Oberwolfach

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[Hi and welcome to Oberwolfach. Mike Hopkins will be our honorary helper with the organization. I'll try to stagger some of the organizational items. It would be nice if we can post some of our stuff on that wall in the library on the left. Those of you who submitted an abstract, those are posted there. That gives ideas of what's up, and then you can grab someone and talk to them. Take advantage of that and also post things. This helps us to decide who might fit in in terms of a talk. Our first talk will be on CFTs and graded tensor categories by Christoph [unintelligible]]

I will start with the overview, and is already apparent in the title, I will do something that, well, we'll do algebra and representation theory in braided tensor categories. I'll state theorems and give definitions. I'll also explain motivations and applications, which will involve interpretation.

Let me start the first chapter anbout modular tensor categories and CFTs, and I'll be talking about rational CFTs.

There's a mathematical structure that I won't explain, which is the structure of a vertex algebra. You don't know this, you won't need to for this talk, I'm actually talking about conformal semisimple vertex operator algebras and also a few more properties hidden in the word rational.

These have representations, and we'll work in the category $\mathscr{C}$ of representations. This category gets a structure that's involved, and you can get vector bundles over the moduli spaces of confromal structures, and this comes with a projective flat connection. In general this gets the structure then of a braided tensor category, but in our case this should be a modular tensor category and the definition of the modular tensor category will be the first one I'll write on the blackboard.

Definition 1 *A modular tensor category is:*

- *Abelian*

- *$\mathbb{C}$-linear*
- *a semisimple tensor category. I will assume that tensor categories are strict*
- *Noetherian, only finitely many isomorphism classes of simple objects,* $\mathbf{1}$ *is simple, and let me choose representatives including* $\mathbf{1}$ *of the isomorphism classes.*
- *then a ribbon category, meaning there is a braiding and a duality meaning* $?^{\vee}$ *and* ${}^{?}\vee$ *coincide.*
- *There is a nondegeneracy condition on the braiding. First of all,* $K_0(\mathscr{C})$ *is a ring. Let me turn it for convenience into a complex algebra, then there is like a Fourier transform,* $K_0(\mathscr{C}) \otimes_{\mathbb{Z}} \mathbb{C} \to End(id_{\mathscr{C}})$ *where you use the twofold braiding [picture] in* $End(V)$*.*

  The braiding is indicated by these crossings, and I will read up from the bottom. The nondegeneracy condition is that this should be an isomorphism of algebras.

It is a fact that i am going to use that goes back to [unintelligible]-Turaev that for any modular tensor category you have a tensor functor which I will call $tft_{\mathscr{C}}$ from cobordisms to finite dimensional vector spaces. The cobordisms are decorated. I won't use the formalism of extended tfts. I will use decorations however. The cobordisms are three-dimensional and the objects are two-dimensional. I want to highlight two facts about it.

1. The factorization homomorphism, if you have a closed oriented surface with marked points (which are germs of curves) and a Lagrangian (suppressed) with another copy of the same surface, then I have a distinguished cobordism between them, the cylinder, and the points remain as ribbons. I can glue two of them with a little ribbon. So they must be decorated with U and $U^{\vee}$. Then I get a factorization morpphism $tft_{\mathscr{C}}(X_{U,U^{\vee}} \to X)$. There is in Turaev's book a theorem that if you add these togother you get an isomorphism $\bigoplus_{i \in I} fact_{U_i} : \bigoplus tft(X_{U_i,U_i^{\vee}}) \tilde{to}\, tft(X)$.
2. Representation of the mapping class group $Map(X)$ on $tft_{\mathscr{C}}(X)$.

Okay, so this was just introducing structure I need. Now let me talk about the things I am really interested in. Let me talk about the correlators in the conformal field theory. Then we would work with X two dimonsional conformal manifolds. I should emphasize that there is an oriented version and an unoriented version. In any case the manifolds can have boundary, which can be empty. I will only consider the oriented case in these lectures. In the spirit I have just explained, X will become a topological manifold.

Now let me say some words of physicists. You could call these vertex algebras chirally conserved quantities, which gives restrictions on correlators. They take their values in a certain space of conformal blocks. So our strategy will be that we want to decorate X (the worldsheet) such that in the end I have enough data to get an object in the cobordism category and use the tft to find appropriate spaces of functions for correlators. The crucial idea is holomorphic factorization. What I'm going to do is to associate to X its oriented cover, gluing together points over the boundary on X. For example, if X is a disk, I take two copies of the disk, glue them together, and get the sphere. If I have a closed surface, I get two copies of it with opposite orientation. So I get $\hat{X}$ which is acted on it by an orientation reversing involution, with a projection onto X; the fixed points are the boundary.

