TFT workshop

Gabriel C. Drummond-Cole

May 21, 2009

1 Jacob Lurie

I explained that topological field theories should be functors from a bordism category to an n-category. In the second lecture I gave an idea of what an n-category or an $\infty - n$ category is.

Let me look at $Bord_n$, where you have k-morphisms for every k which are invertible for $k > n$. The objects are 0-manifolds, the morphisms are bordisms, and so on. The n-morphisms are bordisms of things with corners. Above n the bordisms are diffeomorphisms and so on.

Our goal for this lecture is to understand $Bord_n$, or in other words understand what it means to give a functor into another $\infty - n$ category. This is symmetric monoidal, so it has a tensor product which corresponds to disjoint union of manifolds. In my first lecture I stated the cobordism hypothesis. If I replace this by its framed analogue, you only need to specify what you are going to assign to one object, namely the point. If our target was vector spaces, you could only see finite dimensional vector spaces.

Let's first think about this when $n = 1$. On Monday we went through an analysis. A functor from $Bord_1$ to $Vect$ showed us that he point was taken to X, and this had to be finite dimensional. X needed to have a dual in our category $\mathscr{C}$. Let's start in the language of ordinary categories. Let $\mathscr{C}$ be a symmetric monoidal category. Let X be an object. Imagine that this is a vector space. We are going to define the notion of what it means to have a dual. A dual consists of another object, which we'll call $X^\vee$, and then a relatioship between these two objects, $e : X \otimes X^\vee \to 1$, (this is a bordism from two to zero points) and dually $coev : 1 \to X \otimes X^\vee$. These sholud be related by Zorro's lemma, that $X \to X \otimes X \otimes X \to X$ should be the identity.

There is a similar relation for $X^\vee$. The mark of Zorro proves that the positively oriented point and the negatively oriented point are dual. It's easy to see that a tensor functor must carry duals to duals. This sounds like additional data, but it's not, but if a dual exists, it is uniquely determined up to canonical isomophism. This is true in category theory, and there is an analogue in higher category theory. The notion there amounts to being dualizable here. It turns out that if X has a dual in the homotopy category, then it has a dual in a very strong sense in the $\infty - 1$-category. So what is the upshot of this discussion. Let me state the cobordism hypothesis in dimension one. The one-dimensional cobordism hypothesis says what? Let $\mathscr{C}$ be an $\infty - 1$ category, now you can look at tensor functors from $Bord_1$ into $\mathscr{C}$, and this can be identified with the collection of dualizable objects. When $\mathscr{C}$ is an ordinary category, the same thing works. This is an argument you can carry through in ordinary categories. In higher categories it's a little harder to make this work.

This is nontrivial, even in dimension one. By this I mean if you're willing to let $\mathscr{C}$ be an $\infty - 1$ category, you won't be able to do this just by eyeballing.

I want to mention an implication. Note that $Bord_1$ itself, this is an $\infty - 1$ category. I don't have a set, but a space of maps. Consider the simplest example, $Hom_{Bord_1}(\emptyset, \emptyset)$. If I was talking about the ordinary category, any pair of diffeomorphic bordisms would be identified. Here I know that this is the disjoint union over all n of classifying spaces for 1-manifolds of the form $S^1 \sqcup \cdots S^1$, where there are n copies. Consider the case $n = 1$. We have the classifying space of S^1, well, of orientation preserving diffeomorphisms, which are just S^1 again, or $\mathbb{CP}^\infty$, which has many names. This is a reasonable space but not a discrete space. If I give you a functor into $Bord_1$, this space of maps will map to an analogous space inside $\mathscr{C}$, so I'll get a map $\mathbb{CP}^\infty \to Hom(1, 1)$. This, on the circle, will assign the dimension of X, the composition of the coevaluation with the evaluation. What is this saying? What this is actually computing, we didn't compute the map on all of $\mathbb{CP}^\infty$, we did this calculation by breaking the circle into two pieces. This is really the classifying space for breaking the circle into two pieces. We no longer understand how the circle is acting. This field theory Z should assign to the circle the dimension of X which should have an action of the symmetry group of the circle. Any time you provide X, there is a canonical circle action on the dimension of X, which is nontrivial in examples. You probably won't produce this circle action just by staring at these pictures and seeing what looks right. The cobordism hypothesis tells us that there is more information, something geometric here.

