West Coast Algebraic Topology Summer School

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August 11, 2010

1 h-principles, I

I'll start with an outline. The goal this morning is to introduce and rigorously define Gromov's h-principle. I'll introduce jet bundles and differential relations and natural fiber bundles and then I'll tell you what Gromov's h-principle is.

Okay, so if we have, we can consider maps $f : \mathbb{R}^n \to \mathbb{R}^q$, and pick some coordinates $y_j = f_j(x_1, \ldots x_n)$ for $j = 1, \ldots, q$ and we have a differential operator

$$D^\alpha = \frac{\partial^{|\alpha|} f_j}{\partial x_1^{\alpha_1}, \cdots, \partial x_n^{\alpha_n}}$$

and we say $\psi(x, f, D^\alpha f)$ is a collection of differential equations or inequalities, let's call this collection R. This is a "differential relation." If we have a system of differential equations or operators like this, we can replace the differential operators with algebraic variables z_α and get an algebraic version R'.

If we have a solution to R we call it a genuine or holonomic solution. If we have a solution to the algebraic system, this is called a formal solution. The h-principle says that the space of genuine solutions is weakly homotopy equivalent to the space of formal solutions.

This is good because formal solutions, the existence is necessary for the existence of genuine solutions, and it tells you the topology for formal solutions gives you the topology for genuine solutions, which usually involves some geometry.

Now to define what this stuff means. This is the gist of an h-principle, in this context. There are other ways to state h-principles, but they usually are ways of showing that formal solutions that involve only topology are the same in some sense as genuine solutions that need some geometry

So $f : \mathbb{R}^n \to \mathbb{R}^q$ can be viewed as a section of the trivial bundle $\begin{array}{c}\mathbb{R}^n \times \mathbb{R}^q \\ \downarrow \\ \mathbb{R}^n\end{array}$. So $\bar{f}(x) = (x, f(x))$. So let me define slowly using examples the jet bundle and I'll give a rigorous definition at the end.

The 1-jet of f at $x \in \mathbb{R}^n$ is

$$J^1_f(x) = (x, f(x), \left(\frac{\partial f_i}{\partial x_j}\right))$$

in $\mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^{nq}$. We can take the jet bundle $J^1(\mathbb{R}^n, \mathbb{R}^q)$, the space of these over $\mathbb{R}^n$, and look at sections of these.

So in the simplest case we have $J^1(\mathbb{R}, \mathbb{R}) = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and you can think of the variables as x, y, and y'. So if $f(x) = ax + b$ then $f'(x) = a$. So the section is $y = ax + b$ shifted up to be at $z = a$. This line represents the one-jet of f, $J_1 f$. If I draw a section, it looks like a graph of a function along with tangent planes. I want to draw this because I want to compare it to an arbitrary section. If I just pick any section of the jet bundle, it's not a given that the "derivatives" come from the second coordinate in $\mathbb{R}^q$. They'll be in general unrelated to our curve. So this [picture] is an example of a section. So if they happen to be tangent, then they will be holonomic or genuine solutions, and if they are not necessarily, they will be formal solutions.

So we have to define the jet bundle of a smooth fiber bundle, so if we have a smooth fiber bundle $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$, then I'd like to define a space of 1-jets of sections into E. To do this, I'll first say that sections over U will be denoted $\Gamma(U) = \{U \xrightarrow{s} p^{-1}U | \text{sections}\}$. I'll also define the space of germs, for $x \in E$,

$$E^1(x) = \{(U, s) | U \subset X, s \in \Gamma(U)\} / \sim$$

where $(U, s) \sim (V, t)$ if there exists $W \subset U \cap V$ with $x \in W$ and $ds = dt$ restricted to W from $TW \to Tp^{-1}(W)$.

[Some discussion of this definition]

Let's change the conditions, this is too strong, let's let $x \in X$ instead of E and say that two germs are equivalent if $s(x) = t(x)$ and $ds(x) = dt(x)$.

Hopefully this is okay now. So $\pi([U, s]) = x$ for $[U, s] \in E^1 x$.