Our goal will be to find a decoration of X such that $\hat{X}$ is an object in $cobord_{3,2}^{\mathscr{C}}$. Then I want to specify the correlator in such a way that $Cor(X) \in tft_{\mathscr{C}}(X)$. If I start with something closed and oriented, I get something [unintelligible]which corresponds to left moving and right moving degrees of freedom.

There are two constraints. $Cor(X)$ should be invariant under the elements of $Map(\hat{X})$ which are invariant under σ. The mapping class group for the torus say, is two copies of the mapping class group of the torus, and it would be those invariant under the diagonal action.

We require compatibility with factorization homomorphisms. This links the correlators for worldsheet X with different topology.

We want to find elements in these conformal blocks which are invariant and obey the factorization rules.

Let me state an insight; the decorations for this are furnished by the bicategory of special symmetric Frobenius algebras in $\mathscr{C}$.

An algebra in the tensor category is an element A together with a unit η and $m : A \otimes A \to A$, and the most convenient way to state a Frobenius algebra is that there is a coproduct Δ, a counit ϵ, and the coproduct is a map of A-bimodules.

Symmetric means that two [pictures] coincide,

I also require that $m \circ \Delta = id_A$ If I consider $\epsilon \circ \eta$, I get a multiple of the identity, the dimension of A.

The objects are then the special symmetric Frobenius algebras. My next task will be to explain to you how to decorate our surface with these data. Let me write down what a typical worldsheet looks like. It can be of any genus, with boundary and another boundary component, and more structure, it can have defect lines, distinguished lines that can start, for example at a boundary, which can disappear but at a marked point, which can carry marked points on them, that can be closed cycles, and I will assume that if the worldsheet is cut into different connected components, you can have different Frobenius algebras.

To the two-cells I'll associate a special symmetric Frobenius algebra A. There are boundaries and defect lines. Boundaries will have either Frobenius algebras on the left or the right; they will correspond to left and right modules, and the defect lines correspond to bimodules.

[question]. If I were doing Σ-modules, I would allow maps into $SU(2)$ or $SO(3)$ and glue them together. [note: huh?]

Let me start with the easiest pieces for 0 dimensions. If I have a junction in the middle of A_1, A_2, and A_3. So it's an element of $Hom_{A_1,A_3\text{-bimodules}}(D_1 \otimes_{A_2} D_2, D_3)$.

What about points, on the boundary, $x \in \partial X$ I get a single point in $\pi_*^{-1}(x)$, so I'll attach U in $\mathscr{C}$. Somehow if I have a boundary which changes from M to N, I should have a chiral insertion. which would be in $Hom_A(M \otimes U, N)$.

I can also have insertions in the interior of X, and I have two preimages in the double cover, so I attach (U, V) as my markings, ordered because the cover was oriented, and the morphism I will use is, I can have an insertion on a defect line labelled with D_1 on one side and D_2 on the other. I have U and V, and I will insert a homomorphism of bimodules $Hom_{A_1|A_2}(U \otimes D_1 \otimes V, D_2)$, where I need to give $U \otimes D_1 \otimes V$ a bimodule structure. To act on the left I will do an overbraid over U and to act on the right an underbraid under V

Now I want to specify an element in this space of blocks. Correlations from cobordisms. I want $M_X : \emptyset \to \hat{X}$ and then $Cor(X)$ should be $tft_{\mathscr{C}}(M_X)1$. That will be in $tft_{\mathscr{C}}(\hat{X})$. Now I should specify a decorated 3-manifold M_X. I take $\hat{X} \times [-1, 1]/(\sigma, t \mapsto -t)$. Then I get an oriented quotient whose boundary is $\hat{X}$ with a distinguished embedding of X, choosing $t = 0$.

For example, take X to be a disk with a defect line. I have an insertion on the boundary W and on the inside an insertion U, V. I've chosen M_2, M_1, and M_3. Then $\hat{X}$ is a sphere and the connecting manifold is a 3-ball. The boundary is $\hat{X}$ and I have the orientation reversing involution, and I have an embedding of the disk. Over each point of the interior I have an interval; over a point of the boundary I have a half-interval.

I still need algebras for the cells here. I will take a dual triangulation. [Many pictures.]

I want to show that this is invariant under the mapping class group and obeys the factorization constraints. The fact that this holds is the first theorem.

[Many questions.]

Let me state things without writing them. This construction furnishes elements that have the factorization and mapping class constraints. Once you assume your correlations have the property that the two point functions on the disk and sphere are nondegenerate, you get any solution of these two constraints, can be obtained by this construction.

[More discussion.] [Ingo's talk will be at 11:15.]