Now I want to ask the question, what if X is greater than one? Now, what happens if $n > 1$. Then we need a finiteness condition. Let's consider $Bord_1$ and $Bord_2$. Because I don't want to deal with the kind of issues that I was talking about at the end of the first lecture, I will add a framing to each one of these. Why have I introduced this more complicated notion? Here's one justification. One strategy I might use is induction. Well, if I've made the definitions, used today's conventions, I get a functor $Bord_1 \to Bord_2$. As Orit explained, any diffeomorphism can be turned into a bordism.

This is not an equivalence of categories, so if we're interested in studying functors, we don't expect the classification to be the same. If you have the belief that giving these functors means giving things that satisfy a finiteness condition, the two dimensional case should need a stronger finiteness condition.

To do this, I need to embark upon a digression. So here's a digression. Let me start by giving you an example of a 2-category that I probably should have given in the previous lecture. The objects are categories, the morphisms are functors and the 2-morphisms are natural transformations. In some sense, category theory is the study of this category. What does that buy you? One of the things that category theory buys you is a common language for mathematical topics that are ubiquitous. For example, the cartesian product can be described by a universal property. What this is supposed to buy you is an opportunity to export things from one context to another. Let's take concepts in one 2-category and apply them to other 2-categories. Let me apply this to a particular concept from ordinary category theory, the notion of adjoint functors.

Let $\mathscr{C}$ and $\mathcal{D}$ be categories. You'll recall a pair of functors $f : \mathscr{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathscr{C}$ are adjoint if $Hom_{\mathscr{C}}(C \to g(D)) \cong Hom_{\mathcal{D}}(f(C) \to D)$ for all objects and functorially. This gets into the details of what a category is. You can reexpress this in a purely 2-categorical language.

Let's say you had a set of functors that were adjoint in this sense? How could you encode this? You could take $C = g(D)$, so $Hom(C, g(D))$ contains the identity. This should give you a map from $f(C)$ to D. So we get a natural map from $f \circ g$ into the identity functor, and conversely, such a natural transformation of functors gives me a map from the left hand side to the right hand side.

So I'll need the opposite too, a map from the identity to $f \circ g$. I need to say also that these maps are inverse to each other. Let me write down that relationship. These two maps should give me $f \to f \circ g \circ f \to f$ should be the identity, and similarly for g. The reason I want to go into this, is to call into your attention the analogy to the notion of duality. The functors that are adjoint match up with the objects, the coevaluation and evaluation are counit and unit for the adjunction, and so on. A tensor category can be turned into a 2-category with a single object, with Hom given by objects and composition by the tensor product.

I'm getting ahead of myself. Now I should make a definition. The notion makes sense in an arbitrary 2-category. If $\mathscr{C}$ is any 2-category, and $f : X \to Y$ is a morphism, a right adjoint to f is a morphism $g : Y \to X$ together with 2-morphisms u and v, so $u : id_X \to g \circ f$ and $v : f \circ g \to id_Y$. Just like in the case of dual objects, there is some redundancy, the right adjoint are uniquely determined. This is like the condition of having a dual, so like being finite dimensional. A morphism having a right and left adjoint is some finiteness condition. This makes sense also in an $\infty - n$-category. You could give the same definition with more coherences, or demand that this holds in the truncated 2-category. These are not obviously equivalent but turn out to be.

Now I want to take advantage of this finiteness notion and do something to objects.

An $\infty - n$ category $\mathscr{C}$ has adjoints if every k-morphism has a right and left adjoint for any k strictly between n and 0. More formally, slightly, if $n = 1$ or if $n > 1$ and $Hom_{\mathscr{C}}(X, Y)$ has adjoints for all X and Y and every 1-morphism in $\mathscr{C}$ has both left and right adjoints. So this is a condition which is automatic when $n = 1$ but this is very strong for $n > 1$, never satisfied unless you arrange for it.

