Oberwolfach

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1 Andre Henriques, Invertible Conformal Nets

[Four things to do in the cafeteria: sign the guestbook, sign in the stand, checking your address, make a cross on your last meal, and then, if you need a taxi, you sign up and they'll organize taxis. Speakers write a handwritten abstract into the abstract book.]

This is part of a joint project with [unintelligible]and Chris Douglas. It's possible that you have heard some aspect of this at some point. Today I want to talk about a particular classification of invertible objects in the three category that conformal nets form.

Theorem 1 *Conformal nets form the objects of a symmetric monoial three category $CN3$.*

The logical order of this talk should have been to tell you what a conformal net is, then which model of symmetric monoidal three category they fit into, and then hint about this, then talk about being an invertible object, and then present the result. For the purpose of this talk, this is not a viable strategy, because I would not reach the results. I'll introduce definitions as much as I need them in parallel to presenting this result. As a consequence, the definition will not be complete, the list of axioms will not be exhaustive, but I hav been talking about this for long enough that I should have something available. There is something on my webpage, half of a paper, with definitions and relevant structure that you might want to see.

Okay. So let me start by telling you what conformal nets are. I had to say I liked Kevin's series of talk very much, so I'd like to build on his notion of a factorization algebra.

Definition 1 *A conformal net is a factorization algebra on the category of one dimensional balls. I will take values in something more interesting than vector spaces, values in von Neumann algebras.*

What does that mean? This is an interval with no coordinate, and whenever I have an interval, I have, well, a conformal net, associates to an interval a von Neumann algebra, and to an inclusion of intervals an inclusion of von Neumann algebras, and to an inclusion of disjoint intervals an inclusion of the tensor product of von Neumann algebras.

It's called conformal, the examples come from, there's a philosophy. I look at unitary full CFTs, and conformal nets, there should be an equivalence of categories. No one will claim that there is a theorem, but all important examples of unitary field theories have corresponding conformal nets.

[some discussion]

So what I want to talk about today is the thing contained in my title.

Theorem 2 *A conformal net $\mathcal{A}$ is invertible in $CN3$ if and only if $\mu(\mathcal{A}) = 1$, the μ index. It's an invariant of conformal nets which takes values in $\{0\} \cup [1, \infty]$, and I will define it in a moment.*

There is a companion result that I want to put down but not focus on:

Theorem 3 *The fully dualizable ones are the ones with μ-index finite.*

So, let's see, I should probably tell you what the μ index is. I need some preliminaries about von Neumann algebras. They form the objects of a two-category. The morphisms in the two-category are bimodules ${}_AH_B$ where H is a Hilbert space and A and B are von Neumann algebras. The unit in the bicategory will be a particular $A - A$ bimodule is called $L^2(A)$. The archetypal example of a von Neumann algebra is $L^\infty(X)$. Then if that were $L^\infty(X)$ then this Hilbert space would be $L^2(A)$.

One word about composition, you compose H with K by Connes fusion $H \boxtimes K$. The underlying Hilbert space uses H, K, and $L_2(B)$ if H and K are morphisms from A to B and B to C respectively.

Associated to a bimodule between von Neumann algebras is a number *dim* which lives in $\{0\} \cup [1, \infty]$, so that the dimension is 0 iff $H = 0$ as a vector space. It is 1 if and only if H is invertible, and is finite if and only if H has a dual in the two category of von Neumann algebras.

This is all I want to say about these. This is additive under direct sum, multiplicative under external tensor product or Connes fusion.

Okay, so now let me go back to the μ index and tell you what that is. It's the quantum dimension of a bimodule constructed from A. Let me tell you a fact, which is an axiom of conformal nets but I did not state, namely the two actions of $\mathcal{A}(\frown)$ on $H_0 = L^2(\mathcal{A}(\frown))$ can be used to define an action of $\mathcal{A}(I)$ on H_0 for every interval in S^1. I want to know how I acts on H_0. I use the lower half for the right action and the upper half for the left action, and in the overlap there is something that needs to be done.

Definition 2 *Consider H_0. Cut the circle into four quarters. I can take $\mathcal{A}$ of two opposite* *quarters, and then the tensor product of $\mathcal{A}$ of these quarters act on H_0. This is a bimodule of two von Neumann algebras, so I can talk about its dimension, nad that's the μ index.*

I can give you some facts. A nontrivial theorem by [unintelligible]says that this quantity is the dimension of the category of representations of the conformal net.

So let me tell you how to prove the theorem. I chose this result exactly because in order to talk about it, I shall be forced to enter into the structure of one and two morphisms, so it will populate the definitions with examples.