2 Ingo

[There are lists. The default of the drink at breakfast is coffee, but you can put your name on the tea list, and then there will be enough tea provided. In the afternoon the default is coffee. For eggs there is a list. Anyway, that's all the technicalities I can think of for now. This evening at 8PM there will be a discussion of these two lectures. Now we will hear the second part.

The second port should be called algebra motivated by conformal field theory. I first want to look at a structure called a bulk algebra. Then I want to discuss a Morita invariant formulation of the data for a CFT, which will be a module category. Thirdly will be an outlook, and I never make it there so I can say what I want. I want logarithmic conformal field theory.

Let me recapitulate how to organize the data in a bicategory.

What did we have? I willput square brackets for conformal field theory to distinguish it. I had [a bicotgery, a vertex operator algebra, its representation category, which we assumed to be a modular tensor category.

The input data was $\mathscr{C}$ the modular tensor category, and then the bicategory whose objects are special symmetric Frobenius algebras. Morphisms are bimodules and 2-morphisms are interwiners. I will only put brackets on the left, I am still inside my bracket.

A Frobenius algebra in $Rep\ \mathscr{V}$ was used to decorate the ribbon graph in the three-manifold which gave the vector to produce the correlator. These lead a double life. If I consider the boundary of the surface, we have learned we should label the boundary by an A-module, and then the data I have to specify for the marked point on the boundary is an element $\varphi \in Hom_A(A \otimes U, A)$. This space is particularly simple, this is canonically isomorphic to $Hom(U, A)$. This tells me how many ways I can put a representation on the boundary. Another way to say this is I have a space of boundary fields, open states, $\mathscr{H}_{AA}$ which is just given by A. Because this is a representation this will be an infinite dimensional vector space.

If I have two different boundary conditions labelled by M and N I will get $\psi \in Hom_A(M \otimes U, N)$, then the object that represents this functor I get that $\mathscr{H}_{MN} = M^\vee \otimes_A N$. Insertions on the boundary I associate special symmetric Frobenius algebras, what happens now I can ask on a disk if I have an insertion on the boundary and one in the interior. We have seen in $\hat{X}$, which is a sphere, that a puncture on the boundary gives a marked point on the sphere, and the marked point on the interior is two marked points on the sphere. Whatever object we associate should have to do with two copies of our category of representations. Ultimately we get a multilinear function $A \otimes_{\mathbb{C}} C \to \mathbb{C}$ where A is in $Rep\ \mathscr{V}$, C is in $Rep\ \mathscr{V} \boxtimes Rep\ \mathscr{V}$. I take tensor products over $\mathbb{C}$ of the morphism spaces, taking direct sums, using $\boxtimes$. One of the two copies of $\mathscr{C}$ I take to be $\mathscr{C}_-$. To make this symmetric I call the other one $\mathscr{C}_+$.

Now the algebraic structure, you can already tell what it should be. We expect to find that it should be a commutative symmetric Frobenius algebra. I will give the algebraic construction which produces the algebra C out of our starting point. (I am still in the bracket). I am in the special situation where the boundary condition of the disk is A itself.

We want to take A and get something in $\mathscr{C}_+ \boxtimes \mathscr{C}_-$, and you can do this with the functor R which takes you to the direct sum over irreducibles

$$V \mapsto \bigoplus_{i \in I} \underbrace{V \otimes U_i^\vee}_{\mathscr{C}_+} \times \underbrace{U_i}_{\mathscr{C}_-}$$

which is the adjoint of the functor $T : \mathscr{C}_+ \boxtimes \mathscr{C}_- \to \mathscr{C}$ which takes $U \times V$ to $U \otimes V$.

So if A is a special symmetric Frobenius algebra (ssFa), then $R(A)$ is a ssFa in $\mathscr{C}_+ \boxtimes \mathscr{C}_-$, but this algebra is typically not commutative.

We could take the center, and because this is braided we have to be careful. One definition is

Definition 2 *The center of an algebra A in $\mathscr{C}$ is the maximal subobject C of A such that left-braiding and then multiplying is the same as multiplying. I'll get different answers for the two braidings, so the left center and the right center. You can look at examples where the left center and right center are not isomorphic. You can even find examples where they are not Morita equivalent. You can find subcategories of their algebras which are equivalent.*

Definition 3 *The full center of a ssFa A in $\mathscr{C}$ is $C_\ell R(A)$, which lives in $\mathscr{C}_+ \boxtimes \mathscr{C}_-$.*

I choose the left center because the choices I have made in constructing the algebra on $\mathscr{C}_+ \boxtimes \mathscr{C}_-$ demand it. Let me call this center $Z(A)$.

