TFT workshop

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1 Jacob Lurie

[This afternoon at 4:10 we have a colloquium by Sasha [unintelligible]on quantization and the quantum dilogarithm. Without further ado, the fourth and final lecture of Jacob]

I described some of the terms last time, like fully dualizable objects. But it might be hard to understand what fully dualizable means. At the same time I'd like to tell you about the action of the group $O(2)$, which acts on the space of fully dualizable objects whenever $\mathscr{C}$ is a tensor $\infty - 2$ category.

What does it mean for $O(2)$ to act? Well, this is like $SO(2)$ and $O(1)$. The $O(1)$ should act by swapping objects with their duals. What does it mean for $SO(2)$ to act on an ∞-groupoid. Let's call that groupoid X. That's something we can really think of as a space. What does it mean in the world of spaces? We have a map $SO(2) \times X \to X$. So for x we have a map of the circle into X, based at x. So this represents a class in π_1 based at x. If X is a category, then X is the fundamental groupoid, and then this is an automorphism of x. So every object $x \in X$ has a canonical automorphism. I've only said what it means to have a map $SO(2) \times X \to X$. There are associativity properties that translate into conditions that these automorphisms need to satisfy.

So what are these canonical automorphisms? In this setting, let $\mathscr{C}$ be a tensor $\infty - 2$ category and let X be a fully dualizable object. What does it mean to be fully dualizable? There exists an $X^\vee$ and evaluation and coevaluation maps $e : X \otimes X^\vee \to 1$ and $c : 1 \to X^\vee \otimes X$. A must ask that these satisfy some finiteness, so I can ask that e has a left adjoint, which goes from $1 \to X \otimes X^\vee$, which is the same ase $X^\vee \otimes X$. So e^L and the coevaluation map, how are they related? They're not the same in general, but they're off by some endomorphism of X. A map from $1 \to X^\vee \otimes X$ is the same as a map $S_X : X \to X$, and S is this canonical automorphism. I'm calling it S_X to make you think of Serre functor. If that doesn't mean anything to you, ignore it. How do you see that this is an automorphism? If c has a left adjoint, then $c^L : X^\vee \otimes X \to 1$, which is adjoint to some map $T : X \to X$, and now as an exercise, axiomatics force $S_x = T_x^{-1}$.

What's the upshot of this discussion? It fives you a way to test whether an object is fully dualizable. If we want to make something have duals, first you want to throw out morphisms that don't have adjoints, you might think this is a multistep process, and you have to throw out objects that don't have duals. Let me give you a simple criterion: it is fully dualizable if it has a dual and the evaluation and coevaluation have left adjoints. c has a left adjoint, which tells you that e has a right adjoint, which are off by this T_x. So that's a twist, and that's the $SO(2)$ action you have to put in on fully dualizable objects.

How do you know that S and T are related to this $SO(2)$ action? The answer is, check in the universal case. It might be hard to think about these in an arbitary $\infty - 2$ category; this is automatically the image of a point under a functor from $nBord$. You can check it in the case where $\mathscr{C}$ is the framed version of $2Bord$. Let me draw some pictures. Let $\mathscr{C}$ be $2Bord^{fr}$, so X is a positive point and $X^\vee$ is the negative point. Then evaluation and coevaluation are cup and cap. Naively, the adjoint is the same picture as the coevaluation. But this is not enough in the framed case, I need to put in a framing. We're luck that a 1-dimensional manifold in the blackboard comes with a framing. Here's how I would have to draw my orientations, if you think about how these things have to be framed, you find that the adjoint to e and c have to be framed differently, and they have to differ by a generator of $SO(2)$?

Now I'd like to do a specific example that you can really get your hands on. Let's fix a field k, and let $\mathscr{C}$ be a 2-category with objects k-algebras, morphisms bimodules, composition the tensor product, and 2-morphisms which are maps of bimodules. You might ask when an object of $\mathscr{C}$ is fully dualizable? Such an object is an algebra. The first question is whether it's dualizable. The first claim is that any algebra is dualizable, and the algebra is the opposite algebra. The tensor product is the tensor product over k. To check this, I need to supply evaluation and coevaluation maps, which are one-morphisms in the category. The bimodule you choose to write down for both of these is the canonical one, namely A. When is A fully dualizable? Both of these maps A have to have adjoints. They translate into the condition that A is dualizable as a k-module, and also as an $A \otimes A^{op}$-module. Now dualizability as a k-module, that's easy to test. That just means A is finite dimensional as a vector space. The finiteness shows up here when you look at morphisms. The $A \otimes A^{op}$-module is stronger. These amount to the same thing as requiring A to be semisimple over k.

