TFT workshop

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1 Ben-Zvi

Thanks a lot to the organizers and all of you. My talks are going to be about gauge theory and representation theory. I'm going to try to say a little bit about what gauge theory is, and start to explain about how it captures structures in representation theory. Today I'll start with two dimensional gauge theory, and we'll see how this captures things about finite groups. In my second and third talks I'll try to get to three dimensional gauge theory, which will have something to do with representations of Lie groups. In the last talk I'll be even vaguer about connections to Geometric Langlands. This is a relatively concrete manifestation of some of the things the others will be talking about.

So what is gauge theory? I'm going to work by example, mainly. It's a physical theory. For us it will be a quantum field theory, one in which for us the fundamental objects (fields) are principal G-bundles with connections on them on a space time manifold M. M will be dimensions two, three, and four. So maybe to say, multimedia talk here, let me put some adjectives. We could say classical, quantum, topological. In the classical case we look for connections satisfying equations, like flatness or the Yang Mills equations.

For quantum theory, we'll look at all connections, and study this by attaching $\mathbb{C}$-linear data, giving a weighting by the Yang Mills action.

Now in topological quantum gauge theory, we'll study coarse features which will depend only on the topology of space-time. This is about as much as I want to say on this level of vagueness, so let me say something concrete about two dimensional gauge theory, so let me say a little bit about when G is a finite group. This is called Dijkgraff-Witten theory. So G will be a finite group and we will assign to it Z_G a two dimensional field theory.

We'll start with M, which will be dimension 2, 1, or 0, and all my manifolds will be oriented. We assign to this the space of gauge fields, namely, $\mathscr{M}_G(M)$ will be the collection of all principle G-bundles on M. There should be a connection but because G is finite this is automatic. Perhaps it would be better to say that it's the set of G-Galois covers of M. It's also, up to connectedness and pointedness, representations of $\pi_1(M)$ on G up to conjugation.

This is not just a set, but let's think of this as a finite orbifold, meaning we will have to keep track of symmetries or automorphisms.

This is very simple because the manifolds are of small dimension. The topology is really boring. So $M_G(\cdot)$ is a point with automorphism group G, namely BG.

What about the circle? That's my one manifold. So I get an element of G as I go around a circle. This depends on a basepoint, so I get G up to conjugation, which I will name by $\frac{G}{G}$. A little more concretely, this is a union over conjugacy classes

$$\amalg_{[g]} \cdot_{(Z_G(g))}$$

of a point with automorphism group the centralizer of G.

For $\mathscr{M}_G(\Sigma_g)$ one gets $A_1, \ldots, A_g$, and $B_1, \ldots, B_g \in G$ such that $\prod [A_i, B_i] = 1$ up to conjugation by G.

Let M be a 2-manifol, then a partition function $Z_G(M)$ will be like

$$\int_{\text{fields}} e^{-S(\varphi)} d\varphi$$

In our case this will be the count of the points in the finite set $M_G(\Sigma)$. But this is a weighted number, weighted by symmetries, so

$$\sum_{p \in \mathscr{M}_G(M)} \frac{1}{|Aut\ p|}$$

One thing we'd like to do is calculate these numbers. Quantum theory would not be so interesting if it just gave a number to every two manifold, but how will we calculate it? By locality. We'll count the number of bundles on each piece, and then glue them together. Before, to a closed two manifold we assigned a number. With a boundary, we'll calculate the partition function on M, and the construction won't make sense without boundary conditions, so φ_0 will be a filed on the boundary of M. Once I specify boundary conditions, the integral

$$\int_{\varphi} e^{-S(\varphi)} d\varphi$$

makes sense. So $Z(M)$ is in $Fun(Fields(\partial M)) =: Z(\partial M)$. Before we had two manifolds with a number, and now one-manifolds will get a vector space, like functions on the space of fields. Maybe I can say more generally, suppose I have a manifold with boundary having incoming parts and outgoing parts. I have natural orientations on the surface and the circles, and some places they match and other places they don't. So fields on M should form a correspondence between fields on the outgoing and incoming parts of the boundary. My operation, I have a vector space $Z(\partial_{in})$ and another $Z(\partial_{out})$, which will be mapped by an element of $Z(M)$ by pulling back and pushing forward. So if $f \in Z(\partial_{in})$, I take this to

$$\varphi_{out} \mapsto \int_{\varphi|_{\partial_{out}} = \varphi_{out}} f(\varphi_{in}) E^{-S(\varphi)} d\varphi$$