So if we have a section $f : X \to E$ then we can define $J^1_f(x) = [U, f|_U]$ and now I trivialize everything and explain where the transition maps come from. Let

$$E = \bigcup_{\lambda \in \Lambda} W_\lambda, \varphi_\lambda : W_\lambda \to \mathbb{R}^{n+q}$$

and

$$X = \bigcup_{S \in \underline{X}} V_\delta, \psi_\delta : V_\delta \to \mathbb{R}^n$$

, assume $p(W_\lambda) \subset V_\delta$. Then $x \in W_\lambda$ and $[U, s] \in E^1_x$. So $ds|_{p(x)} : T_{p(x)} V_\delta \to T_x W_\lambda$. So $ds|_{p(x)} \in Hom(\mathbb{R}^n, \mathbb{R}^q) \cong \mathbb{R}^{nq}$. Then $\pi^{-1}(W_\lambda) \xrightarrow{d} W_\lambda \times \mathbb{R}^{nq}$. So now I have one-jets, and then E^2, the two-jets are one jets of one jets, $E^2 = (E^1)^1$, and so on.

So we have these bundles, with, if $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ is a fiber bundle, then we write ΓE for the space of sections with the C^∞ or C^r topology, and we also have this E^r jet bundle ΓE^r for the space of sections of the jet bundle E^r with the compact open topology.

So the act of taking r-jets gives a continuous map $J^r : \Gamma E \to \Gamma E^r$.

Definition 1 *A differential relation R of order r on sections $f : X \to E$ is a subset of ΓE^r.*

So Gromov's h-principle says that the space of holonomic solutions is all there is of the homotopy, the inclusion of the space into formal solutions is a weak homotopy equivalence. Let's go back into the trivial examples that we can draw, take $\mathbb{R} = \{\frac{dy}{dx} = y\} \subset J^1(\mathbb{R}, \mathbb{R}) = \overbrace{\mathbb{R}}^{x} \times \overbrace{\mathbb{R}}^{y} \times \overbrace{\mathbb{R}}^{z}$ and so if we were to draw this, we would have $\mathbb{R}^3$, and we would have a plane, and the solutions would be of the form $y = Ce^x$ sitting in the plane $y = z$ Any other section $f : \mathbb{R} \to J^1(\mathbb{R}, \mathbb{R})$ having $f \subset \mathbb{R}$ is homotopic to one of these.

We have a more complicated example, $R = \{\frac{dy}{dx} = y^2\}$, and this gives us a surface in $\mathbb{R}^3$ So $y = -\frac{1}{x+c}$ and [unintelligible][h principle does not apply?]

So now let me talk about natural fiber bundles. Let $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ be a fiber bundle, and let $Diff_X E$ be diffeomorphisms h_E of E so that there exists h_X such that

$$\begin{array}{ccc} E & \xrightarrow{h_E} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow[h_X]{} & X \end{array}$$

A fiber bundle is natural if there is a section j from $Diff X$ to $Diff_X E$. So if I have $\begin{array}{c} TM \\ \downarrow \\ M \end{array}$, then $h \in Diff_M$ gives me $dh : TM \to TM$. The observation is that if $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ is a natural fiber bundle, then $\begin{array}{c} E^r \\ \downarrow \\ X \end{array}$ is also natural.

Now we can state Gromov's h-principle. Maybe one more definition.

[Some discussion]

Definition 2 *$R \subset E^r$ is* invariant *if it is invariant under the $Diff X$ action via j*

Theorem 1 *If X is open and $R \subset E^r$ is open (a condition on components) and invariant. Then the inclusion of* $\Gamma(E)|_{\mathbb{R}} \overset{J^r}{\hookrightarrow} \Gamma(E^r)|_{\mathbb{R}}$ *is a weak homotopy equivalence.*

Let $E = M \times N$ over M. Then $E^1 = Hom(TM, TN)$ which contains R which is of rank at least k. Then we have a map $\Gamma(E) \to \Gamma(E^1)$ via J^1 on k-immersions, and this will be a weak homotopy equivalence.

There will be more examples and a proof today, basically by induction on a handle decomposition of the base space.