You have many isomorphisms, and a product of matrix rings, which is what this is over $\mathbb{C}$, is just isomorphic to a product of $\mathbb{C}$.

Let's move to an $\infty - 2$ category, and let's use differential graded algebras, dgbimodules, the derived tensor product, and then 3 morphisms are chain homotopies and so on. This is an example of an $\infty - 2$ category. Every object is dualizable, and for fully dualizable you have the same criterion. This is a condition Bertrand discussed in his lectures, that is, A must be smooth and proper. This is motivated by algebraic geometry, you can find a differential graded algebras so that [unintelligible]sheaves are bimodules, and if the variety is smooth (proper) then the algebra will be smooth, that is dualizable as a k-module (proper, that is, dualizable as an $A \otimes A^{op}$-module).

There are many things that satisfy one of these conditions, but not both. Suppose A satisfies condition 2. It has a dual A^* as an $A \otimes A^{op}$ module. This is a bimodule for A, and that morphism is supposed to encode the $SO(2)$ action. If we only assume A satisfies one of these conditions, it's not invertible. So A^* might not be an invertible bimodule. But suppose it is. It turns out to be interesting to restrict your attention to a class of objects for which it is.

Definition 1 *Let $\mathscr{C}$ be an $\infty - 2$ category. Say X is $1\frac{1}{2}$-dualizable if X is dualizable, and $X \otimes X^\vee \to 1$ has a left adjoint, and that left adjoint, and the morphism induced, $S_X : X \to X$, is invertible.*

There are two claims. The first is that the group $SO(2)$ acts on $1\frac{1}{2}$-dualizable objects, and the second claim, due to Kevin Costello, is that there is an analogue of the cobordism hypothesis that says that there's a description of $1\frac{1}{2}$-dualizable objects. So $(\mathscr{C}^{1\frac{1}{2}\text{-dualizable}})^{SO(2)} \cong Fun^{\otimes}(2Bord^{nc}, \mathscr{C})$, which is the noncompact analogue. So this is defined like $2Bord$, except it has fewer 2-morphisms, now every nonempty component of Σ has nonempty outgoing boundary.

I'm attributing this to Kevin, but any proof of the cobordism hypothesis involves both wrestling with category theory and manifolds, and Kevin has made it so you would not have to touch manifolds.

Let me give an application, the string topology operations. What you want to do, to apply this theorem, you want $\mathscr{C}$ to be the $\infty - 2$ category I mentioned before. Let M be a compact manifold. Choose a basepoint (let M be connected) and let A be the singular chains on the based loop space $A = C_*(\Omega M, k)$. The claim is that A is $1\frac{1}{2}$-dualizable, and is $SO(2)$-fixed.

Hopefully this gets back a bit to Matt's question. What does it mean to be $1\frac{1}{2}$-dualizable and $SO(2)$-fixed. First of all it means dualizable, and then also dualizable over $A \otimes A^{op}$. This has to be invertible, the morphism earlier, but also trivial, that's what it means to be fixed. So $x \to A^* \otimes A$, we need to make sure that A is dualizable as a bimodule, so we need to supply this map and then make sure that A^* looks like A, which means we need to supply a map $k \to A \otimes_{A \otimes A^{op}} A$, which is the Hochschild homology of A, so if $A = C_*(\Omega M)$, chains on the based loop space, then $HH_* A$ is chains on the free loop space $C_*(LM)$. We need a map from k into chains on the free loop space. You need a chain where $SO(2)$ acts, well, you have constant loops, so you have the chains of M, which go in as constant loops, and you take the fundamental class $[M]$. If I take the fundamental class here, the requirement that the map be nondegenerate turns out to be equivalent to Poincaré duality.

If you believe the cobordism hypothesis, what you learn is that you get a functor Z from $2Bord^{nc}$ into this category $\mathscr{C}$. What can you say about this? On the circle it gives the free loop space. The conclusion is that the chains on LM has lots of operations. So you have a pair of pants, and that gives some kind of multiplication. This gives a chain complex of vector spaces, and you have a multiplication here which descends to the homology. So on the homology of the free loop space you get a product, and these are the Chas-Sullivan string topology operations. A consequence of the fact that they can be constructed this way is, that these are defined on homology manifolds.

These lectures rests on or is connected to Baez Dolan, also Costello, and the rest of the time I want to talk about connections to Galatius Madsen Tillman Weiss. These four have done work in homotopy theory which is a homotopy theoretic reflection was a guide toward the correct formulation of the cobordism hypothesis. I want to describe their work, and go back to the origins and the Mumford conjecture.