I will call this $\pi_{out*}(\pi_{in}^* f e^{-S})$. When the boundary is empty, I get a number, the same number I had before. Are there any questions? The more precise axiomatics will appear later today–or not.

In our case, all we're doing is finite sums. What is $Z_G(S^1)$, we're supposed to look at functions on $\frac{G}{G} = \mathbb{C}[G]^G$. So suppose I have a cup, a cap, or a pair of pants, what do I assign to these pictures? To the cup I will get a vector, which I will call 1, the unit, and this is a definition. I look at gauge fields, look at these maps, and this one is just a delta function on the unit in the group. The other way, the cap, I'll have a single functional on class functions, evaluation at 1, the trace, which is $\mathbb{C}[G]^G \to \mathbb{C}$. In the pair of pants, I should get $\mathbb{C}[G]^G \otimes \mathbb{C}[G]^G \to \mathbb{C}[G]^G$. This is convolution. If I look around the two loops of the legs, I see that I have holonomies g_1 and g_2, and then $g_1 g_2$ around the waist. So then I can use the map $G \times G \to G$ then you extend to $\mathbb{C}G \otimes \mathbb{C}G \to \mathbb{C}G$. Since G is finite, functions on the group are the same as measures on the group. If I really want to make sense of what I've been talking about, I need pullbacks and pushforwards, I've been thinking of these as both functinos for pullbacks and measures for pushforwards. When I pass to Lie groups these won't be the same things.

To summarize what this is giving me, I get that functions on a group are an associative algebra, and the class functions sit inside this.

One of the first results is that the amount of structure I've given, this suffices to determine what I can give on any topological structure. The one piece I need to know on top of that is that these three pieces give me a commutative Frobenius algebra. The first thing Ineed is that this algebra is commutative. If I have two group elements f and g with $f \times g$ coming out. As I move around I will get $g * f$. Then I can multiply and cap off, and this is nondegenerate. I cut my surface up into pairs of pants and backwards pairs of pants and cups and caps, and I get the backward pairs of pants from the pairs of pants along with the nondegenerate inner product.

Now I can, with $\mathbb{C}G^G$, 1, and $eval_1$, I'd like to solve this system, let me change my angle again on the pair of pants. Say it's a cylinder with a little branch, another element comes and hits an element of $\mathbb{C}G^G$, and changes it. If I diagonalize these guys simultaneously, this means we want to find the joint spectrum of these operators. Well $Spec \mathbb{C}G^G$ is just maps from $\mathbb{C}G^G \to \mathbb{C}$ which is $\hat{G}$, the collection of irreducible characters of G. If I want to describe what I assign to a surface, I'll do that in the basis of characters. Now calculate what I assign to a closed surface. Think of starting with a cap, glue to the twice punctured torus a bunch of times, and then cap off. What we get is that to a surface of genus g, what we get is

$$\sum_{\chi \text{ irreducible character of } G} \left(\frac{|G|}{\dim \chi}\right)^{2g-2}$$

What I'd like to do now is see how much representation theory we can get to. In order to do that, I want to see what this assigns to points.

To a two-manifold we assigned a number $\int_\varphi e^{-S} d\varphi$. To a one-manifold we assigned a vector space, functions on boundary fields. Now suppose I have a one-manifold with boundary. What should I think of there? I need to fix boundary conditions to get something well-defined. So I want to think of this schematically, once I've fixed boundary values I should get a vector space. I can think that every time I fix a field on the boundary, I should get a well-defined vector space. So $Z(N)$ is a vector bundle (or sheaf) on the space of fields on the space $Fields(\partial N)$. So to ∂N the zero manifold I will assign the collection of all vector bundles on fields on ∂N, which is a linear category.

