TFT workshop

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May 19, 2009

1 Jacob Lurie

So, it would have made more sense to give these lectures in the other order. I want to give an overview of higher category, and I'll start with the definition of ordinary category theory.

Definition 1 *A category $\mathscr{C}$ consists of*

- *A collection of objects X, Y, Z,...*
- *For every pair of objects (X, Y) set of morphisms $Hom_{\mathscr{C}}(X, Y)$*
- *Composition maps $Hom_{\mathscr{C}}(X, Y) \times Hom_{\mathscr{C}}(Y, Z) \to Hom_{\mathscr{C}}(X, Z)$*
- *Associativity and unit (identity) for the hom sets.*

We talked about a category Cob_n. There the objects and morphisms looked similar, they were both manifolds. So there you might ask about weaker notions of equivalence on morphisms, like if the manifolds were bordant.

So this is a natural category to try to promote to a 2-category.

Let me give a bad definition of a strict 2-category:

- We have a collection of objects
- the morphisms $Hom(X, Y)$ form a category
- the composition maps are now functors
- the associativity is strict. If we look at the functors $Hom(W, X) \times Hom(X, Y) \times Hom(Y, Z)$, we can get to $Hom(W, Z)$ in two ways, and we insist that these be equal. This strictness is what is bad about the definition.

Let me give you an example. Let X be a space. You can, fixing a base point x, look at $\pi_1(X,x)$, a group whose elements are homotopy classes of loops. You could consider a slightly more sophisticated object $\pi_{\leq 1}(X)$. The objects here are points in X, and morphisms $Hom(X,Y)$ are paths up to homotopy relative to endpoints. This is a special category where morphisms are invertible. This knows about path components, which are isomorphism classes. It also knows about $\pi_1(X)$ because that is the automorphism group of x in the category.

To get π_2 in the picture, this will be an example of a 2-category, we can do something a little fancier. So again the objects are points, the morphisms are paths, and I will need morphisms between morphisms. So suppose you have paths p and q from x to y, then 2-morphisms from p to q are given by homotopies between p and q fixing the endpoints, up to homotopy.

This is supposed to be one of the basic examples of a 2-category. This should be an example of the type of structure we'd like to define, but the strictness fails. It can be made to hold but it requires some effort. Let's see what goes wrong.

To talk about strictness, I should talk about composition. In pictures what you are supposed to do is glue paths together. More precisely, a path is a map from the unit interval into X, and to compose them, to write $q \circ p$, I have to concatenate paths. You do this from $[0,1] \to X$. On the first half of the interval you could follow p but twice as fast, and then on the second half you could follow q, again twice as fast. You could compose $(r \circ q) \circ p$ and this will be r four times as fast and then q four times as fast and then p half as fast, and you'll get something different if you do $r \circ (q \circ p)$. This isn't a problem for the fundamental groupoid because you're dealing with homotopy classes of paths. But it is a problem for this two-category because you're dealing with paths, and the associativity only holds up to homotopy.

I made a choice by saying that paths have length one. You could let paths have variable lengths, so that lengths add. This gets rid of associativity, but by means of an ad hoc trick that doesn't generalize well.

Now we should go even higher, above 2 to higher categories.

Definition 2 *(bad)*
A strict n-category consists of

- *some objects $X, Y, \ldots$*
- *$Hom(X,Y)$ which is a strict $n-1$-category.*
- *There should be composition functors, which satisfy*
- *strict associativity.*

That's an example of a definition. On this board let's write down an example. We'd like $\pi_{\leq n}(X)$. Let me give you an informal description. Objects are points, morphisms are paths, 2-morphisms are homotopies between paths, 3-morphisms are homotopies of homotopies, and so on. At the last level, n, we mod out by further homotopies. If you try to shove this into the hole described above, you can't get around this in generality. So you can't get $\pi_{\leq 3}$ as a strict n-category, up to equivalence.