Let $\mathscr{C}$ be a tensor $\infty - n$ category. We'll say that $\mathscr{C}$ has duals if $\mathscr{C}$ has adjoints and every object of $\mathscr{C}$ has a dual. Because the theory of duals is a specializiation of adjoints, this should be viewed as a special case. We impose the same condition also when $k = 0$. So now let me state the cobordism hypothesis:

Hypothesis 1 *Assume $\mathscr{C}$ has duals. Tensor functors $Bord_n^{fr} \to \mathscr{C}$ can be identified with objects of $\mathscr{C}$. I put the finiteness condition on the category.*

For any $\mathscr{C}$ there exists what might be called a largest category $\mathscr{C}^{fd}$ such that $\mathscr{C}^{fd}$ has duals. If $n = 1$, having duals means that every object is dualizable. So you can throw out the objects which are not dualizable. Otherwise, you'll have to throw out morphisms that don't have adjoints. This is a process that takes place in stages. You should start at the top and work your way down to the bottom. You might not have much of your category when you're done. There's really nothing to be gained, to be lost, by restricting to the fully dualizable case because any functor, roughly speaking, any k-morphism is a bordism, has left and right adjoints given by regarding the bordism, reading it in the other direction. Because this universal property tells you that a tensor functor factors through the fully dualizable objects.

Let me revise a previous statement. Let $\mathscr{C}$ be an $\infty - n$ category. This is what I obtain when I get the fully dualizable part of $\mathscr{C}$ and then throw out nondualizable morphisms.

Let's restrict to vector spaces. A field theory is a vector space. Any natural transformation of field theories, I should get a map between X and Y, but I should get a map $X^\vee \to Y^\vee$, and I should get that the morphisms are [unintelligible]to one another.

I stated a result for the framed bordism category. Suppose I have a manifold M^m, $m < n$ remember that an n-framing is a trivialization of the tangent bundle to an n dimensional bundle. I have orthogonal transformations on these. I have this orthogonal group action, which give an action on the entire category. Let me go back to this statement, and state it in the following way, you have an action of the orthogonal group on $Bord_n^{fr}$ and therefore on everything on the right-hand side. So for any tensor $\infty - n$-category, the fully dualizable objects, those will have an action of $O(n)$. What is that action? Let me tell you in the case it's comprehensible. In $O(1)$ it's the group with two elements. You have an action of $\mathbb{Z}_2$ on objects, which is swapping with the dual. What is this saying? In the $\infty - n$ category, $n > 1$, you have an entire orthogonal group of ways to take a dual. So let's suppose that G is a topological group with a map from $G \to O(n)$. Then we can talk about manifolds endowed with a G structure, whose structure group has been reduced from $O(n)$ to G. We can use G-manifolds instead of framed manifolds to build $Bord_n^G$, the $\infty - n$ category of G-manifolds, manifolds endowed with a G structure. For the trivial group, this is $Bord_n^{fr}$. If G is $SO(n)$ then you get the oriented bordism category. You could take $O(n)$ and get an unoriented bordisms, or the Spin group and get, I don't know, $Bord_n^{spin}$.

Now you might ask, what is the analogue of the cobordism hypothesis? In the bordism category of G-manifolds, the tensor functors from $Bord_n^G$ into $\mathscr{C}$ which has duals, (otherwise I restrict), the framed bordism category maps to this. So any tensor functor gives me an object, which is the object given by evaluation at a point. So we should get an object of $\mathscr{C}$, which is acted on by the orthogonal group, and in particular, by G, this is rigged so that the object you produce is a homotopy fixed point with respect to the G action.

Theorem 1 *A field theory for all manifolds is an object of $\mathscr{C}$ invariant under a symmetry* *group corresponding to your field theory's symmetries.*

The group action of $SO(n)$ is hard in general, but in dimension one, let's take $O(1)$, then tensor functors from the unoriented bordism category, with values in vector spaces, well, first we restrict to finite dimensional vector spaces, so replacing vector spaces with their duals. So these will be vector spaces identified with their duals, so here vector spaces with a symmetric bilinear form. In the next lecture I'll talk about examples in other contexts.

