

# Oberwolfach

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## 1 Corbett Redden, string structures, 3-forms, and tmf classes

Before I get started, I'd like to write down notational conventions. So  $M$  will be a smooth closed  $n$ -manifold, compact, no boundary.  $g$  will be a Riemannian metric. If I say  $Spin \rightarrow P \xrightarrow{\pi} M$ , this is a principal  $Spin(n)$ -bundle, and if I say  $A$ , this will be a connection on  $P$ , and finally, the only nonstandard notation is that  $\mathcal{S}$  is a string class, which I'll define in a minute. This goes nicely into Konrad's talk next, and we'll use the same conventions.

Let me give you a quick outline of where I'm going. I'll start with

1. String structures and string structures up to homotopy, which will be very concrete
2. I want to look at harmonic representatives, which will involve talking about harmonic 3-forms on  $P$
3. Geometry and tmf? (That's what I'll call this last part) Your motivation will depend on your interest, but these come up in a lot of places, so that's why I want to go into the third, but you can view the first two parts as motivation for the third if you're interested in tmf.

What do I mean by a string structure? I'm going to define a space  $BString(n)$ . I have  $BSpin(n) \xrightarrow{\frac{p_1}{2}} K(\mathbb{Z}, 4)$ , and I'll take  $BString$  to be the homotopy fiber.

Then a string structure on some bundle  $P \rightarrow M$  will be a lift of the classifying map

$$\begin{array}{ccc} P & & BString \\ \downarrow & \nearrow & \downarrow \\ M & \longrightarrow & BSpin \end{array}$$

Other work has said that there's the string group, and then you can take its classifying space, but up to homotopy this is still  $BString$ .

Now, what does this mean? When does a string structure exist? It's easy to see that it exists if and only if  $\frac{p_1}{2}(P) = 0$  in  $H^4(M, \mathbb{Z})$ . Now I want to look at string structures up to homotopy, which as a set is the same as string classes, namely  $\mathcal{S} \in H^3(P, \mathbb{Z})$  such that  $\iota^* \mathcal{S} = 1 \in H^3(\text{Spin}, \mathbb{Z})$ . These are ordinary degree three classes on the total space of a spin bundle, but not all of them, only the ones that satisfy this condition.

I'll describe this canonical isomorphism. Finally, if I think about these string classes, this is a torsor for the integral third homology of the base under the natural action of the pullback  $\mathcal{S} \mapsto \mathcal{S} + \pi^* H^3(M)$ . This is bad notation.

So here's the proof of this identification. Let me look at the universal example:

$$\begin{array}{ccc} \pi^* E\text{Spin} & \longrightarrow & E\text{Spin} \\ \downarrow & & \downarrow \\ B\text{String} & \longrightarrow & B\text{Spin} \longrightarrow K(\mathbb{Z}, 4) \end{array}$$

Because  $E\text{Spin}$  is contractible,  $\pi^* E\text{Spin}$  is the same as the homotopy fiber of  $B\text{String}$  to  $B\text{Spin}$  which is then  $K(\mathbb{Z}, 3)$ .

If I want to think about string structures, I can think about geometric string structures, but universally you can see this. You can tell the average graduate student what a string structure is.

Let me say now why string structures. What's a motivation? Well, first off, string structures, there's sort of a slogan that says string structures on  $P$  transgress to a  $\text{Spin}$  structure on the bundle  $LP \rightarrow LM$ , where the fibers are the loop group. If you want to talk about spinors, the positive integer representations are projective, so you need to take an  $S^1$  extension  $S^1 \rightarrow \widehat{L\text{Spin}} \rightarrow L\text{Spin}$ . If I think about what this means, then you want

$$\begin{array}{ccc} \widehat{L\text{Spin}} & \longrightarrow & \widehat{LP} \\ \downarrow & & \downarrow \\ L\text{Spin} & \longrightarrow & LP \end{array}$$

So  $\mathcal{S} \in H^3(P, \mathbb{Z})$  goes to  $(\pi_i \text{ev}^*)\mathcal{S} \in H^2(LP, \mathbb{Z})$  which goes to  $H^2(L\text{Spin}, \mathbb{Z})$  and degree two coefficients classify  $S^1$  bundles. Topologically this is the obstruction for this central extension.

Related to this is the string orientation of  $\text{tmf}$ , which is known as topological modular forms, constructed by Hopkins, [unintelligible], lots of people, which means I have a map from  $M\text{String}$  (also known as  $MO(8)$ ) to  $\text{tmf}$ , namely

$$M\text{String}^{-n}(pt) \xrightarrow{\sigma} \text{tmf}^{-n}(pt)$$

And this is exactly what you need for a spin manifold and string class before, so you can go

$$M, \mathcal{S} \rightarrow [M, \mathcal{S}] \rightarrow \sigma(M, \mathcal{S})$$

and get a piece of  $tmf$  of a point.