So you might try to correct the bad definition, by dropping the word strict. You might say that you have objects, that the morphisms are some other kind of category, and then you would say that you want this to be associative up to coherent isomorphism. What does that mean? It means you require this to be associative up to isomorphism, but these are part of the structure and they behave well, satisfying further conditinos. Even in the case $n = 2$ it's hard to spell these out. In $n = 3$ it's prohibitively complicated. If this is the definition you really want to work with, there are some strategies, I want to consider this example. This example is very characteristic in the following sense. For any space X, its fundamental n-groupoid should be an example of an n-groupoid, an n-category so that all morphisms at all levels are required to be invertible.

Let me state a thesis: Every n-groupoid should arise this way, not uniquely. When $n = 1$, the fundamental groupoid of any simply connected space is the same. You can try to correct this by letting $n \to \infty$. There should then be a theory of ∞-groupoids, with higher morphisms and homotopies, and if you require everything to be invertible, you can give examples, spaces, paths, homotopies, and so on.

Let me state a more precise version of this thesis. Every space X should have a fundamental ∞-groupoid and every ∞-groupoid should be the ∞-groupoid of a space X unique up to homotopy equivalence (weak). So I called this a thesis and not a theorem because I haven't defined anything. This is something that should become a definition. This should be a requirement for what an n-category is. If I get a definition, I can consider when all morphisms are invertible, and if that models homotopy theory, it passes the bar, otherwise, throw it back.

There's an easy way to construct an ∞-groupoid, I could use this rubric and say this is exactly X, and then I'm not forgetting anything. I can also replace the spaces with simplicial sets. If you then wanted to give the definition of an n-groupoid, you could truncate and say it's an ∞-groupoid with nothing above a certain point. So this is like saying, it's a space where homotopy groups vanish above n. It takes more space on the blackboard to define an n-groupoid than an ∞-groupoid. This tends to be a feature of many of the useful approaches of higher category theory. It's easier to go all the way up the ladder than to make a restrictive definition.

Maybe n-categories are hard, but n-categories where all morphisms are invertible are easy. Let me define a term that Bertrand already defined.

Definition 3 *(sketch) An ∞, n-category, is a higher category in which all k-morphisms are invertible for $k > n$.*

What are some examples? When $n = 0$, saying that all morphisms are invertible, this is the same as an ∞-groupoid, which is then the same thing as a topological space.

Now you'd like to build on this, so let's rewrite the iductive definition I did earlier but starting with ∞-groupoids.

So you might try to say that an ∞, n-category is a collection of objects, for every pair of objects, a collection $Hom(X,Y)$ of morphisms which is an $\infty, n-1$-category with a composition law on these things, and an associative composition law (in some reasonable sense). When $n = 0$ we have a definition, that it is a topological space. You might try to apply this definition here and see what pops out in the case $n = 1$. What is this idea? It should tell us that it is a category where the homs are topological spaces. Let's say for a second that we want the associativity to be strict. The morphisms will be organized into topological spaces, and you get compositions that are continuous. If you prefer your spaces to be simplicial sets, then you can substitute that.

This is a correct definition, because you won't lose anything by adopting it, but it's inconvenient in many respects. The inconvenience is illustrated in that, well, we tried to write down an example. We had to do some work, and what came out naturally was an $\infty, 2$ groupoid where associativity only held up to equivalence. It's hard to construct the strictly associative models; it's a lot easier to talk about associativity up to coherent isomorphisms.

The strategy that Bertrand describes implements one strategy to do this. He required that composition is only defined up to a contractible space of choices. In a Segal category, you have objects, Homs, and $Hom(X,Y)$, but you have no associativity on the nose. You have $Hom(X,Y,Z)$ which maps to $Hom(X,Y) \times Hom(Y,Z)$ as a homotopy equivalence and also to $Hom(X,Z)$. To loosen up you don't have a composition at all, instead you have only something that has induces composition up to coherent isomorphism.