2 Applications and the first part of the proof of Gromov's theorem

Recall notation. We start with $E(M)$ a natural fiber bundle over M. By natural here I mean, if $E(U)$ is the restriction of $E(M)$ to U, then $U \to V$ induces a map $E(U) \to E(V)$.

So we can define the jet bundle $\begin{array}{c} E^{(r)}(M) \\ \downarrow \\ M \end{array}$ where $E^{(r)}(M)_x = \Gamma_{\text{at } x} E(M)/\sim$ where $f \sim g$ iff $\partial^\alpha f(x) = \partial^\alpha g(x)$ for $|\alpha| \le r$.

Then $\Gamma E(M) \overset{J^{(r)}}{\to} \Gamma E^{(r)}(M)$ is the h-principle map which takes s to $(x \mapsto [s])$, and the theorem is

Theorem 2 *Let M be an open manifold, meaning $M \backslash \delta M$ has no compact components, and $E_0^{(r)}(M) \subset E^{(r)}(M)$ open and invariant by the action of local diffeomorphisms of M then $J^{(r)}$ is a weak equivalence from ${J^{(r)}}^{-1}\Gamma E_0^{(r)}(M) \to \Gamma E^{(r)}(M)$.*

Let's start with applications.

A **Theorem 3** *(Phillips) Let M be open, with dimension greater or equal to that of N. $Surj(TM, TN)$ are defined to be fiberwise surjective bundle maps. Then*

$$Sub(M, N) \overset{d}{\to} Surj(TM, TN)$$

is a weak equivalence.

Proof. Take $E(M) = M \times N \to M$. Then $E^{(1)}(M)_X = \Gamma_{\text{local at } x}(M \times N \to M)$ which is local maps $M \to N$ up to the equivalence that $f \sim g$ if $f(x) = g(x)$ and $df_x = dg_x$. Therefore this is the same as the union over $y \in N$ of $Lin(T_xM, T_yN)$ by the map taking f to df_x. Sections of this $\Gamma E^{(n)}(M)$ are maps $TM \to TN$. It is a map from the union over x and y of linear maps $T_xM \to T_yN$, and we can take a bundle map

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & TN \arrow[d] \\ M \arrow[r] & N \end{tikzcd}$$

The top map takes v_x to $s(x)(v_x)$ and the bottom takes x to $pr_{TN}s(x)(\sigma_x)$. So we can identify sections of the one-jets with bundle maps. On the other hand, a bundle map gives a section which takes x to α_x.

Now we know the sections of the one-jets. We need to impose a relation, $E_0^{(1)}(M)$ are maps α which are surjective. So sections of this are fiberwise surjective bundle maps $Surj(TM, TN)$. So we can go $\Gamma_0 E^{(1)}(M)$ is a weak equivalence with target $Surj(TM, TN)$. But what is the domain of this map? Since $M \times N$ is trivial, $\Gamma E(M)$ is maps from M to N. So a map from M to N with a fiberwise surjective induced map is a submersion. So the domain is $Sub(M, N)$.□

B Sphere eversion

Theorem 4 *(Smale) Let $i : S^2 \to \mathbb{R}^3$ to be the standard embedding of the sphere. Let i' be the map given by taking this embedding and inverting the inside and outside, essentially.*

Let M have dimension less than N and take the trivial bundle again $M \times N \to M$. Now we let $E_0^{(1)}$ be the bundle maps which are injective on the fibers. Then for M open we have immersions of M into N weak equivalent to injective maps $TM \to TN$. This is also true (due to Phillips) when M is closed.

If we know that, just take $M = S^2$ and $N = \mathbb{R}^3$. Then by the h-principle, these two immersions are homotopic if and only if the derivatives di and di' are homotopic by injective bundle maps. That's checkable.□

C closed manifolds
For example, $Sub(S^1, \mathbb{R}) \xrightarrow{d} Surj(TS^1, T\mathbb{R}) = Map(S^1, \mathbb{R}) \times Map(S^1, GL_1)$ which is nonempty but submersions of S^1 into $\mathbb{R}$ are maps where $df_x = 0$ for all x which is empty for S^1 by compactness. So you can't always use closed manifolds.

Now let's start a sketch of the proof. The idea is to decompose M into pieces (handles), prove it for handles, and then patch together the handles. That's the idea.