First maybe a few words about what it means to be an invertible object in a three category. X is an object. To be invertible means that there is another object Y and an invertible arrow $X \otimes Y \xrightarrow{\sim} 1$. An arrow is invertible if there is another arrow so that the composition of f and g (also g and f) can be two-morphed to the identity by an invertible two morphism, and so on. For the three morphisms they should be actually invertible.

[discussion]

First of all, I need to define the inverse of an object. Because of the other theorem in yellow, fully dualizable, a whole bunch of stuff will be there without invertibility, so it will be only in the last step that invertible will be used.

The inverse of a net $\mathcal{A}$, so $\mathcal{A}^{-1}(I) = \mathcal{A}(I)^{op}$. Let me say now, intervals have an orientation, so that there is a canonical isomorphism between $\mathcal{A}(I)^{op}$ and $\mathcal{A}(\bar{I})$ Now let me construct this one-morphism here. I need to say a few words about what one-morphisms are. So the definition of a one-morphism will be very similar to, a defect, is the following data: it will be a factorization algebra in von Neumann algebra, with source category bicolored manifolds. It is enough to know to on entirely blue, half-blue, half orange, and entirely orange. I look at this category and define a defect to be an algebra only on these colored intervals.

[lost the thread.]

So, let's see. The first example of the defect. The first example will be a defect between $A \otimes A^{op}$ and the identity, this is the transparent color in Chris' talk earlier today, something that you can delete if you want to. D, I will only evaluate on the interval that is bicolored. What I do is I double the $\mathcal{A} \otimes \mathcal{A}^{op}$ part and forget the 1 part. I'm out of time. What do I want to do, given that fact.

What I want to do is tell you what two and three morphisms are and leave it like that. The two morphisms between defects will be the following data: draw a circle with the colors of the two nets on the sides, and the defects on the top and bottom. A two morphism by definition is a Hilbert space with actions of these algebras. A three morphism is a map of Hilbert spaces commuting with these actions. I didn't get to the point where I used this condition, that's too bad but whatever.

2 Urs Schreiber

[Before we start, there was interest in informal discussions, four different interests, so we thought we would put them in the different rooms. If you're interested, please talk to the people. Since there's lots of informal discussions, we thought we should figure out where it would happen.]

Thank you very much. So I did prepare some notes that contain what I will say. Since there are lots of things to say, I thought I'd give you some preparation. The plan is to start with some motivation and then some content behind the title. The plan is to understand the degree to which quantum theory is given by an n-functor on cobordisms. We heard Chris talk about this in dimension two. Mathematicians may be happy with this but the physicists want to know what classical theory is quantized. What is the quantum field theory that we get from a classical background field. In order to have a chance, a systematic chance, how to get from classical to quantum systematically, we had better put these two things in the same world. We want to represent the classical background fields as n functors. So I want to talk about how this works, what it does, what it achieves. I want to talk today about classical theories as n-functors.

So then to make progress with this we need to set up the place that these live, so bundles with connection. This is generalized by the backgrounds that appear in many other theories. So the first task is to understand the smooth context. These will be modeled by a nonAbelian cocycle. In part two I'll motivate briefly, and then in the next step I'll give an idea of what it means for these to be smooth. This is called differential nonAbelian cohomology. When I get to this point I'll make precise the difference between nonAbelian and standard cohomology.

I think I am not using the standard terminology, I guess.

So then it turns out that there's a little twist to the story, so there are two kinds of twist, the differential cohomology is flat, and the background fields are not general bundles, so we need twisted nonAbelian cohomology. After this preliminary work, the upshot should be that there is a simple succinct abstract nonsense description. The examples will be point five.

I want to say one more word on the motivation part. So we're all more or less familiar with the story that an n-dimensional quantum theory is an n-functor from an n-category of bordisms to some n-category A, and how do we get this from classical data? First of all, if you have some classical theory that is a σ-model, the electron coupled to an electromagnetic field on a Riemannian manifold, this should give a quantum field theory determined by a target spacetime X and differential data on X (the field itself) which I will denote schematically with ∇. Since we have X in the game, we can look at bordisms with maps into X. This will be a model for the space of fields of the classical theory.

