West Coast Algebraic Topology Summer School

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1 Introduction to homological stability

We're definitely switching gears. This talk, the beginning, I'll give some more classical examples for stability theorems to get some idea of how they work, and start in then on the Harer stability theorem which is what we're interested in.

The first example (homotopical stability) is the Freudenthal suspension theorem which says, let X be a k-connected space. Then the suspension map on homotopy groups $\pi_i(X) \to \pi_{i+1}(\Sigma(X))$ is an isomorphism for $i \leq 2k$ and surjection for $i = 2k+1$.

So take the identity on X, suspend it to get $\Sigma id : \Sigma X \to \Sigma X$, then by adjunction, this is $X \to \Omega\Sigma X$, and then by taking π_i you get $\pi_i(X) \to \pi_i(\Omega\Sigma X) = \pi_{i+1}\Sigma X$.

This is not an isomorphism everywhere but it's an isomorphism in a certain range depending on the connectivity of X. But suspending makes the connectivity go up, so you can iterate this. You can map $\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \pi_{i+2}(\Sigma^2 X) \to \cdots$ Then these maps will eventually be isomorphisms, and these are the stable homotopy groups.

We want to look at families of moduli spaces indexed by some degree. In this case our spaces were indexed by connectivity. Then we want to ask if there is a stability range and from now on we will look at stability range meaning isomorphism on homology groups. We would also like to ask, is there a stable homology type as the degree gets large.

There was a survey paper by Ralph Cohen on stability on the reading list, there are a lot more examples than the ones I'm saying here. Here's one example, configuration spaces. The basic idea is, you start with some manifold M, and then we can take $F_k(M)$, different ways of choosing k distinct points in M. This has a free action of the symmetric group on k letters by permuting the k distinct points. We can take the quotient by this action, and define $C_k(M) = F_k(M)/\Sigma_k$.

Our stability theorem in this case says that there are maps $\gamma_k : C_k(M) \to C_{k+1}(M)$ which are induce monomorphisms on homology and isomorphisms in some given range. I want to go a little bit into this theorem, I'm not even telling you what the range is. You end up taking these spaces and comparing them with easier spaces that you know what's going on with their homology. So let's go back to M and let $T^\infty M$ be the fiberwise one point compactifications of TM. The sections $\Gamma(M)$ are smooth sections of $T^\infty M$ with compact support. Now since we've compactified things here, we can talk about sections having a degree. So we can say, let $\Gamma_k(M)$ be sections of degree k, and then $\Gamma(M)$ is the disjoint union of $\Gamma_k(M)$. The sections, make it transverse to the ∞ section and count the intersections, in order to get the degree.

So a fact about these is that these all have the same homotopy type. So $\Gamma_k(M)$ have the same homotopy type. This means that if you're looking at $\Gamma_k(M)$, they all have the same homology, so we can define some maps $\alpha_k : C_k(M) \to \Gamma_k(M)$, where the degree of the section is the same as the number of points in M, and what these maps are going to satisfy, now we have

$$\begin{tikzcd} C_k(M) \arrow[r] \arrow[d] & \Gamma_k(M) \arrow[d] \\ C_{k+1}(M) \arrow[r] & \Gamma_{k+1}(M) \end{tikzcd}.$$

Now we apply homology and get

$$\begin{tikzcd} H_qC_k(M) \arrow[r] \arrow[d] & H_q\Gamma_k(M) \arrow[d] \\ H_qC_{k+1}(M) \arrow[r] & H_q\Gamma_{k+1}(M) \end{tikzcd}.$$

I guess I didn't say what the maps $C_k(M) \to C_{k+1}(M)$ are. M has to have boundary and you push out toward the boundary, assuming your points are away from the boundary.

This is supposed to be the warmup example. The idea is that if we get information on the horizontal maps, we can get information on the vertical map we're interested in. So you want to show that $(\alpha_k)_*$ is an isomorphism for $k \gg q$. You can also then take a limiting value, what happens when k gets large.

If $C(M) = \lim_{k\to\infty} C_k(M)$ then

$$H_q(\mathbb{Z} \times C(M)) \cong H_q(\Gamma(M)).$$

for the special case $M = \mathbb{R}^n$, then $\Gamma(M) = \Omega^n S^n$ and $\alpha_k : C_k(\mathbb{R}^n) \to \Omega^n_k S^n$. This is a special case of the more general framework, saying that you can compare this configuration space to loop spaces.