Let me just say what I want to do to make sense of this in our case. So I will think of the same kind of pictures we have before. I have a one-manifold which is a cobordism from a 0 manifold to another. So I think of fields as things that map from one ∂_{in} to another ∂_{out} part of the boundary. Now $Z(N)$ will give a map from vector bundles on $Fields(\partial_{in})$ to vector bundles on $Fields(\partial_{out})$, so I'll get a map between these. You pull back the vector bundle on the incoming boundary to N and then we need to integrate or push forward. So I'll write schematically $\pi_{out*}\pi_{in}^{*}V \otimes e^{-S}$ where this last bit is a correction factor. To think of this more complicatedly, you could think of two one-manifolds with a two-manifold interpolating. So these should be two-functors and a natural transformation.

Rather than explain what this really mean, let me just return to our example and relate these to representations of finite groups. What do we assign to a point in our case? To a zero manifold, we were supposed to get $Vect(Fields(\cdots))$, so this is a point with automorphism group G, so I want vector bundles with symmetry group G. So we want a vector space with an action of G, so a representation of G.

Now, how does this, if I give you a finite group, that's th same as an action of the group algebra, a $\mathbb{C}G$-module. So group elements give a basis for the group algebra.

Now what kind of pictures can I draw? What structures do I see on the collection of representations. Cups, caps, and lines. So I can take two representations and we should get, not a number, but a vector space. So I need a map from A and B, and that will be $Hom_{Rep_G}(A,B)$.

Suppose I have a triangle with vertices A, B, and C, so I'll have elements in $Hom(A,B)$, $Hom(B,C)$, and $Hom(A,C)$, So I should get by filling in the map $Hom(A,B)\otimes Hom(B,C) \to Hom(A,C)$, and that's just composition.

Now let me draw a couple more interesting pictures. What other pictures can I draw. Suppose I put A and A on my endpoints instead of A and B. Now I'll draw a two-manifold with corners, where I have a cup with a point A on the boundary. At the corner I have A. Then I have endomorphisms of A, and then I have a map $\mathbb{C} \to End_G A$. This is still not so interesting. There's always the identity. If I draw it the other way around, I get a map $End_G(A) \to \mathbb{C}$, so if I take any endomorphism, I have a trace on it. This category of representations has a canonical trace. The identity representation gives me the dimension of A.

Maybe I'll just draw one more set of pictures, and the final set of pictures I'll do is a refinement of those two pictures. Instead of thinking of this disk, I'll expand it and chop off the end, and take a cylinder with a marked point on one boundary. So I get a map both ways between $End_G(A)$ and $\mathbb{C}G^G$. This is actually an isomorphism, and this is the center or Hochschild cohomology of my category, so $\mathbb{C}G^G$ is isomorphic to the center of my category. To me then this is the centor of the group algebra $Z(\mathbb{C}G)$. The center of the group algebra will act by endomorphisms. So the class functions are indeed the center of the group algebra.

That's our toy model for Hochschild cohomology. The dual picture is that there is a canonical map from $End\ A$ to $\mathbb{C}G^G$, meaning that these are also the Hochschild homology of the group algebra, meaning that they are the thing that carries the universal map, the universal trace map. The center is the universal thing that maps in, universal commutative thing mapping into $\mathbb{C}G$, and the dual thing HH_* is the thing maping out, so that $tr(a*b) = tr(b*a)$. Concretely what is this thing? That can't be anything unfamiliar. The endomorphisms of representations are determined by their image of the identity, and so the identity is assigned the character of its representation.

I'm out of time. Next time I'll start even more explicitly with representations of a finite group, and then replace those with a Lie group.

[You haven't used manifolds. When you get to higher dimensions, there could be a gap between the information you've used and that you haven't]

I'll defer that to the next speaker.