Let Σ have genus g. Let's study $BDiff^+(\Sigma)$, which is the base for a universal fiber bundle whose fiber is surfaces like Σ. It would be nice to understand, say, the cohomology of the base. On E you have a canonical rank two vector bundle, with an Euler class in $H^2(E)$. You can take powers of this Euler class and you get something in $H^{2n+2}(E)$, and then you can integrate, pushing these classes down, you get a class $\kappa_n \in H^{2n}(BDiff^+(\Sigma))$, so you get a map $\mathbb{Q}[k_1, \ldots,] \to H^*(BDiff^+(\Sigma), \mathbb{Q})$. Mumford conjectured that this is almost an isomorphism, that it is an isomorphismin a range of degrees that approaches ∞ as g grows. This was proven by Madsen and Weiss.

Let me discuss a little of the modern incarnation of their proof and its relation to the cobordism hypothesis. Let me break it down into four steps. You want to talk about the stable properties that relate to genus g. So we

1. define a space X and a map $BDiff^+(\Sigma) \to X$.
2. Prove that this is an isomorphism in cohomology in a range of degrees, which goes to ∞ as g grows,
3. identify X with another space Y, and then fourth compute the cohomology of Y.

I'd like to describe to you steps one and three. Step two is the hardest, you need the Harer stability theorem and so on, and I will ignore that and look at one and three. Note that $BDiff^+(\Sigma)$, this object $2Bord$ is built out of $BDiff^+(\Sigma)$, this has objects, one morphisms, and 2-morphisms, which are classifying spaces of surfaces. So $BDiff^+$ appears insoide $Hom_{2Bord}(\emptyset, \emptyset)$. What does this say about the relationship? The first thing I want to do is play with $2Bord$ a little bit and get it to live in the world of homotopy theory. If $\mathscr{C}$ is any higher category, you can make an ∞-groupoid or space $|\mathscr{C}|$, which you can think of as the space you make by inverting all morphisms of $\mathscr{C}$. What can you say, what is the classifying space of $2Bord$? It's like a space, and there isn't quite a map from $BDiff^+(\Sigma)$ in, because points there are like two-morphisms, so instead there is a map into $\Omega^2|2Bord| =: X$. You have the hard step two, and now we want to identify X with the other space Y. Here the cobordism hypothesis will help us. What kind of space is X? We took the classifying space of an $\infty - 2$ category, for example if you think of a space as an $\infty - 2$ category, then it is its own realization. This has the tensor product, so there sholud be an E_∞ structure, so X is an E_∞ space where you have not just a multiplication but also an inverse, a grouplike E_∞ space, so X is an infinite loop space. So X can be identified with ΩX_1, and X_1 can be identified with ΩX_2, and so on. Say X' is some other infinite loop space, you might ask, what are $Hom(X, X')$ in the world of infinite loop spaces, think of spaces as categories in which morphisms are invertible and infinite loop spaces as tensor categories where all morphisms are invertible. This is the same thing as tensor functors from $2Bord$ into X', and suppose that we were working with $2Bord^{fr}$, the cobordism hypothesis says that this is freely generated by a single object, so the group completion in spaces in generated by a single point. So that tells you that this is QS^0, so $\lim_{n\to} \Omega^n S^n$. So we need to understand how $SO(2)$ acts on QS^0. This action is roughly via the J homomorphism. So what is the geometric realization of $2Bord$? One way that it was stated, was it's by quotienting by the action of $SO(2)$, which is compatible with passage to the classifying space, and so you divide out by the action of $SO(2)$. The quotient is formed in the world of infinite loop spaces, This is called $\Omega^{\infty-2}MTSO(2)$. That notation is for whatever you get out of these homotopy theoretic constructions. We've completed step three, we got this from the cobordism hypothesis, which says that $X \cong \Omega^{\infty}MTSO(2)$. The remaining step is to compute the homology. Algebraic topologists are very good at doing computations. You might bring this to an algebraic topologist, they might be able to help you. If you say $\Omega^2|2Bord|$, not knowing anything else about the situation they might not be able to help you, but if you give them something like $\Omega^{\infty}MTSO(2)$, then you've given them something where they can get to work. Here you're only asking for the rational cohomology of a space, well, there's, you're guaranteed that if it's go a reasonable presentatino in homotopy theory, that calculation can't be too difficult. I think I will stop there. Thank you.