Let us look at some properties. One is that $Z(A)$ is commutative, and is also a ssFa in $\mathscr{C}_+ \boxtimes \mathscr{C}_-$. We're already on a good track. It contains both of these centers $C_\ell(A) \times \mathbf{1}$ and $\mathbf{1} \times C_r(A)$ as subalgebras. I can also decompose $Z(A)$ into simple objects. So I can write $Z(A) = \bigoplus (U_i^\vee \times U_j)^{\oplus z_{ij}(A)}$ where the sum runs over simple objects. This is called the modular [unintelligible]matrix. This has many nice properties. The number of isomorphism classes of simple A-left modules is $tr\ z(A)$. The little $z_ij(A)$, remember in Cristoph's talk he said a marked point is in a vector space, so $z_ij(A)$ is given by $dimHom_{A|A}(U_i^\vee \otimes A \otimes U_j, A)$.

One thing that $Z(A)$ can do for us is define a closed CFT, for worldsheets without boundary. I've got a worldsheet without boundary and with marked points. I'll eventually get a multilinear map from $Z(A) \otimes \cdots \otimes_{\mathbb{C}} Z(A) \to \mathbb{C}$.

Now we can think of what we actually did. We started from A which was associated to the space of boundary fields. We've come up with the data to define a closed CFT. We've constructed a closed CFT which is characterized by $Z(A)$. The boundaries contain all the data we need. In particular we assign correlators to [unintelligible].

If you start the construction from a different boundary, instead of A we took M, we'd get $M^\vee \otimes_A M$. What is the closed theory that this constructs? You can look at $Z(M^\vee \otimes M)$. It would be good if this depended only on the Morita class of the module you begin with, and this is the content of the next theorem.

Theorem 1 *Kang, Runkel*
If $\mathscr{C}$ is a modular tensor category and A and B are special symmetric Frobenius algebras, then A and B being Morita equivalent implies that $Z(A) \cong Z(B)$ as algebras.

If these were algebras over the complex numbers, the easiest way to see this would be that $Z(A)$ is isomorphic to $End(Id_{A-mod})$. The side on the right is Morita invariant so the thing on the left is. Slightly more interesting is that under these assumptions we also have the converse.

Theorem 2 *Kang*
When can I write an algebra as $Z(A)$? If $\mathscr{C}$ is a modular tensor category? If C is a commutative simple ssFa in $\mathscr{C}_+ \boxtimes \mathscr{C}_-$ with dimension of C equal to dimension of $\mathscr{C}$, then

1. *there exists A in $\mathscr{C}$ so that $C \cong Z(A)$*
2. *$T(C) = \oplus A_i$, where A_i are simple and Morita equivalent, and any one of these will reconstruct C.*

So that again looks like a fairly abstract result. If you give a ssFa, it gives you a closed CFT. You can ask which ones are part of an open closed CFT? Do I miss any? The theorem tells you that the closed ones you will get are the ones with the chiral symmetry from $\mathscr{V}$. So every modular invariant closed CFT with left and right chiral symmetry given by $\mathscr{V}$ is part of an open closed CFT, which are the ones constructed by the construction that Christoph explained.

The next thing that I want to talk about is modular categories. I have fixed a modular category and then looked for a ssFa inside. I've also looked for the algebra of boundary fields. I could have started from an equivalent boundary condition. It would be nice to have a formulation that does not rely on fixing a boundary condition.

Let me describe what a modular category is. This is in a way a categorification of a module over a ring. If you have a ring R you would have a map $M \times R \to M$. I'd like to categorify the ring R to get a tensor category, and then M will be a modular category. Let $\mathscr{C}$ and $\mathscr{M}$ be Abelian $\mathbb{C}$-linear categories. Then $\mathscr{C}$, let it be a tensor category. I call $\mathscr{M}$ a right modular category over $\mathscr{C}$ if I have a bifunctor $\odot : \mathscr{M} \times \mathscr{C} \to \mathscr{M}$ which satisfies the properties of associativity and $\mathbf{1}_{\mathscr{C}}$ acting as the unit, which only hold up to morphism, these satisfy a mixed pentagon and triangle axiom.

For example, I can take $\mathscr{M}$ to be $\mathscr{C}$ and then $\odot$ is $\otimes$. Another more interesting example is that if A is an algebra in $\mathscr{C}$ then A-mod (left A-modules) are a right module category over $\mathscr{C}$. The bifunctor is given by tensoring: ${}_AM \odot U$ is ${}_AM \otimes U$.

This is how we start with an algebra and produce a modular category. The modular category is nice because it remembers all modules. Can you reconstruct the algebra? It involves choices but it can be done.

Theorem 3 *Ostrik*
If $\mathscr{C}$ is a modular tensor category and $\mathscr{M}$ is a semisimple finite indecomposable ($\ncong M_1 \oplus M_2$) then $\mathscr{M} \cong A - mod$ for A an algebra in $\mathscr{C}$.