[Can you see anything interesting about finite group representations from this?]

I don't know about anything, this is very general.

2 Toën

Thanks for the invitation, I'm very glad to be here, I'm impressed but I'm also very jet-lagged. So the purpose of this series of lectures is to introduce secondary K-theory. I'll introduce a first definition today and a better definition in the next set of lectures. What we want to do, this is supposed to a categorified version of algebraic K-theory. By categorified, I mean that the objects used to define K-theory, vector bundles, coherent sheaves, are replaced by categorical versions.

To make precise what I mean by a categorical version, let me give you some reasons why we want to do this, some motivation.

- $K^{(2)}(k)$ encodes interesting higher invariants of $X = Spec\ k$.
- $H^1(X, Gm) \cong Pic(X) \to K_0(X)$
- $H^2_{et}(X, Gm) \to K_0^{(2)}(X)$. So $K^{(2)}(k)$ seems close to a K group of motives over $X =$ *Spec* but in the non-commutative setting.

For example, we will see the deRham realization. Okay, and it's not really motivation, but more like a remark. This categorical replacement makes the K-theory more discrete than the original algebraic K theory. There is an additional discrete part. This has something to do with secondary characteristic classes. It is connected to the Grothiendiek group of some kind of sheaves.

The lecture today is to give you the idea of $K_0^{(2)}$ of a commutative ring, so we will need some notions. This talk and the next one will be mainly definitions. So

Here I will define $K_0^{(2)}(k)$. I should say k is a commutative ring, $C(k)$ is the category of complexes of k-modules.

Definition 1 *The category of $dgCat/k$ is defined to be the category of $C(k)$-enriched categories*

We will use $dgCat/k$ to construct a monoidal symmetric category $H_0^{Mor}(dgCat/x)$ (Mor is for Morita) which will be the categorifed version of the monoidal category k-mod with the usual tensor structure over k.

By definition $H_0^{Mor}(dgCat/k)$ is a localization of $dgCat/k$. Fix an element T in this category. We can look at T-dgmod, left dg modules over T, which is the same as $Hom(T, C(k))$. That is, it is a complex in k for each object x in T and then morphisms of complexes for each morphism in T. That's the data of a dg module. These form a category. Inside are the quasiisomorphisms, which are those for which the induced morphism is a quasiisomorphism. So these are $f : E \to E'$ such that for all $x \in T$, $E_x \to E'_x$ is a quasiisomorphism.

Then $D(T)$ is $[qi]^{-1}(T - dgmod)$, formed by inverting the quasiisomorphisms. So $f : T \to T'$ in $dgCat/k$ is a Morita equivalence if $f^* : D(T) \to D(T')$ is an equivalence of categories.

Definition 2 *$H_0^{Mor}(dgCat/k)$ is [Morita equivalences]$-1(dgCat/k)$*

You have to check that the Hom sets are isomorphic to small sets.

Theorem 1 *If T, T' are in $dgCat/k$ where T has flat homomorphisms over k, then $[T, T']_{H_0^{Mor}}$ is naturally isomorphic to $\{E \in D(T \otimes_k T'^{op}) : \forall x \in T E_x$ is compact in $D(T'^{op})\}$/isomorphisms. We denote the compact objects by $D_c(T'^{op})$.*

An object is compact if $Hom(X, \oplus Y_\alpha) \cong \oplus Hom(X, Y_\alpha)$.

A consequence is that the localization exists, but it also gives you a picture of what this looks like. This is a categorification of [unintelligible]. There is a remark which is important for what Jacob will say, which has something to do with the 2-categorical structure. You see that this is a truncation of a natural 2-category. So $H_0^{Mor}(dgCat/k)$ is the 1-truncation of a natural 2-category, well, $(2, \infty)$-category, where the objects are the same as in $dgCat/k$ and the category of morphisms $Hom(T, T')$ are $\{E \in D(T \otimes T'^{op}) : \forall x, E_x$ is compact$\}$ as full subcategories in $D(T \otimes T'^{op})$.

