiff(W) \arrow[r] \arrow[d] & \Omega^\infty MTO(d) \arrow[d] \\ \Omega BC_d \arrow[r, dotted] & \Omega\Omega^{\infty-1} MTO(d) \end{tikzcd}$$

and so I want to construct a map $BC_d \to \Omega^{\infty-1} MTO(d)$ to make this commute up to homotopy.

To be precise I consider C'_d, so that the objects are (M, a) with choice of a tubular neighborhood. Alternatively take fat embeddings. Morphisms are morphisms along with tubular neighborhoods respecting the collar.

So I have BC'_d and $\Omega^{\infty-1} MTO(d)$. There is a weakly contractible way of choosing tubular neighborhoods, so I'll describe my map instead from $BC'_d \to \Omega^{\infty-1} MTO(d)$. I cannot construct a morphism like this directly, so I construct another model, $BPath(\Omega^{\infty-1} MTO(d))$, which will be the classifying space of the path space whose objects are $\Omega^{\infty-1} MTO(d)$ and whose morphisms are objects in $Map([a,b], \Omega^{\infty-1} MTO(d))$ unioned over appropriate pairs. There is a weak equivalence, an inclusion is given by constant paths.

So if I'm given a morphism W in C'_d, with $W \subset N \subset \mathbb{R}^{n+d-1} \times [a,b]$, then what I want to do is, I use the Pontrjagin-Thom construction and I get a map

$$\begin{tikzcd} {\mathbb{R}^{n+d-1}}^+ \wedge [a,b]_+ \arrow[d] \\ Th(\nu^n) \arrow[d] \\ Th(\gamma_{d,n}^\perp) \end{tikzcd}$$

Finally, this defines a map in $Map([a,b], \Omega^{n+d-1}Th(\gamma_{d,n}^\perp))$. By taking the limit over n I get a map to $Map([a,b], \Omega^{\infty-1}MTO(d))$. For objects you do basically the same thing. Finally, the main theorem of Galatius-Madsen-Tillman-Weiss is that this morphism $BC_d \to Omega^\infty - 1MTO(d)$ is a weak equivalence. This will be proved in the next two lectures.

3 Sheaves

So today, I will prove an intermediate step in this theorem, saynig that the classifying space of the cobordism category is weakly equivalent to $\Omega^{\infty-1}MTO(d)$. The method will be to show that BC_d is weakly equivalent to $|D_d|$, which is weakly equivalent to $\Omega^{\infty-1}MTO(d)$. This is the part I will prove today. So first I will talk about sheaves, and the second part of the talk will be a sheaf model of $\Omega^{\infty-1}MTO(d)$. The first part will be very brief.

Let me start with the definition of sheaf.

Definition 1 *Let $\mathscr{X}$ be the category of C^∞ manifolds and C^∞ maps. I define a* sheaf *to be a contravariant functor $F : \mathscr{X} \to Set$ satisfying the sheaf axiom.*

Namely, if X is a C^∞ manifold and $\{U_i\}$ is an open covering of X, and I have s_i in $F(U_i)$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists a unique $s \in F(X)$ which gives s_i when restricted to U_i.

For example, let us consider the Yoneda embedding $\mathscr{X} \to Sh(\mathscr{X})$. Here we have $X \mapsto C^\infty(- - -, X)$.

Definition 2 *Let $\Delta_e^\ell = \{(t_0, \dots, t_\ell) \in \mathbb{R}^{\ell+1}, \sum t_i = 1\}$ whereas Δ^ℓ is a standard simplex and each coordinate is bounded below by zero. This way Δ_e^ℓ lives in $\mathscr{X}$.*

Definition 3 *Let $F \in Sh(\mathscr{X})$. The assignment $[\ell] \mapsto F(\Delta_e^\ell)$ defines a simplicial set. Let*

$$|F| = |F(\Delta_e^\bullet)| = \amalg_n F(\Delta_e^n) \times \Delta^n / \sim$$

I'm going to give you a few more definitions before going into anything useful. I mean, the definition is also useful. [Laughter]

Definition 4 *Let $pr : X \times \mathbb{R} \to X$. Then t_0, t_1 in $F(X)$ are concordant if there exists $t \in F(X \times R)$ so that t agrees with $pr^*(t_0)$ near $X \times (-\infty, 0]$ and agrees with $pr^*(t_1)$ near $X \times [1, \infty)$.*

Suppose that A is a closed subset of X and S is a germ over this A, and S is in the colimit of $F(U)$. Then I can define $F(X, A, S)$ to be the subset of t in $F(X)$ so that t agrees with s near A. We can also talk about relative concordance. So t_0 and t_1 are concordant if there exists a concordance $t \in F(X \times \mathbb{R})$ that agrees with pr^*s near $A \times \mathbb{R}$. Next, with this definition, we can define the set of concordance classes. I denote it by $F[X]$ to be $F(X)/\sim$, and similarly for relative concordance classes, $F[X, A, s] = F(X, A, s)/\sim$.