How can we find an algebra that does the job?

There's actually a pretty construction via internal homs. If I have a modular category $\mathscr{M}$ over $\mathscr{C}$ then for $M, N \in \mathscr{M}$ I define $\underline{Hom}(M, N)$ to be the object in $\mathscr{C}$ which represents the functor $U \mapsto Hom_{\mathscr{M}}(M \odot U, N)$.

It turns out that internal homs have an associative composition $\underline{Hom}(M,N) \otimes \underline{Hom}(K,M) \to \underline{Hom}(K,N)$. So $\underline{End}(M)$ is an algebra in $\mathscr{C}$.

So we can choose A to be $\underline{End}(M)$ for any M which is not zero. You have to choose a particular object in the modular category.

Now given two modular categories $\mathscr{M}$ and $\mathscr{N}$, I can look at functors which are compatible, so that $F(M \odot U) \cong F(M) \odot U$ with a proscribed set of morphisms. Convince yourself that if A and B are special symmetric Frobenius algebras, then $Fun(A-mod, B-mod) \cong B-A-Bimod$ as tensor categoroids. [Some difficulty. There is a bicategory where the objects are modular categories. these should be equivalent to A-mod for some ssFa in $\mathscr{C}$ and morphisms modular functors]

The bicategory which provides the data for our CFT is $\mathscr{C} = Rep\ \mathscr{V}$, and relating back to the worldsheet, the space of boundary fields is $\underline{Hom}(M,N)$. So $\mathscr{M}$ is the collection of all boundary conditions compatible with $\mathscr{V}$.

Maybe in the last two minutes, let me say that it would be nice to extract the bulk algebra again. It turns out that it's much prettier with these modular categories. If $\mathscr{C}$ is a modular tensor category and $\mathscr{M}$ a good category, then I need braided induction $\alpha^{\pm} : \mathscr{C} \to End(M)$. I have two functors which are tensor functors. These I need the braiding of the category, then I map $\alpha^{pm}(U) \mapsto (M \mapsto M \odot U)$. The $\pm$ enters to making this an endofunctor of M. These are compatible with $\mathscr{C}$'s action on the right. I can act with $\mathscr{V}$, and I have to use the braiding to exchange the two, and I have two choices. I make then $End(M)$ into a modular category over $\mathscr{C} \boxtimes \mathscr{C}$, and you take one of these functors and a pair U, V to $\alpha^{+}(U) \cdot F \cdot \alpha^{-}(V)$. Then the statement is that I can look at $Z_{\mathscr{M}}$ which is $\underline{End}(Id_{\mathscr{M}})$. You can prove a little theorem that $Z_{A-mod} \cong Z(A)$ (as algebras). This is a Morita invariant way to construct the bulk algebra.

Maybe I can summarize–I'm out of time. There is a complex analytic bit and a conformal bit. This leads you to look at algebra in this braided monoidal category. You can apply the method of algebra in this setting which is a very useful tool.

[Questions? Comments?]

3 Freed

[I wanted to tell you some good things. They have an amazing electronic library, all Springer lecture notes online and other things that you don't have]

I want to talk about joint work with Distler and Moore. The word orientafold may not be familiar. Σ will be a compact 2-manifold, and X will be a ten dimensional smooth manifold. The Σ will be the worldsheet and X will be the spacetime. The perturbative string theory is study of maps of the worldsheet into fixed spacetime via ϕ. The second theory is the ten dimensional field theory. This is a long distance approximation to the two dimensional theory. In string theory, one considers orbifolds in the ten dimensional side. A manifold is locally an open set in 10 dimensional space while in an orbifold it's an open set on a quotient. An algebraic geometer would call this a smooth real version of a Deligne Mumford stack.

The construction I want to talk about today is orientafold wher in addition we're given a double cover X_w. I want to mention two special cases. X_w has an involution σ, so one is where σ acts trivially, which is called the type one string. The second case is when we have a section, we don't have an orientafold, that's the type two theory. This is the bosonic string. What I want to talk about today is not the heterodic string but it's basically all the other [unintelligible]ones. I should say I'm not going to talk about physics of this, but it's something that physicists use quite a bit. They think there are 10^{500}.

This project started when Jacques showed me a formula with $i: F \hookrightarrow X_w$, the fixed point set, and the RR ([unintelligible]) charge, which is $\pm 2^{\#} i_*(\sqrt{\frac{L'(F)}{L'(w)}})$, where L' is like a Herzebruch genus,

$$\prod \frac{x/4u}{\tanh\ x/4u}$$

Here u is the Bott element in K theory. The ν is the normal bundle of F, and the formula lives in some [unintelligible]theory. This led to foundational questions, and we got this down to a compact set of data. It's a rather intricate system where the same data is used in different ways in the two dimensional or ten dimensional locations. We defined the fields and the theory, and defined the charge formula over $\mathbb{Z}[\frac{1}{2}]$, and the third thing was the anomaly cancellation in two dimensions. There's a post on the arXiv.