If you take $n \to \infty$ then $M = \mathbb{R}^\infty$ and $F_k(\mathbb{R}^\infty)$ is contractible, and then $\Sigma_k \to F_k(\mathbb{R}^\infty) \to C_k(\mathbb{R}^\infty)$, and so $C_k(\mathbb{R}^\infty)$ is a model for $B\Sigma_k$. So that's a nice fact there, and your α_k go from $B\Sigma_k$ to $\Omega^\infty_k S^\infty$.

[Can I describe this α_k? Given a configuration you have to get a map from a sphere to a sphere. Fatten the points noncanonically. Collapse outside the disks to the basepoint, and then what you get is a wedge of k copies of S^n and then you fold to get to S^n.

[Oops, missed something, braid groups, double loop spaces, Ralph Cohen's paper]

Okay, back to homological stability. This is our main goal, homological stability for mapping class groups. This was first proved by Harer, so it's often called Harer stability. It was proved with different methods by Ivanov, Boldsen, and Randall-Williams. One reference was Nathalie Wahl's paper, consolidating the nicest features of each. Let me give some notation. Let $S_{g,r}$ be the surface with genus g and r boundary components. So $\Gamma_{g,r}$ (not the same Γ) or $\Gamma(S_{g,r})$ is $\pi_0(Diff(S_{g,r}, \delta S_{g,r}))$. I'm suppressing that everything is orientation preserving. Then I'll take π_0. This is the mapping class group. There are a couple different maps. The first one is called α_g, from $S_{g,r+1}$ to $S_{g+1,r}$, by gluing on pants. So we could attach in the other way, $S_{g,r} \to S_{g,r+1}$. Call that β_g. We'll have induced maps α_g and β_g on the mapping class groups.

Theorem 1 *If $g \geq 0$ and $r \geq 1$, then $(\alpha_g)_* : H_q\Gamma(S_{g,r+1}) \to H_q\Gamma(S_{g+1,r})$ is surjective for $q \leq \frac{2}{3}g + \frac{1}{3}$ and an isomorphism for $q \leq \frac{2}{3}g - \frac{2}{3}$ and $(\beta_g)_* : H_q\Gamma(S_{g,r}) \to H_q\Gamma(S_{g,r+1})$ is always surjective and isomorphic for $q \leq \frac{2}{3}g$.*

The goal is to use simplicial complexes with an action of these mapping class groups and then use information from those simplicial complexes to understand the mapping class groups themselves.

What are the simplicial complexes? Let S be a surface, and assume it has a boundary. We want to choose two points on that boundary, b_0 and b_1. So there's two ways that this can look. They can be on the same boundary component or on different boundary components. Then what we're going to do is look at paths between them that don't disconnect the surface. Those are the kind of arcs that we're interested in. Consider collections arcs intersecting δS at the endpoints transversally and only up to isotopy. These should be oriented, from b_0 to b_1, which have disjoint interiors, and the whole are nonseparating. Call them a_i.

Now I can define a simplicial complex $O(S, b_0, b_1)$ is a simplicial complex with vertices the isotopy classes of such arcs. A p-simplex is a collection of $p+1$ isotopy classes that are nonintersecting and nonseparating. Also, there should be a cyclic order around b_0 of incident arcs which is reversed around b_1.

So $O^1(S)$ is the set of complexes where b_0 and b_1 are on the same boundary component and $O^2(S)$ is the set where they are on different components. The action of $\Gamma(S)$ on the surface induces an action on $O(S, b_0, b_1)$.

I started a couple minutes late but I'm out of time. I need to at least write down the ingredients of the proof. I don't have time to prove them, but here they are:

Ingredient 1 This has a couple of parts

(a) For $O^i(S)$, $\Gamma(S)$ acts transitively on the set of p-simplices.

(b) The stabilizer $St_{O^1}(\sigma_p) \xrightarrow{\cong} \Gamma(S_{g-p-1,r+p+1})$ and $St_{O^2}(\sigma_p) \xrightarrow{\cong} \Gamma(S_{g-p,r+p-1})$

Ingredient 2 Given $\sigma_p \in O^2(S)$ we have the following commutative diagram:

$$\begin{tikzcd}
\Gamma(S_{g,r+1}) \arrow[d, "\alpha_g"'] & St_{O^2}\sigma_p \arrow[l, hook'] \arrow[r, "\cong"] \arrow[d] & \Gamma(S_{g-p,r+p}) \arrow[d, "\beta_{g-p}"] \\
\Gamma(S_{g+1,r}) & St_{O^1}\alpha\sigma_p \arrow[l, hook'] \arrow[r, "\cong"] & \Gamma(S_{g-p,r+p+1})
\end{tikzcd}$$

[Some discussion. α is attaching a strip, and doing it right moves case 2 to case 1. β turns a case 1 into a case 2 by attaching a strip as well.]