I wanted to consider extended field theories. Last time I defined Cob_n, which was just an ordinary category, and an extension of it $Bord_n$, which you can think of as a fancy version of Cob_n. Now I can describe another version $\widetilde{Cob_n}$. This will be an $\infty, 1$-category. This isn't fancy because of lower dimension, it's fancy because of higher dimension. If I say that morphisms are diffeomorphism classes, I'm forgetting the diffeomorphism. This will contain information about the diffeomorphism groups. Let me describe it informally and then more formally.

All right, let me describe what I'll call $\widetilde{Cob_n}$. The objects are $n-1$-manifolds. The bordisms are morphisms between $n-1$-manifolds. I'll have higher morphisms too. So the 2-morphisms are given by diffeomorphisms. The space of diffeomorphisms has a natural topology, so you can consider homotopies, paths in that space, isotopies, and now I'll put in ellipses, and this should be like the infinity groupoid applied to the space of diffeomorphisms (relative to the boundary), I'll have isotopies, and then homotopies of isotopies, and so on.

This looks like it should be an $\infty, 1$-category because everything above the first morphisms are invertible. Can you actually describe this structure in a precise way? You might let the objects be $n-1$ manifolds. Then you want $Hom(M,N)$ to be a classifying space for bordisms from M to N, a space over which you have a fiber bundle where fibers are bordisms. You want it to be universal, a pullback of a universal bundle. This space exists, and by general nonsense is uniquely determined in the homotopy category. It's not uniquely determined on the nose, so it won't give a strictly associative topological category. I can choose classifying spaces (M, N) and then composition laws. The product of $Homs$ gives me a classifying space classifying pairs (M, N) and (N, P). Given two such bordisms, I can get something from N to P. This was only defined up to homotopy so you can't check associativity except up to homotopy.

So maybe if you are more careful about your classifying spaces you can get something strictly associative. You should be able to do this if you believe what Bertrand said, and what I have said. But it's easier to describe this using the language that Bertrand introduced, as a Segal category.

So what do you have to do? He called it a bisimplicial set, but for me it will be a simplicial space. So what we want is a simplicial space $\dot{X}$ with X_0 some discrete set. This is the set of objects of the $\infty, 1$-category, which is the set of closed $n-1$-manifolds. We can fixthe problem that this is not a set.

Let me give you the nth space. I want a disjoint union over all things these morphisms can be between, over tuples $(M_0, \ldots, M_n)$, of $Hom(M_0, M_1, \ldots, M_n)$, where $Hom(M_0, \ldots, M_n)$ is, well, I'll tell you the set and you can imagine how to topologize it. I'll want real numbers $t_0 \leq \cdots \leq t_n \in \mathbb{R}$ and m-dimensional closed submanifolds B contained in the interval from $[t_0, t_n]$ such that B is transverse to $t_i \times \mathbb{R}^\infty$, and the intersection is associated with M_i. We'll regard our manifolds with an embedding into $\mathbb{R}^\infty$. I mean B is properly embedded in this manifold with boundary. When $n = 3$ you should specify t_0, t_1, t_2, t_3, and then a 1-manifold, which is embedded in an interval cross a large Euclidean space. The intersection with the slices at the times t_i should give me M_i. This is a sequence of composable bordisms, which is tautologically a simplicial space. The composition law from this point of view, I can erase the line where $t = 1$ and now I've composed bordisms.

Modulo issues about how to topologize this, you've got a description of $\widetilde{Cob}_n$ as a Segal category. Similarly, you can describe more elaborate bordism categories using a more elaborate version of the theory of Segal spaces.