I need the monoidal structure. If I have T and T' two objects in $dgCat/k$, then their product $T \otimes T'$ is in $dgCat/k$, where the objects are $ob(T) \times ob(T')$ and $(T \otimes T')((x, x')(y, y')) =$

$T(x,y) \otimes_k T'(x',y')$. This tensor structure $\otimes_k^L$ makes $dgCat/k$ into a closed monoidal category, which induces a closed monoidal structure on $H_0^{Mor}(dgCat/k)$. This will make the category into a symmetric monoidal category (closed).

There is a model structure on $dgCat/k$ whose weak equivalences are Morita equivalences (Tabuada) but it is not a monoidal model category for this monoidal structure. So you have to prove this by hand.

This will be the main object of study. In a way, this is already a way to construct objects. This is a first construction procedure. So there is a localization construction, which takes an object in $dgCat/k$ and inverts some subset of maps as a universal problem in this internal category. So start with T a $dgCat/k$. I will denote by $[T]$ the homotopy category of T, which has the same objects, but maps $[T](x,y)$ will be $H^0(T(x,y))$. The data will be a subset of morphisms, and a localization of T along S is an object S along with a map $T \to L_S T$ in $H_0^{Mor}(dgCat/k)$ such that the image of C^* is maps $T \to T'$ satisfying

$$\begin{tikzcd} {[T]} \arrow[r] \arrow[d] & {[T']} \\ S^{-1}[T] \arrow[ur] & \end{tikzcd}$$

Proposition 1 *Localizations always exist*

This allows me to give you an example. Look at $C_{\mathrm{pe}}(k) \subset dgCat/k$, complexes isomorphic to complexes of projective modules of finite type, and inside here you have quasiisomorphisms. So then $L_{\mathrm{pe}}(k)$ is $L_{qi}C(k) \in H_0^{Mor}(dgCat/k)$

More generally, if T is a dg Category over k then there are perfect modules in the T-dg modules, which induce compact things in the derived category $D_c(T) \subset D(T)$. So you can then look at $L_{\mathrm{pe}}(T)$ which is $L_{qi}T - dgmod_{pe}$. I will give you one more example and then a way to describe these explicitly. Take X a scheme, and then $Perf(X)$ is the dg category of perfect complexes of $\mathscr{O}_X$ modules on X. Locally they're isomorphic to bounded complexes of vector bundles. Inside of this you have quasiisomorphisms, and then $L_{\mathrm{pe}}(X) = L_{qi}Perf(X)$.

In all these examples you have $[L_{\mathrm{pe}}(X)] \cong D_{\mathrm{pe}}(X)$ and $[L_{pe}(T)] \cong D_c(T)$. Then $L_{\mathrm{pe}}(X)(E,E') \cong \mathbb{R}Hom(E,E')$.

Now let me make an analogy between $H_0^{Mor}(dgCat/k)$ and $k-Mod$. We want to see the first as a categorical version of the second. You have $\otimes_k$ and $\otimes_k^R$, they are both closed. I want to consider projective modules of finite type over k, which are exactly the dualizable objects in k-mod. There is a dual $M^\vee$, and then you have $k \to M^\vee \otimes M \to k$. A module with a dual in this sense is projective of finite type, and vice versa. This is how you identify projective modules of finite type with dualizable objects. Then my version of this in $H_0^{Mor}(dgCat/k)$ are dualizable objects in $H_0^{Mor}(dgCat/k)$.

We are getting close to K-theory. So now on the module side we have short exact sequences. On the other side we have the notion, well, it's exact if it's equivalent to a composition of $T \hookrightarrow T'$, a (dg) fully faithful dg functor. and then T/T', simply the quotient by mapping

T to 0. It exists and can be described more explicitly. It's on the level of $dgCat$ tensors. Anything equivalent to one of these sequences is a short exact sequence.