There are objects in generalized cohomology theories and then twistings. I'll talk about that and then come back to [unintelligible].

So let me say a few words about differential cohomology, and you'll hear more about that tomorrow, but say h is any cohomology theory, then if I look at h with rational coefficients, then I get just a wedge of Eilenberg MacLane spectra, and you just get that this is $H(\bullet, h(pt(\mathbb{Q})))$. I'll actually tensor with the reals and say that I have a map $h(\ ) \to H(\ , h_{\mathbb{R}}(pt))4$, and on a manifold I can also use differential forms, and then a fiber product is what is the differential cohomology.

$$\begin{tikzcd} h^{\vee}(M) \arrow[r, dotted] \arrow[d, dotted] & \Omega(M; h_{\mathbb{R}})_{closed} \arrow[d] \\ h(M) \arrow[r] & H(M, h_{\mathbb{R}}) \end{tikzcd}$$

All the juice is in the homotopy between the two things you get. There are exact sequences

$$0 \to h^{q-1}(M, h_{\mathbb{R}} \otimes \mathbb{R}/\mathbb{Z}) \to h^{\vee q}(M) \to \Omega(M, h_{\mathbb{R}})^q_{\mathbb{Z}} \to 0$$

and

$$0 \to \text{forms} \to h^{\vee q}(M) \to h^q(M) \to 0$$

If this is a ring theory we should get a ring structure, that's one comment. I also want to think of objects that represent these cohomology classes. In ordinary homology, we can think of $h(M)$ as $\pi_0(Map(M, h_q))$, I could think of the fundamental groupoid. To remember something completely local, we need higher homotopies. So these are objects in groupoids, higher groupoids, and spaces. We can think about objects, [unintelligible]

I will be using equivariant versions of these without comment. There are versions of the K-theory in [long list of references] but a full version with [unintelligible]has not been worked out.

If I look at $H^{\vee 1}$ this is $Map(M, \mathbb{T})$, where as $H^1(M) = \pi_0(Map(M, \mathbb{T}))$. What are objects of $H^{\vee 2}(M)$? They are connections on bundles over M of [unintelligible][unintelligible].

There are in $H^{\vee 3}$ other things, that go by the name bundle gerbes. Although I use the differential theory, these turn out to be topological. In physics these have come up for defining terms in an action. These topological terms where one wants to integrate a certain class, these go back twenty years. The second place is in charges and currents. There are fields in the string theory that generalize Maxwell theory. The map that remembers the charge is the map to ordinary cohomology. In quantum theory that charge has to be quantized. In the classical theory one just sees the current. In a quantum theory you get actually an integral charge. It also gives you torsion, additional charges. To set up a quantum theory, you might think that [unintelligible]should live in this realm. So it's not surprising that these would come up in physics.

Let me say a little about the other mathematical preliminary, twistings, so KR theory of X_w. So I have X_w over X, and I can do this more generally where these are groupoids divided by a compact Lie group. We can make the discussion more general, and this is a double cover. An object in $KR^0(X_w)$ consists of a $\mathbb{Z}_2$ graded complex vector bundle E over X_w. If you're used to thinking of differences, this is $\mathbb{Z}_2$ graded so the even part minus the odd part. So it's this together with a lift of the involution $\tilde{\sigma}$. We want to twist that notion, so we pull back the bundle and map to E but it's antilinear, $\tilde{\sigma} : \sigma^* \bar{E} \to E$ with $\tilde{\sigma}^2 = \pm 1$.

The two cases, if σ acts trivially, we get a map in each fiber which is antilinear and it's a real theory, $KO^0(X_w)$. If I have a section I can pull back to X and get $K^0(X)$.

I'm going to tell you what a twisting is. The first ingredient is that we may have to pass to a locally equivalent groupoid Y_w over Y. And, the way I want to think about this, the Y_w, say Y is a groupoid, it has p_0 and p_1 from Y_1 to Y_0, and then three maps $Y_2 \to Y_1$, and so on. I want to have a homomorphism ϕ from the arrows to $\mathbb{Z}/2\mathbb{Z}$. You get a double cover by forgetting the green arrows. Now if I have a vector bundle, I can define this to be either V if ϕ is 0 or $\bar{V}$ if ϕ is 1.