Definition 3 *A manifold N of dimension n is obtained from M by attaching a k-handle if $N = M^\urcorner \sqcup_{\phi_k} D^k \times D^{n-k}$. Here $M^\urcorner = M \sqcup_{\delta M} \underbrace{\{(x,t) \in \delta M \times [0,1] | t \leq g(x)\}}_{=:U}$ for $g : \delta M \to (0,1]$. This is to give a collar. Also $\phi_k : (collar \delta D^k) \times D^{n-k} \hookrightarrow U$.*

[Picture]

Definition 4 *A* handlebody decomposition *for M is*

$$D^m = M_0 \subset M_0^\urcorner \subset M_1 \subset M_1^\urcorner \subset \cdots$$

so that $M \cong \bigcup^\infty M_i$ and $M_{i+1} = M_i$ with a handle attached.

Proposition 1 *For M open, there is a handlebody decomposition with handles of dimension strictly less than the dimension of M*

This is proven via Morse theory. How do we use the handlebody decomposition to prove Gromov's theorem?

The idea now is to go by induction on the dimension of handles and the number of handles attached.

Here is the key proposition that will be assumed, which will be proven in the next talk

Proposition 2 *The h-principle map is a weak equivalence for $M = D^n$*

Proposition 3 *Restrictions*

$$\Gamma E_0^{(r)}(M^\urcorner) \to \Gamma_0^{(r)}(M)$$

and

$$\Gamma_0 E(M^\urcorner) \to \Gamma_0 E(M)$$

are fibrations (and weak equivalences)

Proposition 4

$$\Gamma E_0^{(r)}(D^k \times D^{n-k}) \to \Gamma E_0^{(r)} \underbrace{\{x \in D^k | \frac{1}{2} \leq ||x|| \leq 1\} \times D^{n-k}}_{A}$$

is a fibration, and likewise for $\Gamma_0 E$ for k strictly less that n.

Here is the point of the proof:

Lemma 1 *Suppose that the h-principle map is a weak equivalence for M. Then it works for $M' = M^\urcorner \sqcup_{\phi_k} D^k \times D^{n-k}$.*

The proof is by induction on k. For $k = 0$, attaching a zero-handle is taking the disjoint union with D^n. Then we know this because it is true for $M^\urcorner$ and for D^n.

Now let's see the induction step. Suppose our statement is true for handles of dimension [index] less than k. Consider

$$\begin{tikzcd} \Gamma_0 E(D^k \times D^{n-k}) \arrow[r] \arrow[d] & \Gamma E_0^{(r)}(D^k \times D^{n-k}) \arrow[d] \\ \Gamma_0 E(A \times D^{n-k}) \arrow[r] & \Gamma E_0^{(r)}(A \times D^{n-k}) \end{tikzcd}$$

By our proposition, the vertical maps are fibrations and by proposition one, the top horizontal map is a weak equivalence. Then by induction the bottom horizontal map is a weak equivalence, since $A \times D^{n-k}$ is just $D^{n\urcorner}$ with a $k-1$-handle attached. So $A \times D^{n-k} = (D^k \sqcup_\phi D^{k-1} \times D^1) \times D^{n-k}$.

So by the five lemma, the fibers of these two fibrations are weakly equivalent.

Now consider another diagram

$$\begin{tikzcd} \Gamma_0 E(M^1) \arrow[r, "J^{(r)}"] \arrow[d] & \Gamma E_0^{(r)}(M^1) \arrow[d] \\ \Gamma_0 E(M^\urcorner) \arrow[r] & \Gamma E_0^{(r)}(M^\urcorner) \end{tikzcd}$$

The vertical maps are the pullbacks of the vertical maps of the last square by the restriction of $A \times D^{n-k}$ to $M^\urcorner$. Therefore they are fibrations. Note that this means $\Gamma_0 E(M') \to \Gamma_0 E(M)$ and $\Gamma E_0^{(r)}(M^1) \to \Gamma E_0^{(r)}(M)$ are fibrations.