Let me give now the definition of $K_0^{(2)}(k)$. This is $\frac{\mathbb{Z}[\text{saturated dg Categories}]}{T'=T+T/T'}$. In the last two minutes I will give you an example. If X is smooth and proper over k then $L_{\text{pe}}(X)$ is saturated so $L_{\text{pe}}(X) \in K_0^{(2)}(k)$.

More generally, $K_0(Var/k)$ is $\frac{\mathbb{Z}[Varieties/k]}{X=Y+U}$ where $Y \hookrightarrow X$ is closed.

Theorem 2 *([unintelligible])There exists a map to $K_0^{(2)}(k)$.*

This tells you that this group receives some interesting examples from algebraic geometry. I will give a few more examples in the next lecture. The ultimate goal is the existence of a Chern character $K_0^{(2)}(k) \to ?$ and the spectrum for a scheme X which sholud be $K^{(2)}(X) \to ?$.

3 Jacob Lurie

In these lectures, a manifold will be smooth, compact, oriented, and often with boundary. Manifolds of a fixed dimension can be organized in a category $Cob(n)$, whose objects are closed $n-1$-manifolds and whose morphisms are bordisms of $n-1$ manifolds. A bordism $X \to Y$ is a manifold whose boundary is X and Y with proper orientation. We call two the same if they are diffeomorphic relative to their boundary. This is symmetric monoidal, with $\amalg$ the disjoint union.

Definition 3 *(Atiyah)*
An n dimensional TQFT is a tensor functor $Z : Cob(n) \to \mathbb{C}$-vector spaces.

That it is a tensor functor means that $Z(M \amalg N) = Z(M) \otimes Z(N)$ and $Z(\emptyset) = \mathbb{C}$.

To get us all on the same page, let's review the example where $n = 2$. David already explained this in his lecture this morning. Let me say more explicitly what he explained. He said that two dimensional TQFTs are equivalent to commutative Frobenius algebras. Let's review this. Suppose you have a two dimensional TQFT, you can evaluate this functor on objects, on a finite union of circles. So you should get a vector space $Z(S^1) = A$. There aren't a lot of choices of one manifolds, for a disjoint union of circles you get a tensor power of A. Morphisms are given by bordisms between 1-manifolds. You can evaluate on a pair of pants and you'll get a map from the vector space $A \otimes A \to A$. As David explained, this is a commutative and associative algebra, which has a unit, given by evaluating Z on a disk. There's also, you can read this going in the opposite direction, and you get a map $A \to \mathbb{C}$ which I'll call trace. If you look at $A \otimes A \xrightarrow{m} A \xrightarrow{tr} \mathbb{C}$, this gives you something nondegenerate, a nondegenerate pairing. So this is finite dimensional, and this makes A into a commutative Frobenius algebra. You can recover a unique topological quantum field theory from a Frobenius algebra. You can cut up any surface into reasonable pieces, I'll come back to that. Let me give a more concrete example, David's lecture. To any finite group you get a field theory, and $Z(S^1)$ will be the center of the group algebra. Returning to the general case, here we looked at $n = 2$, by seeing what Z does on objects. There's only one interesting vector space here, but it's also possible to emphasize what Z does on dimension n. This is a closed n-manifold, so it's a bordism from the empty set to itself. This is a linear map from the complex numbers to itself. This is given by multiplication by some number. I'll just abuse notation by calling this number $Z(M)$. It then assigns numbers to closed n-manifolds. He wrote down a formula for what that number was. More generally, you might think, what are the invariants you get. Any commutative Frobenius algebra generates one of these, which we can evaluate on closed surfaces and get numbers. How would you compute those invariants?

What is $Z(\Sigma_g)$? Let's draw a picture. Here's Σ_2, and then we can cut this up along circles to get pieces that are simple. Here I've carved up into six pieces. Two are disks and four are pairs of pants. You know what this field theory assigns to each pair of pants and each disk, so you need to just compose a bunch of linear maps. The hard work is in showing that the number you get doesn't depend on how you drew your circles.

