TFT workshop

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1 Modular Tensor Categories over Number Fields

I would like to thank the organizers for inviting me here. This talk is intended to give a classification of modular tensor categories. Why should one be interested in classifying these?

Theorem 1 *A modular tensor category determines a* 3-2-1 *TQFT, where e is exactly what is assigned to the circle. a* 3-2-1 *TQFT is one where you can only go down to one manifolds. This is found in Turaev's book. We'll use this theorem to discuss key features of a modular tensor category.*

One obvious feature is that a modular tensor category is $\mathbb{C}$-linear, and it also hasa tensor structure. The tensor structure is given by the map associated to the pair of pants, $Z(S^1) \times Z(S^1) \to Z(S^1)$. It has finiteness conditions, which come from Zorro's lemma, which says whatever functor our TQFT associates to the Z-shaped cobordism should be equivalent to the identity.

So this is semisimple, with finitely many isomorphism classes of simple objects.

To discuss more features of this category, let's consider a 2-bordism M and a diffeomorphism f of M relative ∂M. What we can do is realize this diffeomorphism via [unintelligible], we can cosider $M \times I$. Now we can identify the incoming and outgoing boundary components, one via the identity and the other via f. What our TQFT is going to assign to this is, well, $Z(M \times^f I) : Z(M) \to Z(M)$, a natural ismorphism from this functor $Z(M)$ to itself. Consider a particular case where M is the pair of pants and f is a diffeomorphism that interchanges the two incoming circles. This is a natural transformation between $\otimes$ and $\otimes^{op}$. To any pair of objects, X and Y, I have an isomorphism $Z(f)_{X,Y} : X \otimes Y \to Y \otimes X$. This isomorphism is known as the braiding. This is another feature of a modular tensor category.

I should make a remark that I was sloppy about defining what this diffeomorphism is, but given a homotopy between two diffeomorphisms, this homotopy will give me a diffeomorphism of 3-bordisms. Since we're not working in the ∞ world, we are quotienting by these diffeomorphisms, and hence $Z(f_1) = Z(f_2)$. So there are a few more features of this category that I will not dwell on. One is rigidity. We have duals. The other is nondegeneracy of our braiding that I will not explain, and a few more.

Okay, so um, one key property of modular tensor categories, is the following. This is known as Ocneanu rigidity. Fixing the Grothiendieck ring of modular tensor categories, the number of equivalence classes with this Grothiendieck ring is finite. The goal of this talk is to introduce a procedure which will begin with a modular tensor category with a given Grothiendieck ring and then construct new modular tensor categories fixing the Grothiendieck ring.

Theorem 2 *Z. Wang, T. Hagge*
Given a modular tensor category $\mathscr{C}$ over $\mathbb{C}$, there exists a number field K and a $\mathscr{C}_{alg}$ defined over K such that $\mathscr{C} \cong \mathscr{C}_{alg} \otimes_K \mathbb{C}$

Remark 1 *What does is mean to take this tensor product? We can think of $\mathbb{C}$ as a category with one object and $\mathscr{C}$ morphisms. We are doing scalar extension of Hom spaces.*

To define a modular tensor category over a number field, we need extra assumptions, like $End(simple) \cong K$ and $\mathbf{1}$ should be simple, and when I write $\cong$ what I mean is an isomorphism of modular tensor categories, menaing that it respects the monoidal and grading structure.

Let $\mathscr{C}$ be over $\mathbb{C}$, fix $\mathscr{C}_{alg}$ and K as in the theorem, and $\sigma \in Gal(K/\mathbb{Q})$. Then $\mathscr{C}^\sigma = \mathscr{C} \otimes_K^\sigma \mathbb{C}$. So let me give a brief sketch of the proof.

The first sketch is to use Moore-Seiberg data to construct a parameter space $V^{modular}$, a space that parameterizes modular categories with fixed Grothiendieck ring. This parameter space is complex affine variety defined over $\mathbb{Q}$ and equipped with a G-action.