There's another diagram that looks like this but α and β are reversed.

Ingredient 3 $\alpha : \Gamma(S) \to \Gamma(S_\alpha)$ and $\beta : \Gamma(S) \to \Gamma(S_\beta)$ are injective, and for any 0 simplex of $O^i(S)$ there are conjugations c_α:

$$\begin{tikzcd}
St_{O^2}(\sigma_0) \arrow[r] \arrow[d, hook] & St_{O^1}(\alpha\sigma_0) \arrow[d, hook] \arrow[dl, dotted] \\
\Gamma(S) \arrow[r, "\alpha"'] & \Gamma(S_\alpha)
\end{tikzcd}$$

and a similar diagram for β

Ingredient 4 This is the hard condition and we'll have a whole hour on it tomorrow. That is, $O^i(S_{g,r})$ is $g-2$-connected.

2 Homological stability of Mapping Class groups, the spectral sequence argument

So the aim of this talk is to take the four ingredients we had from the previous talk, the proof of one of which is deferred, and apply a spectral sequence argument to prove the theorem. Let me start with a recap of notation. So $S_{g,r}$ is the oriented surface of genus g with r boundary components. Then $\Gamma_{g,r} = \Gamma(S_{g,r})$ and $O^i(S_{g,r})$ for $i = 1, 2$ and the vertices are isotopy classes of nonseparating arcs between two points on δS. The p-simplices are collections of $p+1$ distinct such isotopy classes of arcs such that you can represent them all simultaneously by a set of arcs that are non-separating and satisfy the clockwise-anticlockwise orientation condition.

We know that $\Gamma(S_{g,r})$ acts on $O^i(S_{g,r})_p$ and the stabilizer on O^i of a p-simplex σ_p is denoted $St_{O^i}(\sigma_p)$. The stabilization maps for $r \geq 1$ and $g \geq 0$ are given by gluing a pair of pants either to two or to one boundary component. You can extend mapping class elements by extending by the identity on the new pairs of pants. It's easy to extend to a little bridge and then $O^2(S_g, r)$ goes to $O^1(S_{g+1,r})$ so we have

$$\begin{tikzcd} O^2(S_{g,r+1}) \arrow[r, "\alpha"] & O^1(S_{g+1,r}) \\ \circlearrowleft & \circlearrowleft \\ \Gamma(S_{g,r+1}) \arrow[r, "\alpha_g"'] & \Gamma(S_{g+1,r}) \end{tikzcd}$$

and similarly

$$\begin{tikzcd} O^1(S_{g,r}) \arrow[r, "\beta"] & O^2(S_{g,r+1}) \\ \circlearrowleft & \circlearrowleft \\ \Gamma(S_{g,r}) \arrow[r, "\beta_g"'] & \Gamma(S_{g,r+1}) \end{tikzcd}$$

[Another diagram that he erased immediately]

So the first ingredient above is that $\Gamma(S_{g,r})$ acts transitively on $O^i(S_{g,r})_p$. [Argument in pictures] You take an element of $\Gamma(S\backslash\sigma_p)$. Fixing the boundary, then you get a map to $\Gamma(S)$ This is an isomorphism onto the stabilizer of σ_p.

The induced map on the right in the second ingredient is on a surface of lower genus. When I say I have a square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & H_2 \arrow[d] \\ H_1 \arrow[r] \arrow[ur, dotted] & G \end{tikzcd}$$

If I say I have a conjugation, that means there exists a g in G so that $g : H_1 \to H_2$ is an isomorphism and $g = id$ on K.

So the input is $X \xrightarrow{f} Y$ with actions of G on X and H on Y with a map ϕ from $G \to H$ so that the actions are equivariant. I want the actions transitive and let X and Y be simplicial complexes, with X $c-1$ connected and Y c-connected. Then our spectral sequence will go to zero in a stable range.