If you've not seen this before, here are some examples you can do, you get $Z(S^2) = tr(1)$, and $Z(T^2)$ should be $dim\ A$.

These tools show what happens when you cut along closed submanifolds of codimension one. If n is large this is not very simple. This sort of analysis will not give us a very good handle in high dimension. You want to be able to compute your invariants given decompositions into arbitrarily fine pieces. Your gluing is not along closed submanifolds of codimension one. We need manifolds with corners with higher codimension.

Definition 4 *(Sketch) An extended n-dimensional TQFT is a rule which assigns data to manifolds of dimensions at most n. To n manifolds it assigns complex numbers. To closed $n-1$ manifolds it assigns vector spaces. To bordisms it assigns maps of vector spaces. There should be compatibility between bordisms and closed n-manifolds. The map from $\mathbb{C} \to \mathbb{C}$ should be multiplication by the number given by the other piece of data.*

There are some choices about how to extend this. A lot of people say that for $n - 2$ you should put $\mathbb{C}$-linear categories. This means that the Homs should form complex vector spaces. Bordisms of $n - 2$-manifolds should be assigned the appropriate notion of map, which I will call $\mathbb{C}$-linear functors. There is compatibility between this and what you have higher up in the diagram. You should get a $\mathbb{C}$-linear functor to a closed $n - 1$ manifold, which should be given by multiplication (tensor product) by the given complex vector space.

David was describing an extended two dimensional TQFT. His linear category was representations.

This is a sketch; these invariants should obey certain rules that describe what happens as you glue these together. You could formalize these in an efficient way originally using category theory. Saying that this is a functor encodes a gluing rule. In the case $n = 2$, you have bordisms, and gluing maps, so we should get composition of linear maps when we glue together. You can do this in a similar language. You want to say that these invariants "glue nicely." We can formulate this with category theory, but you need fancy category theory. An extended n dimensional TQFT is a tensor functor between n-categories, where on the left we get $Cob(n)$ but more elaborate, and on the right side we see something more elaborate than vector spaces. I'm not going to define the notion of an n-category in this lecture. If you're not familiar, you can take it as a black box, it's supposed to axiomatize the ways that manifolds with corners can be glued together.

When you make a definition, you might think it's a complicated thing. You need n-categories, and then you need these two particular ones, meaning we need to assign invariants to n-manifolds with corners, things with low dimension, and more abstract things to assign to low dimensional manifolds.

But there's another idea out there, which is that this notion should be much much simpler. This was articulated by Baez-Dolan in what is called the cobordism hypothesis. Let me paraphrase it:

Hypothesis 1 *Extended TQFTs are "easy," that is, easy to describe, construct, and classify.*

Before I go on, let me say why you might believe this. You need to assign an invariant to every n-manifold with corners. But locally you get just something that looks like Euclidean space. We should be able to calculate what is associated to a complicated manifold by looking at what it does on a piece. Because any manifold can be built out of these constituents, you should be able to describe it by describing it on the simple pieces.

Let me describe this on the example we can handle, where $n = 1$. Let's now consider an example where $n = 1$. If you thought I was doing an easy example before then now this should be even easier. So $Z : Cob(1) \to Vect(\mathbb{C})$. Let's start by evaluating this on objects. This is just a finite union of points. This needs an orientation. Some of the points are positive and some are negative. I can evaluate Z on a positive point and I should get X. I could also evaluate on a negative point and get Y. The axiomatics force us to have X and Y be dual. Consider the interval as a bordism from two points to the empty set. Then we get a map $X \otimes Y \to \mathbb{C}$. This map ev should exhibit X as a dual to Y. We can construct an inverse, a map from $\mathbb{C} \to X \otimes Y$, and I'll leave it as an exercise (Zorro's lemma). Once I know this, I know that X is finite dimensional because Y is a dual to X. So I can forget about Y and just remember that X is finite dimensional. You know now what Z is doing on all objects. So I could have three points, two positive and one negative. This should be taken by Z to $X \otimes X \otimes X^\vee$.