We will use Ocneanu rigidity to say that G has finitely many orbits in $V^{modular}$. I will explain what G is later.

So all we are left to do is take a modular category $\mathscr{C}$ located on an orbit and find an algebraic point $\mathscr{C}_{alg}$ in that orbit.

So $K = Q(\mathscr{C}_{alg})$ and we've set up things so that by construction $\mathscr{C} \cong \mathscr{C}_{alg} \otimes_K \mathbb{C}$. I'm confusing notation, using $\mathscr{C}_{alg}$ to mean both a point in a variety and a modular tensor cotegary from that data.

Now there are a number of intersting questions that one can ask.

1. One possible question is, how much of the moduli space of modular tensor categories with fixed Grothiendieck ring does the Galois action "see?" For example, is this action transitive.

2. Can we find examples where we have fixed points? If we can, then this would produce a representation of $Gal(K/\mathbb{Q}) \to End(id_{\mathbb{C}})$. What do you know about the geometry of parameter space?

3. It's easy to convince ourselves that by forgetting some of the structure of the modular tensor category, we have a map into another parameter space V^{fusion}, the space of fusion structures. Again this is $\mathbb{C}$-linear, a tensor category, semisimple with finitely many objects and rigidity but no braiding.

4. What can we say about K? In all known examples, $Gal(K/\mathbb{Q})$ is Abelian. Is this true in general?

What I want to do in the remaining time is talk about the parameter space $V^{modular}$, explain how we construct it.

Let me give an example, $\{\mathbf{1}, \epsilon\}$, two isomorphism classes of simples. Then the Grothiendieck ring will be $\epsilon \otimes \epsilon = \mathbf{1} \oplus \epsilon$. We know that our category is strict, meaning that any sequence of objects can be parenthesized and tensored in different ways, and strictness means that these two objects are in fact the same.

The monoidal structure is captured in the tensor product of morphisms. Let me fix notation: I will call $Hom(a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_m)$ for a and b simple as the (n, m) morphism space. So we start with the $(1, 1)$ morphism space, that's $Hom(a, b)$, which should be 0 if $a \neq b$ and $\mathbb{C}$ if $a \cong b$. Given two of these, we can consider tensoring them $Hom(a, b) \otimes Hom(a', b') \to Hom(a \otimes a', b \otimes b')$. Now let's consider the $(2, 1)$ morphisms. Here we have $Hom(a \otimes b, c)$. The dimensions of these Homs are fixed by our Grothiendieck ring. In particular, in this example, we have $(2, 1)$ morphism spaces of interest, namely, one is $Hom(\epsilon \otimes \epsilon, \mathbf{1})$ and the other is $Hom(\epsilon \otimes \epsilon, \epsilon)$. These are one-dimensional. We claim that all other hom spaces are tensor products of these.

Consider one level up $Hom(\epsilon \otimes \epsilon \otimes \epsilon, \epsilon)$. Now here I can take two $(2, 1)$ morphisms and compose to get an element here. I have three incoming legs. The vertices are labelled by the morhpisms g and g'. The internal edge is labelled with ϵ. When we look at this diagram, it is apparent that we can also consider tree diagrams where we label the internal edge by the unit object. This will correspond to taking a morphism from $Hom(\epsilon \otimes \epsilon, \mathbf{1})$ and composing into a morphism $Hom(\epsilon \otimes \mathbf{1}, \epsilon)$. What these really tell me is that I have an isomorphism where I take $Hom(\epsilon \otimes \epsilon, a) \otimes Hom(a \otimes \epsilon, \epsilon)$, and sum over all simple objects, and that is an isomorphism into $Hom(\epsilon \otimes \epsilon \otimes \epsilon, \epsilon)$. We can also get another isomorphism if we draw our trees with the other association. Now the relations capture the strict monoidal structure up to coherence one level up. I can draw a pentagon diagram where I start with such a diagram with four branches and consider all ways of flashing it right. One way would be to forget about these two and then forget about these two... [standard pentagon].