2 Ben Zvi

So I wanted to finish off the story we were talking about last time. We've been drawing this same picture a lot, and we interpreted this picture as relateing $Z(S^1)$ with $EndA$ where A is what we assign to a point. So there was an action but we have the map going the other way, charge or character which goes from $End(A)$ to $Z(S^1)$, and I want to see what we get on the identity. So we'll get $Ch(A)$, which generalizes both the character of representations of finite groups and also the Chern character of a vector bundle or a dg category. This lives in $HH_*(Z(\cdot))$, or the dimension of $Z(\cdot)$. The cobordism hypothesis means that this is rotation invariant.

I don't want to get into more of this but we had this topological field theory that had to do with $\mathcal{D}$ modules, but when our fields are G bundles for G a reductive group over $\mathbb{C}$ and B a Borel subgroup. What does this give in the case of a reductive group? we get on one side an element of $\frac{G}{G}$, and an interval with two reductions at the ends, $B\backslash G/B$, and in the middle $\frac{G}{B}$. This is very familiar, we sometimes write this thing in the middle as $(G \times G/B)/G$.

What is my map? If I give you an element in g and a flag or Borel subgroup, we can project it to the class of the group element, so (g, B') can be projected either to $[g]$ or to $(B', g \cdot B')$. We'll restrict to the diagonal and look at its inverse image. We'll get a very famous diagram, the Grothiendieck Springer correspondence, we'll get $\tilde{G}/G$, so g, B but $g \in B$ now. So we can project to one side or the other. This is more familiar in the Lie algebra version. This is worth thinking about. For example, we can look at the locus where the group element is semisimple, and this maps to H^{reg}/W, the eigenvalues up to conjugation, but I can find another piece in here as an affine cotangent bundle of the flag manifold $T^*(G/B)$ resolving the singular variety N/G, where N is nilpotents. The whole diagram here, the whole set looks like the Weyl group. The collection looks like W-twisted versions. What does this have to do with our character theory? Let me give a paraphrased version of Lusztig's definition.

Definition 1 *A character sheaf on G is a $\mathcal{D}$-module on $\frac{G}{G}$ in the image of this Grothiendieck springer correspondence $\pi_*\delta$ on the Hecke category $\mathcal{D}(B\backslash G/B)$.*

Maybe I should give you the basic object in the theory, the prime example, the springer sheaf itself, so (here π maps to $\frac{G}{G}$ and δ to $B\backslash G/B$). This is $\pi_*\delta\mathbb{C}_{diag}$ which is $\pi_*\mathbb{C}_{\tilde{G}G}$. This is used to construct irreducible representations of the Weyl group, but this guy (S) as a $\mathcal{D}$-module, is the Harish-Chandra's system of invariant differential equations that are satisfied by characters of representations of G thought of as a real Lie group. What he showed is that these characters satisfy things that are very powerful.

So, this is a very beautiful object. Harish-Chandra studied this, and then Lusztig studied this, including a version. These are geometric avatars of representations of G over finite fields. These let you construct characters of representations all at once. In these examples we see that sheafs are some kind of characters, so let me rephrase the theorem from last time: what we explained last time is that character sheaves are exactly characters of these categorified representations of the Hecke category. The theorem was on the board last time, but it says that $\mathscr{H}$-mod satisfy the conditions of the cobordism hypothesis, and defines a two dimensional unoriented topological field theory, and $Z(S^1)$, the Hochschild homology or dimension of the Hecke algebra, and also the center, is exactly Lusztig's category of character sheaves.

If you give me a module for the Hecke algebra, the place where you expect [unintelligible]to live is the character sheaves.

Maybe I should say that more strongly is that not just is there an abstract relation, I don't just have maps, the universal maps, the trace map and the action map are exactly given by this correspondence. The action of the Hecke algebra is the dual of the adjoint, going in the opposite direction.

So in the remaining time, I wanted to give a very brief introduction to the geometric Langlands program. This is obviously going to have to be very sketchy. I want to move to four dimensional gauge theory. One reason is that we'll see a lot more structure.