So now turning to bordisms, we just need to think of what happens to connected one-manifolds, because disconnected ones should correspond to tensor products. So I have five of these, corresponding to $Z(\to) = \mathrm{id}_X$, $Z(\leftarrow) = id_{X^\vee}$, and then evaluation $X \otimes X^\vee \to \mathbb{C}$, coevaluation $\mathbb{C} \to X \otimes X^\vee$, and then the circle, which should be taken to the dimension of X, the trace of the identity.

What's the upshot? A one dimensional TQFT Z is uniquely determined by $Z(*) \in Vect_{\mathbb{C}}$. What is the n-dimensional analogue. To guess at that, I need a little bit of terminology. I said you should think of this as a functor between n-categories, so I will introduce one of these called $nBord$. In this lecture, I'll only explain what this is by example. The objects here are oriented 0-manifolds. Then I'll have morphisms, which will be bordisms of 0-manifolds, and 2-morphisms, which are bordisms between bordisms, and you're supposed to continue along these lines until you hit bordisms between bordisms between ... At stage n I consider these to be the same if they're diffeomorphic relative to their boundaries.

Let me make an incorrect guess at the answer. You might think, take $\mathscr{C}$ a tensor n-category, and you can look at $Fun^{\otimes}(nBord, \mathscr{C}) \to ob(\mathscr{C})$, determined by a point.

Remember that for a vector space to appear as $Z(*)$ it needs to be finite dimensional. To fix this you need to impose some finiteness condition, namely that this should be in bijection with things that are finite dimensional. So let me say fully dualizable objects. This is a finiteness condition that makes sense in an n-category with a tensor product. When the category is vector spaces it reduces to finite dimensional.

The reason we believe this is because locally every manifold looks like $\mathbb{R}^n$. What this means is that locally you look like $\mathbb{R}^n$. You can compare the tangent space, or a neighborhood of zero in it, to a neighborhood. Manifolds of dimension one, you get a framing, an orientation gives one. Giving an orientation and a framing, well, those are different for $n > 1$, because of non-trivial tangent bundles. Now I wish I had not erased this board. I wish I could point to the definition of $Bord_n$. I described for you an n-category that I called $Bord_n$. Now I want to replace $Bord_n$ by a framed version which I will call $Bord_n^{fr}$. If M is a manifold of dimension $m \leq n$, then an n-framing of M is an isomorphism, $T_M \oplus \mathbb{R}^{n-m} \cong \mathbb{R}^n$. Now $Bord_n^{fr}$ is defined like $Bord_n$ but using n-framed manifolds.

If I restrict my attention to the n-framed bordism category, this should kill off the orientation versus framing issue. The precise statement says that for any n-category with a tensor product, giving a tensor functor $Bord_n^{fr} \to \mathscr{C}$ is the same as giving an object, and then the objects that so arise are fully dualizable ones.

So you might say that's not so good, because restricting to the framed case, when $n = 2$, I've thrown away most closed surfaces. You could ask many questions. What does any of this mean? I've used terms I haven't defined here, like n-category, fully dualizable, and $Bord_n^{fr}$. You might ask for a precise definition. A less formal question is to ask what happens if we allow non-framed manifolds. The short answer is that there's a good theorem in that case too, but they require ideas that I don't have time for in this lecture.

Also applications, let's look at interesting n-categories, then you look for fully dualizable objects, and you find topological field theories. The field theory he was describing arise as an application of this theorem for a convenient choice of $\mathscr{C}$ and convenient fully dualizable object. A followup question is, what do you get in that case? This is what I will try to address in this lecture series. Here's the rough plan. On Tuesday I'll address some of the questions of the first type, like sketch the definition of some of the terms like n-category, so that people can get ideas for how thes things are defined. On Thursday, I'd like to answer question two, and see how we can generalize to the non-framed case. On Friday I'll probably talk about answering the question, what about examples. All right, that's all I got. Thank you.