So we have a choice of bases of the $(2, 1)$ Hom space. Now F is the matrix of basis changes in $Hom(\epsilon \otimes \epsilon \otimes \epsilon, \epsilon)$ and any other Hom space. So the pentagon tells us that $FF = FFF$, sort of, with the appropriate indexing.

In this case, this would look like $F_{\mathbf{1}}^{\epsilon\epsilon\epsilon}$ which is the matrix (1) and then $F_{\epsilon}^{\epsilon\epsilon\epsilon}$ which is $\begin{pmatrix} -\phi & z \\ -\phi/z & \phi \end{pmatrix}$ where $\phi^2 = \phi + 1$ and $z \in \mathbb{C}^\times$.

G is the change of bases in $(2,1)$ Homs, and $G = GL(1) \times GL(1)$ where $(\lambda, \lambda') : z \mapsto (\lambda^2/\lambda')z$. I hope you believe me that one can parameterize the rest of this structure that I put on the blackboard. Now is a good time to stop.

2 Ben-Zvi

So the goal for today is to discuss three dimensional gauge theory on a point. Now let me recall some things. I'm going to recall that, I'll denote Γ to denote a finite group. For Γ a finite group, and then complex representations of Γ will be a $\mathbb{C}\Gamma$-module. I can look, for any ring k at $Rep_k\Gamma = k\Gamma$-modules. So I'll think of representations as modules for an algebra.

If I study representations for infinite groups, one way to specify which representations you're interested in, if you say topological groups, you have many choices of what the group algebra is, and each different one will specify a different kind of representation, with different levels of continuity or smoothness or what have you. If G is a complex affine group (later reductive, meaning like $GL_n(\mathbb{C})$ and $E_8(\mathbb{C})$, so what we'd like to look at is notions of G actions on a category, which will be dg categories. So how do you specify this? You might say you get a functor from $\mathscr{C}$ to $\mathscr{C}$ for every group element, and maybe $a_{g_1 g_2} = a_{g_1} a_{g_2}$, maybe with weaker things if you're in the homotopical setting, and you 'd like to say that these are algebraic on g in some sense. So I'll define different types of G-categories, circumventing this, by defining different group algebras.

So for Γ my finite group I will look at the monoidal structure $Vect\ \Gamma$, I have a monoidal product $*$ from $\Gamma * \Gamma \to \Gamma$. I can define a Γ action on $\mathscr{C}$ is a $Vect\ \Gamma$-module category.

For G an algebraic group, let me say I have QG and $\mathcal{D}G$, these are the group algebras valued in quasicoherent sheaves or $\mathcal{D}$-modules. In the finite case both of these are vector bundles. So singular vector bundles and singular vector bundles with flat connection. Both of these are monoidal dg categories. There's a monoidal structure there, I will use this, I have a map $G \times G \to G$, if I have two sheaves, I put them on the two factors, tensor them externally, and push forward along the multiplication map. These will be my multiplication map.

Maybe I'll think of this as a quasicoherent group algebra, and the other one smooth or flat. The adjective smooth is in analogy with [unintelligible].

We'd like now a version of topological field theory where to the point I assign $Vect\ \Gamma$ modules, QG modules, or $\mathcal{D}G$ modules, quasicoherent G-categories or smooth G categories, so we need as well a module structure $QG \times \mathscr{C} \to \mathscr{C}$. The point goes to the two-category of such modules, and these module categories will be examples of $\infty, 2$-categories.

Then you can ask why we care, we'll try to develop that, but the first thing you might ask is, what does this give us with the cobordism hypothesis. This needs some strong finiteness conditions, and let me elaborate more later, but for finite groups, things work the way you expect. Here we have a two-category for a point, and you get a $3D$ topological field theory. The QG setting gives only a 2 dimensional TFT. It's not finite enough, not dualizable enough. The bottom one is only a one dimensional TFT. There's a modified version which is two dimensional.