Bertrand described Segal categories. We have an approach that gives ∞, n-categories for any n. Instead of slicing things in only one direction, you could slice along other sorts of slices. I don't want to get into the details, but just mention that we can use this idea, use these ideas to define an object that I will call $\widetilde{Bord}_n$. Recall Cob_n could add manifolds of lower dimension to go to $Bord_n$ and also higher dimension to go to $\widetilde{Cob}_n$, and now you can do both at the same time. The objects will be zero manifolds, 1-morphisms are bordisms, and so on, and then the n-manifolds will be bordisms with corners, and then all higher morphisms are diffeomorphisms, isotopies, and so on. These ∞, n-categories are the objects of study for extended topological field theories. This is more elaborate, and enjoys a similar property, which I will take up again in the next lecture, connecting to Galatius-Madsen-Tillman-Weiss on the Mumford conjecture.

2 Ben Zvi

Last time we talked about a toy example of a two dimensional extended topological field theory. This was associated to a finite group. We know that $Z_G(\cdot)$ was $Rep_{\mathbb{C}}(G)$. We got S^1, which gave us a Frobenius algebra $\mathscr{C}\frac{G}{G}$ and this had maps to and from $End_G(A)$, the action and class map. The action was $\mathbb{C}\frac{G}{G} = Z(\mathbb{C}G)$ where here this means the center. The character was that the identity could be assigned the character $\chi_A \in \mathbb{C}\frac{G}{G}$

So where will I find representations of G? I can take a subgroup, and get a natural representaiton, and this will be $\mathbb{C}[G/K]$. We can plug this in and see what we get. What are our action and character? I will repeat all this story later on for complex Lie groups. In order to describe it, I need to describe the endomorphisms of the representation. This is $\mathscr{H}_{G,K}$, the endomorphisms of $V_{G,K}$. One thing this is is a representation of a functor $Rep\ G \to Vect$ which assigns to A the K-invariants in A. So $Hom_G(V_{G,K}, A) = A^K$. In particular, this tells me, if I look at endomorphisms of this representation, it acts on the space of K-invariants in any representation.

This allows us to describe the Hecke algebra. It's given by K-invariants in $V_{G,K}$, hich is $\mathbb{C}[G/K]$. So this will be double cosets $\mathbb{C}[K\backslash G/K]$. This is an algebra and has a multiplication. I can think of this as functions on double cosets, this is a subalgebra of K-biinvariant functions. We had the multiplication $G \times G \to G$. If I want to put $\mod K$, then I put the K everywhere. All these pictures come naturally from the TFT picture. We're doing G gauge theory so studiying principle G bundles. I could study these on the interval, and I have only one bundle, $\cdot/G$. But what if I mark the two endpoints with the subgroup K. I have G bundles, but at the two points at the end I do reductions on the two ends. This gives $K\backslash G/K$. The Hecke algebra is what happens when we mark the endpoints of an interval with the subgroups. Suppose that K is trivial. Then we're looking at G bundles on an interval that are trivialized on the endpoints. We'll get the group algebra. So then you could draw two of these, and glue them along the endpoint and get a single interval, this is the convolution product.

That's a toy picture for the Hecke algebra. Now I'd like to write down my character and action maps. I'm going to draw the same picture I did last time.

I'll think I have an interval with K along an interval as part of the boundary of a cylinder. Fields $\mathscr{M}_G$ are fields on the cylinder with a reduction to K on a contractible set. I can map his to the fields on a circle, which is $\frac{G}{G}$. On the other side we get $K\backslash G/K$, and in the middle it's $\frac{G}{K}$. Now I'll take functions on this and pull back and push forward. I want you to show that if I pull back and push forward, I'll get endomorphisms.

What about the character, I have the same picture, and then I have the function 1 at the identity double coset, which I pull back and push forward, I get the character of the induced representation. You didn't need topological field theory to do this but it will be crucial in the future.

For now I'd like to replace G with a complex Lie group, or maybe even an affine algebraic group over $\mathbb{C}$, perhaps $Gl_n(\mathbb{C})$ or $Sp_n(\mathbb{C})$. The first thing to do is look at the space of fields. For me, to keep things clear, my fields $\mathscr{M}_G(M)$ will be G-local systems, which are flat G-bundles or $\pi_1(M) \to G$ maps.