You have this map

$$\begin{array}{ccc}
 & & tmf \\
 & \nearrow \sigma & \downarrow \\
 MString & \xrightarrow{\text{Witten genus}} & Modular\ forms
 \end{array}$$

so  $\sigma$  is a lift of the Witten genus, which is the  $S^1$  index of the Dirac operator on the loop space  $LM$ , well, at least in quotes, and  $\sigma$ , if you want to think about index theory on loop spaces, you need to go to  $tmf$ .

I'll return back to this picture. Is there any questions? If you replace *Spin* with a compact semisimple Lie group,  $p_1/2$  with another class, this whole thing carries over.

So now I've said what, how I can think about string structures, and now what I want to do is say what is the harmonic representative of this class. The answer should make you say this was a good question even if you're not a geometer. A quick reminder, if you have a manifold and choose a metric  $g$ , it picks out the adjoint  $d^*$ , and you can get the Hodge Laplacian  $\Delta = dd^* + d^*d$ , and  $H^k(M, \mathbb{R}) \cong Ker \Delta_g^k \subset \Omega^k(M)$ . Once you pick a metric, you get a canonical form. I want to put a metric on my principal bundle, and then pick a metric and see what this is.

**Construction 1** *Start with  $(P \xrightarrow{\pi} M, g_M, A)$ ; if you want, you can let  $A$  be the Levi-Civita connection. Choose a bi-invariant metric  $g_{Spin}$ , and there is a one-parameter family of these. The connection gives me an orthogonal splitting of the tangent space,  $\pi^*g_M \oplus g_{Spin}$ , and I use the connection to define this. The picture is, well, [picture], and if I look at the tangent space of the total space, I can take the tangent space of the fiber and the choice of the horizontal space is given by the connection.*

One thing I had to do was choose a biinvariant metric which is only unique up to a scaling factor. I want this unique so I'll scale it away. I'll introduce a scaling factor  $\delta > 0$ , and I do the same exact thing as above, but I rescale the fiber direction,  $g_\delta := \pi^*g_M \oplus \delta^2 g_{Spin}$ , and then take the adiabatic limit, as  $\delta \rightarrow 0$ .

Now I've got a one-parameter family, where I can think of the kernel of the Laplacian, which is ordinary homology, but when  $\delta$  goes to zero, the metric becomes singular, so you have to be careful, but fortunately others have already been careful.

**Theorem 1** *Mazzeo-Melrose, Dai, Forman*

*$\ker \Delta_{g_\delta}^k$  extends smoothly to  $\delta = 0$ , and comes from a filtration that is isomorphic to the Serre spectral sequence for the fibration  $Spin \rightarrow P \rightarrow M$*

This is a really nice theorem and it would be a nice talk, but this is a situation where the two definitions of spectral coincide, the spectral sequence is related to the spectrum of the Laplacian.

[Kevin: what do you mean comes from?]

There's something finite after  $E_2$  and you can identify it [I miss something], so the filtration is a geometric thing, and you can write down a Hodge theory spectral sequence.

Okay, I can't really talk any more about this, but this means that

$$H^k(P, \mathbb{R}) \xrightarrow{\cong} \lim_{\delta \rightarrow 0} \text{Ker} \Delta_{g_\delta}^k =: \mathcal{H}^k(P)$$

Now I have the string class in the third cohomology of this bundle, and now I can pass to the real class and see what I have in this isomorphism

**Theorem 2** (*Redden*)

Given  $P \rightarrow M, g, A$  and  $\frac{p_1}{2}(P) = 0$ , then what do I get with  $H^3(P, \mathbb{Z}) \rightarrow H^3(P, \mathbb{R}) \rightarrow \mathcal{H}^3(P)$ ? Well,

$$\mathcal{S} \rightarrow \underbrace{CS_3(A)}_{\text{Chern-Simons 3-form}} - \underbrace{\pi_* H}_{\in \pi^* \Omega^3(M)}$$

So the way you describe  $H$ , you subtract your string form from the Chern Simons form and you get a pullback.

Now what I want to talk about is, what does this mean and what are the properties of  $H$ ?

[Dan Freed: If I don't scale to zero, what do you get, is it still a pullback?]

That's a very good point. In general, I can talk about the harmonic representative as a one-parameter family of forms, and  $[S]_{g_\delta} = CS_g(A) - \pi^* H + O(\delta)$ , but the  $O(\delta)$  terms are not in the pullback of the base. You get this splitting of the tangent bundle, the purely vertical and purely horizontal, the Chern Simons has some in various pieces, but if you don't go to zero you'll get other things in the mixed degrees.

[Dan: Is there a more general theorem about subtracting the Chern Simons from anything like this in the adiabatic limit?]

Yes, but I want to be careful because I just worked this out.

I'm running slightly short and I wanted to say what is  $H$ , I want to make a slight digression.