Before I go to four dimensional field theory, there's one key theory that we've ignored, the notion of local operators, let's just say very briefly that you calculate in physics $\int e^{-S} D\varphi$, and taking a point at (x,t) in space time, so you might think that you want to measure $\int \mathscr{O}_{x,t} e^{-S(\varphi)} D\varphi$ and that's an expectation value. Let me draw the following picture. If I give my manifold M, I have M cross the interval. If I pick a point x and make a measure at time $\frac{1}{2}$, how do I formalize this? I pick a ball here and remove it. This gives me a map of $Z(Sn-1) \otimes Z(M) \to Z(M)$. What structure do these have? If I look at $Z(S^n)$, I could draw this picture. Here I have the 2-sphere, and this operation tells me that there's a map $Z(S^2) \otimes Z(S^2) \to Z(S^2)$. This defines a multiplication that depends on a configuration, not commutative but braided. This is called an E_n multiplication. That's the kind of multiplication that you see on local operators in an n-dimensional field theory.

All I want to get out of this is a multiplication that gets more commutative as n gets bigger and $Z(S^{n-1})$ is an algebra, and we get an E_n module in an appropriate sense.

Why am I bringing this up? I'd like to talk about four dimensional gauge theory. Now we have room for interesting local operators. Even in 2-dimensional gauge theory, you saw disorder operators. You take a measurement at a point of time, but the measurement was made with a ball around the point. We could look at a singularity at this point. In two dimensional gauge theory, with Γ my finite group and a conjugacy class C with an element of a group algebra $\mathbb{C}\frac{G}{G}$ that we assign to S^1 We could take a surface, but we could also insert the correlation 1_C. This changes my number from the number of G bundles, but we insist on a particular kind of singularity at the point we insert the correlation disk.

Now these are what physicists call order and disorder operators. In four dimensional gauge theory we'll see that these match up with [unintelligible].

Now we'll jack up one more dimension. So these are $4d$ gauge theories. The first one makes a lot of sense will be the B model. I'll just, the B model will be a $4d$ analogue of Z_G^Q. I'm going to tell you some of what this field theory assigns. To a three manifold this assigns the space of flat G-bundles on the manifold, and some derived version of functions on this, $R\Gamma(\mathscr{M}_G(N^3), \mathscr{O})$. What about to Σ? This will be $\mathscr{O}(\mathscr{M}_G(\Sigma))$, coherent not quasicoherent sheaves, so like finite rank vector bundles.

Just for fun, so to continue, this will be the main object, but to see, what's $B_G(S^1)$? before it was sheaves on $\frac{G}{G}$, now it's sheaves of categories on $\frac{G}{G}$. I'll think of this as sheaves of dg categories.

Finally, if you really want to apply the cobordism hypothesis, this will be coherent sheaves of $\infty - 2$-categories over BG, so $\infty - 2$ representations with an action of G.

Okay, what I wanted to focus on, is as a three and a little bit dimensional field theory. What do we want to do with this? We will ask different questions. We'll look at a surface. What kind of structure do you have? I'll look at my Riemann surface is moving along in time and I make a measurement. To know my operators I'll need to know G-bundles on the two-sphere, and then I'll use coherent sheaves on that. So the 2-sphere is two disks, so I need two disks glued together along $\frac{G}{G}$. Ignoring some dg nonsense, I'll say this is namely $\cdot/G$. I'll roughly approximate this as $\cdot/G$, so sheaves, coherent sheaves on this will be representations of G. For $x \in \Sigma$ I get $RepG$ acts on $\mathscr{O}(\mathscr{M}(\Sigma))$. I get a very large collection of commuting operators.

So this should be an E_3-category. I'll get that each point I get a ring action.

What do I want to say? What this field theory assigns to Σ is a copy of $|[unintelligible]$ for each point in the surface.

Let me assume for a second that G is finite. Then here is $\Sigma \times S^1$. On this cross section, I have, well, I can't realliy draw, and then cross a time interval, and so instead of having a point operator, I have a loop, $\Sigma \times S^1 \times I$. This is what a physicist would call a Wilson operator.