I wanted to state this theorem:

Proposition 1 *Let $X \in \mathscr{X}$ and let $F \in Sh(X)$. Then*

$$F[X] \xrightarrow{\cong} [X, |F|].$$

There is also a relative version. Let $s \in F(\Delta^0_e) = F(pt)$. Then s is a zero simplex in $|F|$. But by looking at the map $X \to pt$, then $f^*(s)$ is a constant germ near A. Then the relative version is

$$F[X, A, s] \xrightarrow{\cong} [(X, A), (|F|, s)]$$

An observation: If $X = S^n$ and A is a point, then $F[S^n, pt, s] = \pi_n(|F|, s)$.

One more definition.

Definition 5 *$\tau : F_1 \to F_2$ is a weak equivalence if $|\tau| : |F_1| \to |F_2|$ is.*

Lastly, there is a criterion, which says that $\tau : F_1 \to F_2$ is a weak equivalence if for all (X, A, s), the map $F_1[X, A, s] \to F_2[X, A, \tau(s)]$ is surjective. For example, if you pick S^n and a point, it has to be a surjection on homotopy groups. To show that it is injective on homotopy groups, consider the pair $X = S^n \times \mathbb{R}$ and $A = pt \times \mathbb{R} \cup S^n \times \mathbb{R} \backslash (0, 1)$.

I think that's all I wanted to talk about for sheaves

I'm going to give a sheaf model now for the infinite loop space.

Definition 6 *Let $D_d(- - -, n) \in Sh(\mathscr{X})$ be the set of closed submanifolds W inside $X \times \mathbb{R} \times \mathbb{R}^{d-1+n}$ with projections π, f, and j, and so that $\pi : W \to X$ is a submersion with d-dimensional fiber. That means over each point W is a d–dimensinoal manifold. Also, the map $(\pi, f) : W \to X \times \mathbb{R}$ is proper. At each fiber over x you have an $\mathbb{R}$ direction and an $\mathbb{R}^{d-1+n}$ direction. It's a long thing in the $\mathbb{R}$-direction, but it's compact in its slices.*

Then D_d is the colimit of $D_d(---, n)$ so $W \in D_d(X)$ if $W \subset X \times \mathbb{R} \times \mathbb{R}^\infty$ satisfying the same conditions and so that for all compact $K \subset X$, we have $\pi^{-1}(K) \subset K \times \mathbb{R} \times \mathbb{R}^{d-1+n}$.

Let me state the main theorem of today:

Theorem 6 *$|D_d|$ maps via weak equivalence to $\Omega^{\infty-1} MTO(d)$.*

So the method of proof is to construct a bijection between the homotopy classes of maps

$$\begin{tikzcd} {[X, \Omega^{\infty-1} MTO(d)]} \arrow[r, shift left, "\sigma"] & D_d[X] \arrow[l, shift left, "\rho"] \end{tikzcd}$$

for all compact X in $\mathscr{X}$.

I will concentrate on constructing this. I want to start by constructing ρ. Given $W \subset X \times \mathbb{R} \times \mathbb{R}^{d-1+n}$. The normal bundle of this embedding N is the pullback of a diagram, since W has a tangent over the fiber.

$$\begin{tikzcd} N \arrow[r] \arrow[d] & U_{d,n}^{\perp} \arrow[d] \\ W \arrow[r] & Gr(d,n) \end{tikzcd}$$

So now I have this diagram. I choose a regular value a of $f : W \to \mathbb{R}$. We have the normal bundle of M which is $f^{-1}(a)$ in $X \times \{a\} \times \mathbb{R}^{d+n-1}$ is the further pullback

$$\begin{tikzcd} N|_M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & U_{d,n}^{\perp} \arrow[d] \\ M \arrow[r] & W \arrow[r] & Gr(d,n) \end{tikzcd}$$

Now I can apply the Pontryagin-Thom construction and get a map from

$$X_+ \wedge S^{d-1+n} \to Th(N|_M) \to Th(U_{d,n}^{\perp})$$

In the limit I will get an element which is the definition of $\rho(W)$. You have to check that this is independent of the choices, the embedding, the regular value. I will check that it is independent of the regular value. If b is another regular value with $a < b$, then we can consider $W \cap f^{-1}[a, b]$, call that $W_[a, b]$. Then it is inside $X \times [a, b] \times \mathbb{R}^{d-1+n}$ and the same Pontryagin-Thom construction gives me $X_+ \wedge [a, b]_+ \wedge S^{d-1+n} \to Th(U_{d,n}^{\perp})$, showing that you get homotopic maps if you choose different regular values.