Definition 4 *A twisting of $KR(X_w)$ is a locally equivalent $Y_w \to Y$ and $\tau = (d, L, \theta)$ where $d : Y_0 \to \mathbb{Z}$ is continuous (so locally constant), L is a Hermitian line bundle, $\mathbb{Z}_2$-graded, over the arrows, and the θ is an isomorphism, if I have a composition of arrows $a \xrightarrow{f} b \xrightarrow{g} c$ then we get $\theta :\ ^{\phi(f)}L_g \otimes L_f \to L_{gf}$, and then the cocycle condition is that $d(b) = d(a)$ for $f : a \to b$ and then a cocycle composition for composition of three arrows.*

We should make this a higher groupoid. That's what a twisting is. And as I said, it's important to recognize these as classified by a cohomology theory. There is a cohomology group. In the special case that we're doing $K(X)$, we have three homotopy groups π_0, π_1, and π_3, which are $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}$. This extra grading gives a cohomology class in π_1 and d gives the class in π_0

For KO theory you get $KO(X_w)$ are for $\pi_{0,1,2}$ are $\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$. For $KR(X_w)$ the isomorphism classes are $H^0(X,\mathbb{Z}) \times H^1(X,\mathbb{Z}/2\mathbb{Z}) \times H^{w+3}(x,\mathbb{Z})$ as a set.

So we can also [unintelligible]and get a twisted 3-form. That's one comment. Another comment is that you can define a twisted [unintelligible]in KR but there's one thing that I do need, there's this element u, that you think of as K^2 of a point, and multiplication by it is an isomorphism. This is also in $KR^{\tau_1+2}(pt)$, and $4\tau_1 = 0$. I'll give a concrete model for u. I told you a KR class is a complex vector bundle over [unintelligible], and so you get the vector space $\mathbb{C}^{1|1}$, the parity reversed complex. Since it's in degree [unintelligible]it has an action of the Clifford algebra, and now I have to lift the involution. If I have a z_+ and z_- I have to say where that goes, it goes to $(-\bar{z}_-, \bar{z}_+$. Bott periodicity for KR says that this is an isomorphism as well.

We can take τ and map to $\tau + \tau_1 + 2$.

That's enough about this background. Now I'm going to come back to this string theory and give you the central definition, tell you what a piece of this string background is. In the two dimensional theory, then this data is background. In the ten dimensional theory these are actually fields. The definition is that an $NSNS$ superstring background. So NS stands for Nova and Schwartz, repeated twice, and this is four pieces of data,

1. X a smooth ten-orbifold with metric and a real function
2. a double cover $X_w \to X$, an orientafold
3. the B-field $\beta^\vee$, a differential twisting of $KR(X_w)$, and the fourth piece of data
4. $\kappa : R(\beta) \to \tau^{KO}(TX - 2)$, which is an isomorphism of twistings of $KO(X)$, a twisted version of a spin structure. That's it. So, I haven't really explained everything, but let me make some comments.

The B field in the bosonic string would be an object in differential H^3. Adding these extra pieces is really adding torsion.

[I can't follow any more.]

4 Kevin

[email Martin Olbermann a 1 to 3 page abstract if you are a speaker.]

What is the analogue of deformation quantization for quantum field theory? The classical mechanical system is described bya commutative algebra A^{cl} with a Poisson bracket. This should be thought of as the algebra of classical observables. The deformation quantization approach, you need to find an associative algebra A or A^q over the ring $\mathbb{R}[[\hbar]]$ such that:

- $A^q/\hbar A^q = A^{cl}$
- if a and b are in the classical observables and $\tilde{a}$,$\tilde{b}$ are lifts of them, then $\{a, b\} = \frac{1}{\hbar}[\tilde{a}, \tilde{b}]$ mod $\hbar$

In these lectures, I want to give an analog of this picture for quantum field theory. So I

1. need to explain what plays the role of commutative, Poisson, and associative algebras
2. need to explain how classical field theory is encoded in commutative and Poisson
3. need to explain how to quantize the classical field theory.

Since everyone is tired, I want to go slow, and I want to explain the piece that takes the role of associative algebras. I want to say what plays the role of associative in this picture. The structure that plays the role of associative algebras is a factorization algebra. You may not have seen this word or might associate it with a terrifying word. This is a C^∞ analog of a chiral algebra.

Let M be a manifold (on which we do quantum field theory). Let $B(M)$ be the set of all balls in M. This is an infinite dimensional manifold. You can put a metric and then it will be finite dimensional. Let $B_n(M)$ be the set of n disjoint balls embedded in a larger ball. I can draw a picture. [Picture]

A factorization algebra is the following data: firstly a vector bundle F on $B(M)$. I should have said that there are obvious projection maps $B(M) \xleftarrow{q} B_n(M) \xrightarrow{p} B(M)^n$, and the vector bundle should have maps $p^*(F^{\boxtimes n}) \to q^*(F)$ satisfying some evident compatibility.