What about the circle? A flat bundle will be G up to conjugation, $\mathscr{M}_G(S^1) = \frac{G}{G}$. Or $\mathscr{M}_g(\Sigma_g)$ will be sets A and B in the group satisfying $\prod[A_i, B_i] = 1$, which [unintelligible]

In order to get some kind of invariants out of this, I need to figure out how to count the points in the set. One answer you might say, you should, looking at a finite field, I can look at $\mathscr{M}_G(\mathbb{F}_q)(\Sigma_g)$. If you'd like, I get a family of theories depending on q. This in algebraic geometry, this counting, is an avatar for looking at cohomology. Let me point out that this point of view is at the heart of Hausel and Rodriguez-Villegas. Counting points over $\mathbb{F}_q$ corresponds to vector spaces, and this is an avatar for some kind of sheaves which are now a category. Instead of assigning a number of points, I'll assign a vector space, instead of a vector space for a circle I'll assign a category.

So this is a passage from $2d$ topological field theory (extended). That's how these field theories come out of physics. In physics, I'd write down an action S but here I'll start with a point and build up.

So let's move to function theory, but we will need to replace $\mathbb{C}G$-modules when G is more than a finite group. I'll start with a complex algebraic variety (or some kind of stack) and assign to it a categorification. I'll be using a lot of ideas from Bertrand's talk. The two variants I'll give are quasicoherent sheaves and $\mathcal{D}$-modules. Our first solution, to X we assign $Q(X)$, which will be quasicoherent sheaves on X, which will form a *dg* category. These, well, we have the basic example, a holomorphic vector bundle, of fiite or infinite rank. I can take direct sums of infinitely many of them. I can take these basic objects and generate others by taking things like kernels, cokernels, complexes of such vector bundles.

A quasicoherent sheaf is just a module over R. Everything is in the derived category, so this will be compelexes in this. Then *Spec* R I can assign to X and I'll invert quasiisomorphisms and invert these. Localize by quasiisomorphisms as Bertrand explained. I don't want to get into the technical condition, but, uh, and more generally, with any variety or stack X you define these by gluing $Q(U)$ for an open cover of X. I think it's not helpful for me to say this.

For the purpose of function theory it is more useful to turn to $\mathcal{D}$modules. So turn from X to $\mathcal{D}(X)$. What are $\mathcal{D}$-modules? The basic objects are vector bundles with flat connection.

This is a map $\nabla : \mathscr{F} \to \mathscr{F} \otimes \Omega^1$ with $\nabla^2 = 0$. That's the same as a map $T \otimes \mathscr{F} \to \mathscr{F}$, so this extends to an action of all polynomial differential operators.

That's what a $\mathcal{D}$-module is. Why am I introducing these? One motivation is that $\mathcal{D}$-modules are closer to counting points over a finite field. So I can take $f \in \mathbb{C}^\infty(X)$, for example $e^{\lambda x}$, you might say, how do I capture this algebraically, I can multiply f on the left by all polynomial differential operators. That's a subspace of $C^\infty(X)$, but it's a $\mathcal{D}$-module, because I can act on f. If my function is nice, then the $\mathcal{D}$-module almost captures it. What is $\mathcal{D}e^{\lambda x}$? This looks like $\mathcal{D}/\mathcal{D}(\partial - \lambda)$, so that $(\partial - \lambda)e^{\lambda x} = 0$. So similarly, $D\delta_x$ is $\mathcal{D}/\mathcal{D}(x - \lambda)$.

Now some algebra. We can replace functions with $Q(X)$ or $\mathcal{D}(X)$. I want to say that $Q(X)$ and $\mathcal{D}(X)$ are both kind of commutative. They're commutative algebra objects of $dgCat/\mathbb{C}$. To make sense of this I need to say a lot. The collection of categories over $\mathbb{C}$ has a notion of symmetric monoidal $\infty, 1$ categories. Then you can do algebra there. The foundations are in Lurie's Derived Algebraic Geometry II and III.