What is a loop operator? Nothing fancy, I just cut out a neighborhood and see what I can insert along the neighborhood. I have a 4 manifold, with a knat missing., so I need to calculate what I can insert here, but the simplest thing, [unintelligible]$Rep(G)$, and now I have a vector space. What are the operators I can insert? Given a loop and a representation, I can construct the Wilson loop operator. Let me just say very briefly, if I give you P as a flat G-bundle, its associated vector bundle, a flat vector bundle, now I can measure the holonomy of my connection and take its trace. This I'll call $W_{L,R}$.

Instead of an element in the representation ring, I give you an honest representation, so if

P is a G-bundle on Σ and R a representation of G, with $x \in \Sigma$, you get a vector space $P_R|_x \in Vect$. As P varies you get a vector bundle, that's a functor, that's, tensoring with $W_{x,R}$.

This is a long way to spell out what the local operators are.

This category carries a tensor category for every point in Σ. These are silly operators, they're multiplication operators. I have a bunch of commuting operators, they're all diagonalized, they're just multiplication operators. This is an answer to a question we haven't asked. You have a family of commuting operators, you want to diagonalize them, it's done.

[Are these like skyscraper sheaves?]

Yeah, they're like an eigenbasis for these operators. So what do we do, what is geometric Langlands about? This is a topological picture, due in physics to Kapustin-Witten and in math to Beilinson-Drinfeld. Now I will call this theory A_g. What is this theory? It is a four dimensional gauge theory of bundles with connections, but it's closer in spirit to the theory that depended on $\mathcal{D}$-modules. Let me just describe some of the features. What does this assign in three or two dimensions? Now Σ will be an algebraic curve, a Riemann surface. My space of fields will be holomorphic, $Bun_G\Sigma$, the stack of holomorphic G bundles on Σ. Okay, and now, what, this is my space of fields, and to build my field theory I'll use $\mathcal{D}$-modules. This category $A_G(\Sigma)$ will be the category of $\mathcal{D}$-modules on $Bun_G(\Sigma)$.

The local operators here in the physics language are called t Hooft operators or Hecke operators. There's no local information on a holomorphic G bundle, so I'll have to introduce disorder operators. What are my local operators? The physics picture, in four dimensions, I'm going to think of what kind of singularities I can insert along a loop in four dimensions. So I'll insert a kind of singularity which is a G version of a magnetic monopole along a line in four dimensions. One can work out what those are, we believe they come from linearizing the 2-sphere. So $Bun_G(S^2)$ are the possible local singularities, which set theoretically is just I get $Hom(\mathbb{C}^*, T)$ up to conjugation. If you look at G bundles on the 2 sphere is that they all come from the torus. What you discover is that they're in bijection with irreducible representations of the Langlands dual group $G^\vee$. You build a group with the dual torus. So maybe just kind of summarizing, you can make this analysis in several ways, what are the local operators I can do? You discover what your theory assigns to the 2-sphere is something roughly called $\mathscr{H}_{sph}$, the spherical Hecke category, it's $\mathcal{D}$-modules on a double coset space, for loops into G and positive loops.

Okay, so, this is, these are again a monoidal infinity category, $\mathcal{D}$-modules on a double coset space. You can directly say why this acts on $\mathcal{D}$-modules on Bun_G. Now let's just have one theorem, and the one theorem that motivates this program is called the geometric Satake theorem, and there's a long list of names, which I'll say, [unintelligible], which is that this spherical Hecke category $A_G(S^2) \cong \mathscr{O}(\mathscr{M}_{G^\vee}(S^2))$ which is $B_{G^\vee}(S^2)$, representations of the Langlands dual group, these categories are equivalent as E_3 categories. Maybe a better way to paraphrase it is that the local operators coincide.

Now that we're out of time, we can state a conjecture

Conjecture 1 *(math) $\mathcal{D}$-modules on $Bun_G(\Sigma)$ with the action of $\mathscr{H}_{sph,x}$ is equivalent to $\mathscr{O}(Loc_{G^\vee}\Sigma)$ which carries an action of Rep $G^\vee, x \in \Sigma$.*

Conjecture 2 *Electric-Magnetic duality, or S-duality. The entire field theory A_G is equivalent to $B_{G^\vee}$. There is some duality that identifies these. This encodes an incredible amount of structure, as topological field theories.*