The lower horizontal map is a weak equivalence by hypothesis, supposing that the h-principle holds for M, and then the fibers of this diagram are the same as the fibers of the other diagram, and those fibers were equivalent, so by another five lemma, the top map is a weak equivalence.

The end of the proof is, I have two minutes. Take a decomposition $D^n = M_0 \subset M_0^\urcorner \subset M_1 \subset \cdots$ for M By our assumption, the h-principle holds for the disk. By our lemmas, it holds for each M_i. If the sequence is finite, we are fine. We notice that $\Gamma E_0^{(r)} M$ is the inverse limit of $\Gamma E_0^{(r)} M_i$. The second thing is the same for $\Gamma_0 E(M)$ and $\Gamma_0 E(M_i)$. We also know that these maps $\Gamma_0 E(M_{i+1}) \to \Gamma_0 E(M_i)$ and $\Gamma E_0^{(r)}(M_{i+1}) \to \Gamma E_0^{(r)}(M_i)$ are fibrations. So it holds for the limit as well.

3 Second part of the proof of Gromov's theorem

Proposition 5 $\underbrace{\Gamma_0(D)}_{C^r\text{-sections of } E \text{ landing in } E_0^{(r)}} \to \underbrace{\Gamma(D)}_{\text{continuous sections of } E^{(r)}}$ *via $f \mapsto j^r f$ is a weak homotopy equivalence.*

Proposition 6 *$\Gamma(M^{\urcorner}) \to \Gamma(M)$ and $\Gamma_0(M^{\urcorner}) \to \Gamma_0(M)$ are Serre fibrations and weak homotopy equivalences.*

Proposition 7 *$\Gamma(A) \to \Gamma(B)$ or $\Gamma_0(A) \to \Gamma_0(B)$ are Serre fibrations for $A = D_\lambda^{[0,2]} \times D_{n-\lambda}$ and $B = D_\lambda^{[1,2]} \times D_{n-\lambda}$*

Let's prove the first proposition. We want first to see that $\Gamma(D) \to \Gamma(0)$ is a weak equivalence, where the map is evaluation at 0. First we see $D \subset \mathbb{R}^n \subset M$, and have maps $T_y : \mathbb{R}^n \to \mathbb{R}^n$ mapping $x \mapsto y + x$, just translation by the vector y in $\mathbb{R}^n$. We try to construct a homotopy inverse to evaluation at zero using these. We want to try to start with a section over zero and make a section over the disk. More concretely, I define $I : \Gamma(0) \to \Gamma(D)$ just by $I(j)(x) = \overline{T_x}(j)$. Now by naturality, this $\overline{T_x}$ is

$$\begin{tikzcd} E \arrow[r, "\overline{T_x}"] \arrow[d] & E \arrow[d] \\ \mathbb{R}^n \arrow[r, "T_x"'] & \mathbb{R}^n \end{tikzcd}$$

So $ev_0 \circ I = id_{T(0)}$. On the other hand, $I \circ ev_t \cong id_{\Gamma(D)}$ where $H(g,t)(x)$ is an explicit homotopy defined by

$$\begin{cases} g(x) & x \le t \\ T_{x - \frac{tx}{|x|}} g(\frac{tx}{|x|}) & \end{cases}$$

Now to prove the statement it's enough to show $\rho = ev_0 \circ j^r$ is a weak homotopy equivalence.

To see that, we first trivialize and F is the fiber of $\begin{matrix} E \\ \downarrow \\ D \end{matrix}$ and embed $F \hookrightarrow \mathbb{R}^N$ with a tubular neighborhood $W \to F$. So ρ goes from $\Gamma_0(D) \to \Gamma(0)$. So let's see surjectivity. Given $f : S^i \to \Gamma(0)$, we want to show that it's the restriction of an element in a homotopy group of $\Gamma_0(D)$. Using several identifications, we see $\Gamma(0) \subset J_0^r(D, F) \subset J_0^r(D, \mathbb{R}^N)$, which because these have polynomial representatives, we can take these as in $C^\infty(D, \mathbb{R}^N)$. I call this total map here G. It associates to a jet at zero, a polynomial function defined on the whole disk, in $\mathbb{R}^n$. So for each $s \in S^i$, $G \circ f(s)$ is a polynomial function on D.