Concretely, F assigns a vector space to every ball B in M, and if we have a configuration of balls like this picture [Picture], several disjoint balls within a ball, then there is a map $F(B_1) \otimes F(B_2) \to F(B_3)$, which vary smoothly as our configuration varies. The compatibility condition, which should be clear if you have seen the little disks operad, is that composition,

$$\begin{tikzcd} \bigotimes F(\text{small balls}) \arrow[r] \arrow[dr] & \bigotimes F(\text{bigger balls}) \arrow[d] \\ & F(\text{biggest balls}) \end{tikzcd}$$

This is an algebra over a colored operad, where the colors are $B(M)$, the manifold of balls, and the n-ary operationsare $B_n(M)$, with the extra conditions so that the vector spaces we assign to each color form a smooth vector bundle. The maps should be compatible with this structure.

There are several specializations that I will talk about now. Vector bundles come in three flavors, C^∞, holomorphic, and locally constant sheaves. The notion I've described corresponds to C^∞. You can modify this to make sense in either of the other two categories. So factorization algebras exist in the other settings. In the third case it's a vertex algebra.

Definition 5 *A locally constant factorization algebra is like a factorization algebra except that instead of being a vector bundle, F is a locally constant sheaf on $B(M)$, and the structure maps are maps of locally constant sheaves. These are sheaves of, I've drunk the derived Kool-Aid, so everything is a cochain complex. This is true even in the C^∞ case or the holomorphic case. In this case you could talk about topological spaces or anything.*

The case we'll consider first is the locally constant case on $\mathbb{R}^n$. Since the space of balls in $\mathbb{R}^n$ is contractible, and locally constant sheaves only depend on the homotopy type, you can think of F as a trivial locally constant sheaf with fiber V a cochain complex. There's some extra structure to put on V. If we have a picture like [Picture] we get a map $V \otimes V \to V$. In this setup, as I vary my configurations of disks, these maps vary by homotopies. So as the configurations of disks vary, the products change by homotopies. So what we see is a locally constant factorization algebra on $\mathbb{R}^n$ gives us an E_n algebra.

I'm channeling Jacob and Kevin Walker with this locally constant stuff.

[The E_n spaces aren't contractible.] The spaces of the single disk is contractible.

The next specialization I want to consider is holomorphic factorization algebras. So I'm only going to discuss this in the case of Riemann surfaces. Let Σ be a Riemann surface and we can make the same definition as before and I want to make it in a holomorphic context. We can look again at balls in Σ. We know what it means for a map to from a complex manifold to $B(\Sigma)$ to be holomorphic. So we can speak about holomorphic objects on $B(\Sigma)$. For a family to vary holomorphically, the center must vary holomorphically, the radius can do what it wants.

[Discontent about the holomorphic maps]

Maybe this definition is too weak. Then I apologize. Let me, it will still be nontrivial for the map to vector bundles to be holomorphic, as we will see in the example.

[Does it help if M must be locally holomorphically trivial?]

Can we not worry? This is for one example for $\mathbb{C}$ where things are translation invariant. Let's consider a holomorphic factorization algebra on $\mathbb{C}$, and also assume it's translation invariant, and also dilation invariant. If you only consider round balls, the vector bundle has no structure. Let V be what F assigns to any round disk. We just have a vector space. If we have a configuration of disks, a disk of large radius centered at zero, and a disk of a smaller radius centered at zero, and another one centered at z. Then there is a map $m_z : V \otimes V \to V$. Fix the radii to be R, ϵ, and δ. As z varies, keeping the radii fixed, the map should vary holomorphically.

So what does this mean? m_z is defined for z in an annulus, so m_z is a holomorphic map from an annulus to $Hom(V \otimes V, V)$ so it has a Laurent expansion $m_z = \sum z^k a_k$, and the coefficients of z^k are completions of $Hom(V \otimes V, V)$. The vector spaces should be the duals of nuclear Frechet spaces and the tensor product is the completed projective tensor product.

If I'd gotten the definition right, you would have seen that this was very reminiscient of a vertex operator algebra. If you read Beilinson and Drinfeld's book, they make this definition in the algebraic setting, and what they show is that in this setting, the axioms for a chiral algebra are almost the same as those for a vertex algebra. Okay, so

[Somehow there are locality and finite [unintelligible]condition] I haven't thought it through in detail here, but Beilinson and Drinfeld get the exact same things, with certain conditions.

Maybe I'll just say the last thing. I want to say that a factorization algebra is a good way to encode observables of quantum field theory. One is vertex operator algebras, and you get an associative algebra in quantum mechanics, factorization algebras on the real line are associative algebras. You consider observables on M, you should consider a factorization algebra.

By the way, everything is joint with O. Guillam (?)