Let me move to pullback and pushforward. If I give you a map $\pi : X \to Y$ I get a map $\pi^* : Q(Y) \to Q(X)$ and $\mathcal{D}(Y) \to \mathcal{D}(X)$ but then also pushforwards $Q(X) \to Q(Y)$ and $\mathcal{D}(X) \to \mathcal{D}(Y)$. This is very general. I don't need any kind of compactness. This means take cohomology along the fibers. This is very general, although you may have to restrict based on what you need.

Once I have this, why am I going through this so quickly? I want to convey the idea that this is similar to what you might do on finite sets. You can do all of the basic things you did on finite sets and all the basic results hold. For example, let me say, let me make the analogy morp precise, if we had finite sets X and Y, if we look at $X \times Y$, which maps to X and Y, if you look at $Fun(X \times Y)$, this looks like $n \times m$ matrices, this is $Fun(X) \otimes F(Y)$, or I can write it as $Hom_{\mathbb{C}}(Fun(X), Fun(Y))$. If X and Y both map to Z, you can look at $X \times_Z Y$, and then $Fun(X \times_Z Y) = Fun(X) \otimes_{Fun(Z)} Fun(Y)$ and then this is $Hom_{Fun(Z)}(Fun(X), Fun(Y))$, so block diagonal matrices where this is linear over "scalars" in $Fun(Z)$. Suppose I give you a sheaf K on $X \times Y$, so $K \in \mathcal{D}(X \times Y)$, this induces a functor $\mathcal{D}(X) \to \mathcal{D}(Y)$, which sends $\mathscr{F} \to K \star \mathscr{F}$, which is $\pi_{Y*}\pi_X^*\mathscr{F} \otimes K$ which I'll write maybe to indicate path integrals

$$\int_{\pi_Y} \mathscr{F}(x)K(x,Y)dx$$

Theorem 1 *$Q(X) \otimes_{Q(Z)} Q(Y) \cong Q(X \times_Z Y) \cong Fun_{Q(Z)}(Q(X), Q(Y))$ Then $\mathcal{D}(X) \otimes_{\mathcal{D}(Z)} \mathcal{D}(Y) = \mathcal{D}(X \times_Z Y) = Fun_{\mathcal{D}(Z)}(\mathcal{D}(X), \mathcal{D}(Y))$. So the same things we had before for finite groups are true. All functors are linear transforms, and you can also represent them this other way. So what are the hypotheses? Bertran Toën for X and Y schemes, and Ben Zvi, Francis, Nadler for perfect stacks, which I won't say what that is exactly. The theorem for $\mathcal{D}$ is me and Nadler for smooth schemes.*

Let me now motivate what I want to do next time. I want to build TFTs using the cobordism hypothesis. I want modules over the various notions of group algebra. So $Q(G)$ and $\mathcal{D}(G)$. I'll try to build modules over group algebras. The basic motivation for this kind of story is Beilinson Bernstein. If you look at $\mathcal{D}(G/B)$, you get a module for $\mathcal{D}(G)$, we'll write down the diagram, and take a pushforward. That will be a module, this B.-B. tell me that this is basically complex representations of $\mathfrak{g}$, with the parameter 0, forget it, studying these for flag manifolds captures the representations for the Lie algebra. I'll get a lot of interesting things about complex Lie groups and Lie algebras, therefore by looking at these.

[So among these functors, there are symmetric monoidal ones]

There are not many. $Q(X)$, for example, that recovers X. You don't have a lot of flexibility. You don't have a lot of interesting operations.

3 Wockel

Thank you for the introduction, and also for the opportunity. This is about higher dimensional covers. Perhaps to motivate, [missed several sentences]

The approach I want to take is via the categorification of the structure group for gague theories. So let me in the beginning fix the notion of cover. This, when you're working with topological spaces, it's clear what you mean. If X is an $n-1$ connected space, you consider a space which is same, but with some homotopy group trivial. It should have all homotopy groups equal to the homotopy groups of X except that bottom one. This should be a fibration $Y \to X$ with π_k isomorphisms for $k \geq n$ and $\pi(Y) = 0$.