So the construction is, we have X, so we can have $\tilde{C}_.(X)$, so we have $\mathbb{Z}$ in dimension -1, then $\mathbb{Z}X_0$ in dimension 0, and so on. This is a chain complex in $\mathbb{Z}G - mod$.

We also have E_*G, a projective resolution of $\mathbb{Z}$ in $\mathbb{Z}G - mod$, which is $E_0G \leftarrow E_1G \leftarrow \cdots$

So I've got two chain complexes, and I want to form a double complex. So we will let $p \geq -1$ and $q \geq 0$. An object in the double complex will be $\tilde{C}_p(X) \otimes_G E_{q-1}G \bigoplus \tilde{C}_p(Y) \otimes_H E_qH$. Then this maps to $\tilde{C}_{p-1}(X) \otimes_G E_{q-1}G \bigoplus \tilde{C}_{p-1}(Y) \otimes_H E_qH$. So the components of the horizontal differential are just the differentials from $\tilde{C}$ with the identity on E.

The differential in the other direction goes to $\tilde{C}_p(X) \otimes_G E_{q-1}G \bigoplus \tilde{C}_p(Y) \otimes_H E_qH$. The differential in this case acts as the identity on $\tilde{C}_p$ and by the differential of E_q, along with components $\tilde{C}_p(X) \to \tilde{C}_p(Y)$ and $E_qG \to E_qH$ from our initial data, via f_* and ϕ_*.

You check that this is a double complex, and so as with any double complex, you get a spectral sequence filtering it either horizontally or vertically and it converges to the second thing. The horizontal spectral sequence gives $E^1_{p,q} = 0$ for $p + q \leq c$ because X and Y are connected, X was $c - 1$ connected and Y was c-connected.

The point is, in one direction we'll work out that it converges to zero in a range.

[What are we doing? I've lost track]

[We're calculating the [unintelligible]of a relative mapping class group.]

The first step is to show that this converges to zero, we just need to show that the horizontal differential is exact. So when $p \leq c-1$, then X and Y are both p-connected, so the differentials are exact. We then tensor with a projective resolution and still get something that is exact. What about when $p = c$ and $q = 0$. In this case, the lower part is still exact by connectivity, and the upper part is $E_{q-1}G$ so that's zero for a trivial reason. So then $E^\infty_{p,q} = 0$ for $p+q \leq c$ so $E^1_{p,q} \to 0$ for $* \leq c$. This will also be true for the vertical spectral sequence.

So now let's look at E^1 of the vertical spectral sequence. The formula is:

$$E^1_{p,q} = H_q(\tilde{C}_p(X) \otimes_G E_{*-1}G \bigoplus \tilde{C}_p(Y) \otimes_H E_* H)$$

So let's start with $p \geq 0$. Here we have $\tilde{C}_p(X) = \mathbb{Z}X_p \cong \mathbb{Z}(G/G_p)$ by the orbit-stabilizer theorem. So G_p is the stabilizer in G of σ_p. The theorem says $G/G_p \cong X_p$ (using transitivity).

The second thing is $\mathbb{Z}(G/G_p) \otimes_G$ — is the same as $\mathbb{Z} \otimes_{G_p}$ —, where $[g] \otimes m \mapsto 1 \otimes gm$. Then we can simplify our formula, and

$$E^1_{p,q} = H_q(\mathbb{Z} \otimes_{G_p} E_{*-1}G \bigoplus \mathbb{Z} \otimes_{H_p} E_* H)$$

Now we can use that we originally had the mapping cone in the vertical direction. This is the mapping cone of the map $\mathbb{Z} \otimes_{G_p} E_*G \stackrel{1\otimes\phi_*}{\to} \mathbb{Z} \otimes_{G_q} E_*H$.

Now E_*G is a projective resolution of $\mathbb{Z}$ in $\mathbb{Z}G - mod$? So it's also a projective resolution in $\mathbb{Z}G_p - mod$. So tensoring this with $\mathbb{Z}$ and taking homology gives the homology of G_p. We have on the other side the complex that would give homology of H_p. So then we get as our formula, by a slightly waffly argument

$$H_q(G_p \stackrel{\phi}{\to} H_p)$$

This is for $p \geq 0$. So for $p = -1$ you get $E^1_{-1,q} = H_q(G \stackrel{\phi}{\to} H)$. So everything is realized as group homology.