Next, we want to construct a map $\sigma : [X, \Omega^{\infty-1} MTO(d)] \to D_d[X]$. Given that an element g starts from something compact, I can assume it is of the form $g : X_+ \to \Omega^{d-1+n} Th(U_{d,n})^{\perp}$. Then take the adjoint $X_+ \times S^{d-1+n}$, and assuming that $g \pitchfork Gr(d, n)$ so we take $M = g^{-1}(Gr(d, n) \subset X \times \mathbb{R}^{d-1+n})$. Since W is the inverse image of a coset and this one is compact, M is also compact. Let this inclusion be $\pi_0 : M \subset X \times \mathbb{R}^{d-1+n}$. I have a subset of a particular slice, and I want it to lie in something with one more coordinate. Let $W' = M \times \mathbb{R} \subset X \times \mathbb{R} \times \mathbb{R}^{d-1+n}$. This doesn't satisfy all of our conditions. Let me name the projections π_1, f, and j. Notice that (π_1, f) is proper, so there is nothing to prove since X is compact. But it doesn't satisfy the second condition, π_1 may not be a submersion. So what I need to do is make it into a submersion. We only have one method. We have one good method, I mean. Let me try to draw the picture. [Picture] These three coordinates are supposed to be X, $\mathbb{R}^{d-1+n}$, and $\mathbb{R}$, but the projcetion here is not a submersion. So we try to perturb this by the h-principle. So I get a perturbation so that it is a submersion.

To use the h-principle I will need to construct a bundle epimorphism $TW' \to TX$ which goes by

$$\begin{tikzcd} TW' \arrow[r] \arrow[d] & TX \arrow[d] & \overset{\text{Phillips}}{\rightsquigarrow} & TW' \arrow[r, "d\pi_2"] \arrow[d] & TX \arrow[d] \\ W' \arrow[r] & X & & W' \arrow[r, "\pi_2"'] & X \end{tikzcd}$$

with π_1 homotopic to π_2. So the construction of $\hat{\pi_1}$, we take the normal bundle of the inclusion of M into the Grassmannian. This normal bundle in $X_+ \wedge S^{d-1+n}$, since it is a submersion near the zero section, so it is a pullback of the normal bundle of $Gr(d,n)$ in the $\perp$-bundle, so it is, we have

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, phantom, "\cap"] & Gr(d,n) \arrow[d, phantom, "\cap"] \\ X_+ \wedge S^{d-1+n} \arrow[r, "g"'] & Th(U_{d,n}^{\perp}) \end{tikzcd}$$

Let $T^\pi M = g^*(U_{d,n})$, then $N \oplus T^\pi M = \epsilon^{d+n}$, but also $N \oplus TM =\cong \pi_0^*(TX \oplus \epsilon^{d-1+n})$, so $TM \oplus \epsilon^{d+n} \cong \pi_0^*(TX) \oplus T^\pi M \oplus \epsilon^{d-1+n}$, and then obstruction theory tells me that this map is induced from

$$TM \oplus \epsilon \to \pi_0^*(TX) \oplus T^\pi M \twoheadrightarrow \pi_0^*(TX).$$

In this way we have constructed $\hat{\pi}_1$. At the end, so we get $W' \overset{\pi_2, f}{\to} X$, and we keep the f, and we can lift it to an embedding

$$\begin{tikzcd} & X \times \mathbb{R} \times \mathbb{R}^{d-1+n} \arrow[d] \\ W' \arrow[ur, dotted] \arrow[r, "(\pi_2, f)"'] & X \times \mathbb{R} \end{tikzcd}$$

for a big enough n (n doesn't stay the same), and I define the image of this lifting to be my W, which is $\sigma(g)$. There is a final step of showing that it is a bijective map. I will just draw some pictures.

The last thing to do is to check that this correspondence

$$[X, \Omega^{\infty-1}MT(d)] \underset{\rho}{\overset{\sigma}{\rightleftarrows}} D_d[X]$$

is one to one. Let me first check $\rho \circ \sigma$. I start with a map g, use the adjoint, $X_+ \wedge S^{d-1+n} \to Th(U_{d,n}^{\perp})$. [Argument in pictures]. So $g|_M$ and $\rho \circ \sigma(g)|_{M'}$ define normal bundles of these embeddings, but M and M' are isotopic if I increase the dimension, so that these two maps define the same stable normal bundles and so lie in the same homotopy class in $[X, \Omega^{\infty-1}MT(d)]$.

So finally, let's look at $\sigma \circ \rho$. So given $W \subset X \times \mathbb{R} \times \mathbb{R}^{d-1+n}$, choose (a, b) an interval of regular values of $f : W \to \mathbb{R}$ and you can show that this is concordant to the manifold which has a nice simple form, like a cylinder. So then I can assume by Morse theory that $W \cong M \times \mathbb{R}$ where M is the slice at the zero section. Then my construction is to collapse the slice and get a homotopy class of maps, and then look at the zero locus, basically the same one, and then I do some perturbation and get the element I want, $\sigma\rho(W)$, but I can avoid using the h-principle, I can define an isotopy map from this new cylinder back to W, and write down the precise perturbation. This shows that the process is invertible. I think that's all for my talk.