How can you construct these spaces? Because X is $n-1$-connected, you have a map $X \to K(\pi_n, n)$ which is an isomorphism on π_n. Then let $Y = f^*(P(K(\pi_n, n))$. This is unsatisfactory when you're interested in topological groups or Lie groups, if you started with a topological group X you lose the group structure. The P is the path space.

What are important examples? Let's look at $n = 1$. The simply connected cover of a group, say $Spin \to SO$. For $n = 2$ you get for instance $\Omega Spin$, so we can get one with $\pi_2 \neq 0$ because it's infinite dimensional. The failure of π_2 to vanish gives certain counterexamples.

For $n = 3$ we would like to map into spin as well to kill the π_3. So what is the most basic example we can consider? This is the case $n = 1$. No a simple but instructive example. In this part of the talk, G will be connected and probably a Lie group (topological or simplicial group). If G is connected you have a simply connected cover with $\pi_1 \subset \tilde{G}$. This is a π_1 principal bundle. This is a twisted version of a direct product of G with π_1. It's a central extension of G by π_1. What you know now from group cohomology is that this central extension can be realized in a very specific shape as $\tilde{G} = \pi_1 \times_{\theta_1} G$. The 1 in θ_1 is [unintelligible]. That's a group. The map is $G \times G \to \pi_1$, so the multiplication is $(a, g)(b, h) = (a + b + \theta_1(g, h), gh)$.

Associativity requires that $\theta_1(g, h) + \theta_1(gh, k) = \theta_1(g, hk) + \theta_1(h, k)$ and $\theta_1(g, e) = \theta_1(e, g) = 0$. This defines the group structure but how about the smooth structure? If I assume that θ_1 is smooth on some unit neighborhood $U \times U$, so if I assume that this is smooth, since the target space is π_1, the smoothness here means a particular constant. Because of the second condition I had, this means that the constant actually vanishes. What you can derive from this is a Cech cocycle τ (for transgression) in $H^{\vee 1}(G, \pi_1)$, and this is a good candidate for a principal bundle, I want this cocycle to be the classifying cocycle for the bundle.

On $\pi_1 \times_{\theta_1} G$ we now have the structure of a central extension of a Lie Group. Let me give a construction. How do you get this cocycle? For this example you can write it down very explicitly. For $g \in G$, choose a path from e to the group element which is smooth. Now from e you can travel to g via γ_g, or to h via γ_h, or to the product via γ_{gh}. You can construct an element in π_1, you can construct θ_1, you can get a loop by going along γ_g then $g\gamma_h$ and then γ_{gh}^{-1}, and this can be given by the group differential, this is $d_{gp}(\gamma)(gh)$, which si $\gamma(g) + g\gamma(h) - \gamma(gh)$. This is closed so defines an element in π_1, and you want $\theta_1(g, h) = q(d_{gp}(\gamma))$. This is a map from $Z_1(G) \to H_1 \cong \pi_1$. I claim it's a cocycle, it's already written as d of something. It's $d_{gp}\theta_1$ which is $d_{gp}(q \circ d_{gp}\gamma)$, and now q is linear, and we get $qd_{gp}^2\gamma = 0$.

Theorem 2 $[\theta_1]$ *is universal which amounts to saying that it describes the universal cover of G for 2-cocycles which vanish on some unit neighborhood. We had the requirement that if θ_1 vanishes, it defines a topology on the simply connected cover, and the statement here now is that each other such cocycle with values in an abelian group actually factors over this cocycle. That means that we can take $[f] = [varphi_f \circ \theta_1]$ for a group homomorphism $\varphi : \pi_1(G) \to A$. Then $Hom(\pi_1, A) \to H_{gp}^2(G, A)$ via $\varphi \mapsto [\varphi \circ \theta_1]$*

You can use standard covering theory, in particular the path lifting property. We understood now how to rephrase this in terms of group cohomology. So let me say that $H_{gp}^n(G, A)$ will be locally smooth.