The input is as I said, the output is a spectral sequence which converges to 0 in total degree at most c. So now we get as our next differential $H_q \begin{pmatrix} G \\ \downarrow \\ H \end{pmatrix} \stackrel{i}{\leftarrow} H_q \begin{pmatrix} G_0 \\ \downarrow \\ H_0 \end{pmatrix}$.

It's finally time to state and hopefully prove most of the theorem.

Theorem 2 *$\Gamma(S_{g,r+1}) \xrightarrow{\alpha_g} \Gamma(S_{g+1,r})$ and $\Gamma(S_{g,r}) \xrightarrow{\beta_g} \Gamma(S_{g,r+1})$ have $(\alpha_g)_*$ is an isomorphism for $* \leq \frac{2g-2}{3}$ and surjective for $* \leq \frac{2g-1}{3}$, while [unintelligible]for β_*.*

The relative homology $H_q(\alpha_g)$ is 0 for $q \leq \frac{2g+1}{3}$ and $H_q(\beta_g)$ is 0 for $q \leq \frac{2g}{3}$. So for the base case of the induction, BG is path connected so $H_0(\alpha_g) = 0$ for all g and likewise for $H_0(\beta_g)$. This gives us some base cases for both $\alpha(1_0)$ and $\beta(2_0)$ and (2_1). So step 1 we want $(2_{\leq g}) \implies (1_g)$ for $g \geq 1$ and for step two, $(1_{\leq g-1})$ implies (2_g) for $g \geq 2$.

So step one, $H_*(\beta_{g-p}) = 0$ for $p \geq 0$ and $x \leq \frac{2}{3}gp$. Wo want to prove that $H_*(\alpha_g)$ for $*$ at most $\frac{2g+1}{3}$. The inputs in this cas are $O^2(S_{g,r+1}) \to O^1(S_{g+1,r})$ with actions by $\Gamma(S_{g,r+1})$ and $\Gamma(S_{g+1,r})$. We have equivariance, and we have the connectivity assumption for $c = g-1$ by our ingredient 4. Then ingredient 1 gives us transitivity. We have the E^1 page. The first claim is that we have a bunch of zeros in a triangle on this page [picture] The corners are $0, 0$, $(\lfloor \frac{2g+1}{3} \rfloor, 0)$, and $(0, \lfloor \frac{2g+1}{3} \rfloor)$. The top corner is X, not 0. The second claim is that $E^{\infty}_{-1,q} = 0$ for $q \leq \frac{2g+1}{3}$ and claim three is that $E^1_{-1,q} \overset{d^1}{\leftarrow} E^1_{0,q}$ is 0 for all $q \geq 1$, and lastly that $E^1_{-1,q} = H_q(\alpha_g)$.

Then nothing can be killed in the leftmost column and yet they must be killed in E^∞ so they are 0 in E^1. Then our last claim tells us that we have found the groups we are interested in.

The first claim is, we have a formula for the entries of the E^1 page. Ingredient two identifies this with β of a surface of lower genus, and then the inductive hypothesis gives us that this is zero.

For the second claim, you recall that it converges to zero in the described range just as a specialization.

The last claim, when we identified the entries of the E^1 page, these for $q = -1$ were the homology of the entire group. Then the third claim, we want to show that the differentials induced by the square of group homomorphisms is zero.

In our case we have $G \to H$ is $\Gamma(S_{g,r+1}) \to \Gamma(S_{g+1,r})$, and you want to show that the map to the left induces zero on homology:

$$\begin{tikzcd}
\Gamma(S_{g,r+1}) \arrow[d] \arrow[dr, dash] & St(\sigma_0) \arrow[l] \arrow[d] \\
\Gamma(S_{g+1,r}) & St(\sigma_0) \arrow[l]
\end{tikzcd}$$

Slightly technical and that's the end of step one, step two is pretty much the same. I'll just finish with a picture, I think.

The last thing we need to prove to get the full result is that $(\beta_g)_*$ is always injective. This is much nicer to prove. You need a one-sided inverse. So you apply β_g, gluing on a pair of pants, then apply β_g, and then apply the map δ, applying a cap, and the composition is the identity, so you can see that it should always be injective. You isotope this to itself but on the new copy of $S_{g,r}$. You can slide the diffeomorphism up to the new boundary. The composition is 1, so it's injective on homology. It's only a one-sided inverse because you can't move the diffeomorphism around from the cap to the pair of pants.