So there existss an $\epsilon > 0$ so that for all $s \in S^i$, $G \circ f(s)|_{D_\epsilon(0)}$ has values in W and its r-jets lie in E_R^r. Then we can look at $k(s) = r \circ G \circ f(s)$ has the correct r-jet at 0. Now we just have to make this globally defined, we will take h from the unit disk to the ϵ-disk an isomorphism which is the identity on a small neighborhood of zero, then $s \mapsto \overline{h^{-1}}k(s) \circ h$ can be chosen as $\overline{f}$. This is $S^i \to \Gamma(D)$.

Let's go to proposition two, we want to show that the restriction from the collared neighborhood is a Serre fibration.

I want to identify $M^{\urcorner} = M \cup (\delta M \times I)$ and we want to show that $\Gamma_0(M^{\urcorner}) \to \Gamma_0(M)$ is a

Serre fibration. That means, you have some diagram

$$\begin{tikzcd} Q \times \{0\} \arrow[r] \arrow[d] & \Gamma_0(M^{\urcorner}) \arrow[d] \\ Q \times I \arrow[r] \arrow[ur, dotted] & \Gamma_0(M) \end{tikzcd}$$

and you want to lift the homotopy to the dotted arrow. So $H : Q \times I \times M \cup Q \times \{0\} \times M^{\urcorner} \to E \times Q \times I \to M^{\urcorner} \times Q \times I$ and I want to extend it to $Q \times I \times M^{\urcorner} toE \times Q \times I$.

So if I have this H, I can extend it still to a small neighborhood of the domain. I use here an exponential property. The function and derivatives are continuously parameterized. I first claim that I can extend to $\tilde{H} : Q \times I \times (M \cup \delta M \times [0, \epsilon]) \cup Q \times [0, \eta] \times M^{\urcorner} \to E \times Q \times I$.

You use, I guess, that $C^{\infty}(M^{\urcorner}, \mathbb{R}) \to C^{\infty}(M, \mathbb{R})$ are Serre fibrations, and use openness and so on, adapted tubular neighborhood.

Now, if one has that, it is reasonably easy, one takes isotopy of the large domain into this domain. Should I write that down?

For the other thing, it's much easier, because $\Gamma(M^{\urcorner}) \to \Gamma(M)$ is a Serre fibration just follows from the covering homotopy property of locally trivial fiber bundles, with E^r_R over $M^{\urcorner}$.

So the weak homotopy equivalence is similar to the last one. These are the beginning, now let's get really involved with the third proposition.

Again, show that these two maps $\Gamma(A) \to \Gamma(B)$ and $\Gamma_0(A) \to \Gamma_0(B)$ are Serre fibrations. The first one is again easy, so I'll concentrate on the holonomic sections. So again I'll write down

$$\begin{tikzcd} Q \times \{0\} \arrow[r] \arrow[d] & \Gamma_0(A) \arrow[d] \\ Q \times I \arrow[r] \arrow[ur, dotted] & \Gamma_0(B) \end{tikzcd}$$

Again, you can transform that into $f : A \times Q \times \{0\} \cup B \times Q \times I \to E \times Q \times I$. We are given such an f and now the task is to extend it, extend the domain to $A \times Q \times I$. That's much more difficult. Happily, first a tubular neighborhood, at least we can extend it to some small neighborhood. So we can assume f is defined on a slightly larger domain which is $D^{[\alpha,2]}_{\lambda} \times D_{n-\lambda} \times : \times I \cup Q \times \{0\} \times A$. Here $\alpha < 1$.

Now, we have f on a larger domain but it's still not easy. We construct a family of additional functions μ_i with a partition $0 = t_0 < t_1 < \cdots < t_N = 1$ where $\mu_i : D^{[\alpha,2]}_{\lambda} \times D_{n-\lambda} \times Q \times [t_i, t_{i+1}] \to E|_A \times Q \times I$.

I should also choose $\alpha < \beta < \gamma < 1$. These μ_i can be chosen so that $\mu_i(x, y, p, t) = \mu_i(x, y, p, t_i)$ for $|x| \in [\alpha, \beta]$ and $t \in [t_i, t_{i+1}]$. We forget the Q and I factors and project to $E|_A$, it's time independent for $|x|$ in this range.