So let me generalize the notion, understand the notion of an n-cover of a topological group. Let me look at $n = 2$. You want to understand the covering of $\Omega Spin$. From now on G is simply connected. The assumption that G was connected was important because I needed a path from the basepoint to each group element. I want to use simply connectedness to describe the similar thing for $n = 2$. Because I don't have any theory that I have to match with, I can start directly with a construction.

It's construction of $\theta_2 : G \times G \times G \to \pi_2(G)$. For all g, h in G I can write triangles as before, and choose an $\eta_{g,h}$ so that the topological boundary $\partial\eta_{g,h}$ is $d_{gp}\gamma$. (The choice of γ is unique up to equivalence because of simple connectedness) So now $C_*(\Delta^2, G)$ is where this map lives, with $*$ meaning that it's pointed. Now $d_{gp}\eta$ evealuated on g, h, k describes a closed thing in G.

[I miss a calculation where this is shown to be a tetrahedron.]

So you define $\theta_2 = qd_{gp}\eta$ for $q : Z_2 \to H_2 = \pi_2$. The theorem now is that this cocycle here, and this is joint with [unintelligible], the cocycle that I have here is in $H_{gp}^3(G, \pi_2(G))$. This is universal for locally constant 3-cocycles, that is, and then the same statement I had before. The proof here is by making use, remember that in the case $n = 1$ I used covering theory and the path lifting property, but here with parallel transport on 2-bundles. The questions now is to what extent describes θ_2. At the beginning I said a 2-connected cover is [unintelligible], and that's now encoded in this cocycle. If I now take topology into account, then here I get a central extension of 2-groups, a weak group object which is [unintelligible]. The extension I get is $B\pi_2 \to \mathscr{G} \to \underline{G}$, so this has one object and morphisms labeled by π_2.

What is the topological interpretation? So now I get $[\tau\theta_2] \in H^{\vee 2}(G, \pi_2)$, what I will get is the following theorem:

Theorem 3 *Principal $\mathscr{G}$ 2-bundles over G are classified by Cech cohomology of G with coefficients in the two-group $\mathscr{G}$ and in particular if $\mathscr{G}$ is now this very basic 2-group $B\pi_2$, then the coefficients $H^\vee(\mathscr{G}, B\pi_2)$ give $H^{\vee 2}(G, \pi_2)$, and you would hope that you can do the same thing, get a Lie 2-group and get an extension of Lie 2-groups. That's a bit too much. What would be nice is a Lie 2-group structure on the principal bundle associated to τ.*

When you classify ordinary principal bundle, you take a direct limit, but doing the same thing here leads you to classifying principal bundles not up to isomorphism but a weaker notion of equivalence, Morita equivalence. The point here is exactly the same. An example is a Cech group coming from an open cover, and you know that this groupoid is Morita equivalent to the direct limit, the union. That's not good, you don't have a smooth map backwards. You cannot come smoothly back, choose representatives, and that's the same problem that occurs here. But if it's discussed this way, the process still works. When you convert these equivalences here, the ones leading to the same stack, [unintelligible].

Now what I can do here is realize $P_{\tau\theta_2}$ by obstruction of stacks. So we invert Morita equivalences and pass to smooth stacks. Now I will get a group object in the 2-category of stacks, a "stacky Lie group."

Let me shortly say what this is good for. The way I stumbled into this was methods of integrating Lie algebras which do not integrate to Lie groups. Infinite dimensional Lie algebras may not have Lie groups. Using θ_2 we can write down, the failure comes from the nontriviality of π_2, killing this π_2 can show the failure in this situation. Other interesting questions involve killing $n = 3$, but [unintelligible]. Thank you.