So what do we want to say? Maybe, um, maybe I can spell out, maybe it's worth spelling out quickly without spelling out the finite group case, we have for finite groups in dimension 0 we have $Vect\ \Gamma$ modules, here for S^1 we get $Vect(\frac{\Gamma}{\Gamma})$, class functinos again, here a vector space, and then for surfaces it's $Fun(\mathscr{M}_\Gamma(\Sigma))$ and for dimension three it's $\#M_\Gamma(N)$.

The first thing to say here, where do I find examples of these kinds of categories. If I give you a variety X with an action of G, then we get modules of this kind, I can look at $Q(X)$ and $\mathcal{D}(X)$ which will naturally be modules. You can ask, what do I mean, you know that G acts on a variety, you can take a bundle and pull it back, a bundle with connection and pull it back, and you should be able to pull it back. This is because you have maps $G \times X \to X$. You have G and F, put them on each factor, and push forward. In this ∞ setting, I need to give you higher things too, $G \times G \times X$ and so on, but I won't bother with that.

So, um, what we'd like to do is develop a little bit of this. What kind of structures do we have here?

If I look at an algebraic subgroup of G, I can look at $Q(G/H)$, which live in $Z_G^Q(\cdot)$. These are natural examples. You can ask what are the endomorphisms. This allows you to manipulate, the endomorphisms should be a Hecke algebra, $End Q(G/H)$ should be $Q(H\backslash G/H)$, which is $\cdot/H \times_{\cdot/G} \cdot/H$.

So for example $End\ Vect$, the case $H = G$, this is kind of big, this is $Q(BG)$ where BG is $\cdot/G$, so this is representations, not necessarily finite dimensional of G, with their usual tensor product.

Let me now say something about Morita theory. Let me return to the finite set setting. We can look at functions on $X \times X$, that's square matrices, so it's the algebra of square matrices, which is Morita equivalent to $\mathscr{C}$ which is $Fun(\cdot)$, which means $Fun(X \times X)$-mod$\cong \mathbb{C}$-mod. So for $X \to Y$ you get $Fun(X \times_Y X)$-mod is Morita equivalent to $Fun(Y)$-mod which is $Vect(Y)$. Now in the G setting, the more orbifoldy setting, the Hecke algebra $Q(H\backslash G/H)$, this is like an algebra of block matrices, but with quasicoherent sheaves, you can find Morita theory.

Theorem 3 *(Ben-Zvi, Francis, Nadler)*
$Q(H\backslash G/H)$-mod is equivalent to $Q(G)$-mod which is equivalent to $(Rep\ G)$-mod, they are the same for any pair of H. So representations are insensitive to the subgroup. This is due to [unintelligible]for finite groups.

I should have put "dualizable modules." These are equivalences of $\infty, 2$-categories.

Maybe I should say, there are two directions I'd like to go. In the $\mathcal{D}$-module setting, these statements are really false, and that's where a lot of really interesting things come from. The prime example will be G a reductive group, say, $GL_n(\mathbb{C})$, and K a Borel subgroup, say, upper triangular matrices. Then G/B is the flag variety, complete flags in $\mathbb{C}^n$. This is a natural module for $\mathcal{D}G$, this $\mathcal{D}(G/B)$. This is interesting because Beilinson-Bernstein tells us that this keeps track of $\mathfrak{g}$-representations where we've set some parameter to zero. A good question to ask is how the action G acts on representations of the Lie algebra, that's because it acts on the algebra by conjugation.

What are the symmetries? On one side, we have $\mathcal{D}(G)$ acting on $\mathcal{D}(G/B)$, and on the other side we have $\mathcal{D}(B\backslash G/B)$, which I will call $\mathscr{H}$ a finite Hecke category. The orbits of these double cosets are called Schubert cells, and they are in bijection with the Weyl group W. What kind of flat bundles will you write on double cosets. The basic objects $K(\mathscr{H})$ looks like the group algebra $\mathbb{Z}W$. The convolution structure looks like the Weyl group. There's somehow not quite a good way to think about it.