**Theorem 3** *Cheeger-Simons, possibly Chern's name should be here*

If I have  $P \rightarrow M, A$ , then  $\frac{p_1}{2}(A) \in \check{H}^4(M)$ , you have this differential class. Now remember that when you have a differential theory you have the sequence

$$\Omega_{\mathbb{Z}}^3(M) \longrightarrow \Omega^3(M) \longrightarrow \check{H}^4(M) \longrightarrow H^4(M, \mathbb{Z}) \longrightarrow 0$$

$$H \longrightarrow \frac{p_1}{2}(A) \longrightarrow \frac{p_1}{2}(P) = 0$$

So  $H$  is the this pullback. In particular, you know that thinking about this as a differential character, you get a map to  $\mathbb{R}/\mathbb{Z}$ , and integrating  $H$  gives me a lift of this up to  $\mathbb{R}$ . The different lifts are what you get modulo torsion from a string structure. It needs to be topologically trivial for this to exist, but this gives exactly a lift to  $\mathbb{R}$

[The differential character gives you this without the metric, what is the role of the metric?]

You also have  $d^*H = 0$ , and so the first condition determines  $H$  uniquely up to  $\mathcal{H}_{\mathbb{Z}}^3(M)$ , and this is the kernel of  $\Delta_g$ . This is where the metric condition comes in. When you set  $d^* = 0$ , this is like in standard Hodge theory, once you set  $d^*$  to zero it determines it uniquely. In fact, there is a natural equivariance, and this is that

$$H_{S+\pi^*\psi} = H_S + \pi^*H_\psi$$

where  $\psi \in H^3(M, \mathbb{Z})$  and the pullback will be harmonic on the bundle, which will only be true if you take the limit. That was the construction, and overall what this tells me is that I go from  $M \times \{\text{String Class}\} \rightarrow \Omega^3(P) \rightarrow \Omega^3(M)$ , and I should have said, I have

$$\begin{array}{ccc} \text{Met}(M) \times A(P) \times \{\text{string classes}\} & & g, A, \mathcal{S} \\ \downarrow & & \downarrow \\ \Omega^3(P) & & CS_3(A) - \pi^*H_{g,A,\mathcal{S}} \\ \downarrow & & \downarrow \\ \Omega^3(M) & & H_{g,A,s} \end{array}$$

It's not clear yet that we're getting anything additional. Let me do an example, it's too abstract. Again, I have this map

$$\begin{array}{ccc} & & tmf^{-n} \\ & \nearrow \sigma & \downarrow \\ MString & \longrightarrow & MF \end{array}$$

**Conjecture 1** (Stolz) *If  $M$  is string and admits positive Ricci curvature metric, then the Witten genus of  $M$  is zero*

This should remind you of this theorem from index theory.

So playing wishful thinking how about also  $\sigma(M, \mathcal{S}) = 0$ ? This thing here, no way, this couldn't be true. There's lots of torsion in  $tmf$ , and I'll show you an example, but there's maybe something else that we can do.

A hypothesis, this is even less than a conjecture, but it should be, like, tested and stuff, say  $M$  is a spin manifold that admits a metric and string structure  $(g, \mathcal{S})$ , and  $P$  is  $Spin(TM)$

with  $A$  the Levi-Civita connection, then there is a simultaneous pair of conditions, what if the Ricci curvature is strictly positive  $Ric(g) > 0$  and  $H_{g,S}$  is zero in  $\Omega^3(M)$ , then  $\sigma(M, S) = 0 \in tmf^{-n}(pt)$ . The condition of  $H$  being zero is very strong.

Let  $M = S^3 = SU(2)$ , and let  $p_1 \in H^4(S^3) = 0$ . Then  $H^3(S^3, \mathbb{Z}) = \mathbb{Z}$  is the set of string classes, and  $dH = d^*H = 0$  implies just  $H \in H^3(S^3, \mathbb{R}) \cong \mathbb{R}$ . So you have these string classes and I want to go down to  $MString^{-3} = \pi_3^S = tmf^{-3} = \mathbb{Z}/24$ . So one, this is a three-sphere, so it's the boundary of a four-ball. I can also do a left invariant framing because it's a Lie group, and there's also a right one. So  $\partial D^4$  goes to zero, and the left and right ones go to generators, and you can see how this works. The first thing to note is that there's no way that [unintelligible]. Now what I want to do is just say, consider, there's a one parameter family of Berger metrics on  $S^3$ , which is just rescaling the fiber in the Hopf fibration. Now I've got some variables I can play with, I can draw a graph, with one axis the family of metrics  $g_x$ , nad the other dimension  $H_{g,S} \in \mathbb{R}$ , and then there are some special values of  $g_X$ , and I get a picture like [Picture].

This equivariance says all these graphs are translates of each other. I can put in  $\partial D^4$ , left, and right, and I can point out that the minimum is the round metric, and where you cross zero, the metric is not Ricci positive. I start looking at when this is satisfied, and I see that it's zero and Ricci positive in the trivial string class. Where else is the form zero? It's not when it's Ricci positive for the right invariant metric. In fact, in the limit as it goes to infinity, it is all Ricci positive as well. It's not quite a stupid hypothesis. If you weaken the condition, it's evidently false. That's all.