The other property is that μ_i restricted to $D_\lambda^{[\gamma,1]} \times D_{n-\lambda} \times Q \times [t_i, t_{i+1}] \cup D_\lambda^{[\alpha,2]} \times D_{n-\lambda} \times Q \times \{t_i\} = f$ restricted to the same domain.

The distance between the t_i one has to choose very small, and between f at the beginning and f at the end, one takes these sufficiently close to each other, so that the values of f are sufficiently close. One flows along the geodesic, but only at that one end, on the other end there is no flow at all. f traces a path, and for each $\tilde{t}$, if we take the geodesic between it and the next one, and fixing $\tilde{t}$ and vary x y and p, define the function where you take the endpoint of the geodesic to $f(x, y, p, t)$. You can also say, take the beginning point of the geodesic. In between one takes a fixed portion of the geodesic, and walks along it to the endpoint.

So assume you have such μ_i. We can still construct a global function only stepwise. So we introduce a new name, namely g_i. We want $g_i : D_\lambda^{[0,2]} \times D_{n-\lambda} \times : \times [0, t_i] \to E$. These should have the properties that $g_i|_{t=0} = f|_{t=0}$ and

$$g_i|_{D_\lambda^{[\epsilon_i,2]} \times D_{n-\lambda} \times : \times [0,t]} = f|_{\text{same domain}}$$

for $1 > \epsilon_N > \cdots > \epsilon_1 \cong \gamma$. So we lose ground but it doesn't matter, as long as we stay the same between 1 and 2.

If we can construct these g_i then g_n will do the job. It is everywhere defined, agrees with f for $t = 0$ and then is okay from a little before 1 to 2. We'll construct this by induction. Start with $i = 1$ and

$$g_1(x, y, p, t) = \begin{cases} \mu_1(x, y, p, t) & |x| \geq \alpha \\ f(x, y, p, 0) & x \leq \alpha \end{cases}$$

for $t \in [0, t_1]$. We can extend this by time independence to the left. Okay? Assume by induction that we have constructed g_i. Now we want to construct g_{i+1}. To construct g_{i+1} consider the following isotopy of $D_\lambda^{[0,1]} \times D_{n-\lambda}$. Here is the first and last time we use $\lambda < n$, to construct this isotopy. I will draw a picture, slightly asymmetric. Now the isotopy, we care what it does with an ϵ-tube around 0. The image of this tube at the end of the isotopy looks like this [picture]. What are the important things about the picture? The first property is that $H_t = id$ for $t \leq \frac{t_{i+1}}{2}$. The second is that on a neighborhood of the boundary, $H_t = id$. On a neighborhood V of $S_\lambda^{\epsilon_i} \times D_{n-\lambda}^{[0,\epsilon]}$ also $H_t = id$. Next $H_t(D_\lambda^{[\alpha,2]} \times D_{n-\lambda}^{[0,\epsilon]} \subset D^{[\alpha,2]} \times D_{n-\lambda}$, and the most important property, $H_{t_{i+1}}(H_\lambda^{[a,b]} \times D_{n-\lambda}^{[0,\epsilon]}) \subset Int(D_\lambda^{[\alpha,\beta]} \times D_{n-\lambda})$. The important thing is that the part that was between a and b is now over the part between α and β, which was important because our μ_i was time independent. That only works if we have a dimension in $D_{n-\lambda}$.

If we have the homotopy, I can write the definition of g_{i+1} on the region that I erased. So g_{i+1} on $D_\lambda \times D_{n-\lambda}^{[0,\epsilon]}$ is by the following picture. [picture] So g_{i+1} is defined as $g_i, f \circ H_t$, $\mu_{i+1} \circ H_{t_i}$ and a trivial deformation in different parts. It's not everywhere g_i, we always change it. So it's g_i until t_i and ϵ_i, $f \circ H_t$ from ϵ_i to 1 until t_i, $u_{i+1} \circ H_{t_i}$ from b to 1 and from t_i to t_{i+1}, elsewhere being the trivial deformation.