There should be natural bases of $\mathscr{H}$ labelled by $w \in W$. So I have ways to extend, and I will get a basis. Let me call the standard basis, you extend $i_{W*}\mathbb{C}_W$. But these don't satisfy the Weyl relations. This is really a version of the group algebra of the braid group $\mathbb{Z}B_G]$, with $n-1$ strands? I guess up to a shift, for GL_n. So if you look at these guys, you get that they satisfy the braid relations $T_{S_i}T_{S_j}T_{S_i} = T_{S_j}T_{S_i}T_{S_j}$ but $T_{S_i}^2 \neq id$.

Why am I giving you all this? If I act by $\mathscr{H}$ on $\mathscr{C}$, that's not really an action of the Weyl group, but the braid group on $\mathscr{C}$. This is how most of the braid groups in, say, Khovanov homology, arise, as the action of this kind of category.

I'd like to shift emphasis. I'd like to sell you on $\mathcal{D}(G)$ modules is a bad thing, not well-behaved. If I assign $\mathscr{H}$-modules to a point, then I do get

Theorem 4 *Ben-Zvi, Nadler*
This is a 2-dimensional extended TFT

So this is better behaved. But in some sense this is somehow the right analogue. You could ask for examples. One answer is Khovanov homology. How will I get a module for $\mathscr{H}$, I look at $\mathcal{D}(K\backslash G/B)$, this gets an action of $\mathcal{D}(B\backslash G/B)$. So this is $(\mathfrak{g}, K)$-modules, so catogory $\mathscr{O}$ [unintelligible]and categories of representations of $G_{\mathbb{R}}$.

Let me say now, there's a family of these guys, and at some points the family captures all of $\mathcal{D}(G)$. This is captured, it doesn't exist, by the ones that do exist, the $\mathcal{D}(G/B)$.

So now maybe I should, you want to do categorified representation theory. So let's look at the circle. In a topological field theory, $Z(S^1)$, I'll say dimension at least two, then $Z(S^1)$, then let me draw the pictures, for how this relates to $Z(\cdot)$. We got two different maps, a map $Z(S^1) \to End\ A$ and back $End\ A \to Z(S^1)$. What are these telling me? They tell me that $Z(S^1)$ is the Hochschild cohomology or center of $Z(\cdot)$ and at the same time the Hochschild homology, or dimension.

Here center means endomorphisms of the identity functor of $Z(\cdot)$. In my case $Z(\cdot)$ is modules for some $\mathscr{A}$, and the center means, in terms of $\mathscr{A}$, the center of an algebra are maps $\mathscr{A} \to \mathscr{A}$ that commute with the $\mathscr{A}$ action on the left and right.

What is Dim, or Hochschild homology? If you have an object that's dualizable? If I write the circle as a composition of two half-circles, that composition is $Z(S^1)$, that's the dimension of the thing that I assign to a point.

If $Z(\cdot)$ is modules for an algebra, then Dim is $\mathscr{A} \otimes \mathscr{A}$ as modules over $\mathscr{A} \otimes \mathscr{A}^{op}$.

This is something that we had briefly [unintelligible], what is $A \otimes_{A \otimes A^{op}} A$, this equalizes $X \times Y$ and $Y \times X$ in a universal way.

The thing you assign to a circle should be the dimension and here also the center.

One example appeared in the talk this morning, $Vect\ \Gamma$-mod was $Z(\cdot)$, and to the circle we got $Z(S^1) = Cent(Vect\ \Gamma)$, which are those V in $Vect\ \Gamma$ so that $V* \cong *V$. So $Z(S^1)$ is the Drinfeld center, it's a braided tensor category.

Theorem 5 *QG has dimension and center equal to class functions, $Q(\frac{G}{G})$, which is $Z(S^1)$.*

You can check for a Riemann surface, this is just the Dolbeaut cohomology, $R\Gamma(\mathscr{M}_G(\Sigma), \mathscr{O})$. The version of functions is the derived version of functions.

So let me say something about character theory [missed]. I was able to get maps to the Hecke algebra and class functions, we'll look at this next time and see [unintelligible].