So you can do something more trivial, using just disk shaped bordisms, forming something like the fundamental groupoid of X. The idea is to understand what these things $\Pi(X) \to A$ are like. Onec we know this, we try to extend to bordisms over X and then extend further to bordisms. There are toy models. From a category theory point of view, you want a Kan extension, a pushforward of functors. I linked to something with examples. The Dijkgraf-

Witten example is a Kan extension of [unintelligible]. I have a point of view of why I want to extend [unintelligible]. Now I want to understand the smooth bit. We want to understand what it means to be smooth, so I want some notion of smooth nonAbelian cohomology.

Okay, so, to motivate the smooth bit, let us mention the toy example we already mentioned, for example Dijkgraaf-Witten theory, which is a σ-model with target space the one object groupoid BG with one object and a morphism for g in a finite group G.

Then the background field is an n-functor from BG to $B^nU(1)$. This is trivial everywhere except [unintelligible].

How can we understand this setup? This wants to live in the context of infinity-groupoids, which we model as Kan complexes, a simplicial set with a condition on it so that the cells have a suitable notion of composition and inversion. It behaves like something with n-cells which can be composed if adjacent and inverted. The collection of ∞-groupoids forms a category enriched over ∞-groupoids. Such an ∞-groupoid enriched category is a model for an $\infty, 1$ category. this is even better, it's an $\infty, 1$-topos, so it behaves like we'd expect topological spaces to behave. The point of this is that we have an obvious notion of cohomology on Top

So what is $H(X, A)$? In topological spaces, this is $\pi_0(Maps(X, A))$. You can say the same thing for these being ∞-groupoids. So you say $\mathbf{H}(X, A) = Hom_{\infty-groupoids}(X, A)$ which is an ∞-groupoid whose objects are cocycles and morphisms are coboundaries, so that the objects in $\pi_0(Maps(X, A))$ is just the cohomology classes. Then Dijkgraaf Witten is one of these for a particular choice.

[discussion of calling this nonAbelian cohomology]

Let me, maybe, continue. Suppose I just take the liberty of calling it this way. The background field for Dijkgraaf Witten theory is a cocycle in this sens, $\mathbf{H}(BG, B^nU(1))$ is the same thing as $H^n_{group}(G, U(1))$. The idea is to generalize this from coefficient objects that are discrete groupoids to something that knows about smooth cocycles. Whatever you are doing, the physical theory will not be discrete, it will be a manifold or something stranger. How can we understand this? We have some control over which properties it is that let us speak of mapping spaces as cohomology. The general nonsense tells us that the things that feel like Top are $\infty, 1$-topoi, which are ∞-stacks on a site. So this means make this category $\mathbf{H}$ an ∞-topos, which means that $\mathbf{H}$ is a collection of ∞-stacks on some site S. So what is this? An $A \in \mathbf{H}$ is a functor $S^{op} \to \infty - groupoids$. How can this be smooth? Take S to be the category $Diff$ of smooth manifolds. Then this contravariant assignment $Diff^{op} \to \infty - groupoids$, well, you would be able to take the manifold and [unintelligible], so you can take $M \mapsto Maps(M, A)$ will be this thing. So you can pull this assignment back along the domain object. This being a stack means this knows about gluing of manifolds, so that the assignment knows about the fact that littler ones fit inside bigger ones, and so this is the ∞ version of a sheaf, of course. The upshot here is that we now pass, I want to interpret this in a context where we have smooth ∞-groupoids, so I want to pass to what I want to call smooth ∞ groupoids, meaning ∞-stacks on $Diff$.

The upshot now is that there is a nice concrete model for this thing that everyone has seen some aspect of, which is [interrupted]. So this is modeled by ∞-groupoid valued sheaves, $A : Diff^{op} \to \infty - groupoids$ whih are Kan complexes in simplicial sets. So you could just put the ordinary sheaf condition on this, and [unintelligible].

And then you equip the sheaves with extra information that remembers which morphisms, which ∞-functors, between two such things, would have homotopy invariance. You add this information, and this would have weak inverses in the full $\infty - 1$ category of ∞-stacks. This is, of course, standard modeled by either a model category structure on $[S^{op}, SSet]$, which was developed by Joyal and [unintelligible], but there is also a Brown category strcutres, a simpler thing, due to Kenneth Brown, 1973, and so in this context here, essentially, we single out and remember class of morphisms which behave locally as if they were surjective equivalences of ∞-groupoids. So P is locally a surjective equivalence of ∞ groupoids. Then the upshot is that what you get from this model, this ∞ groupoid of morphisms between two such objects is that you get, the corresponding $\mathbf{H}$ that you are after is the colimit over all possible acyclic fibrations, using these things, hypercovers, of sheaves from Y into A, this colimit over all diagrams

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & A \\ X & \end{tikzcd}$$

The collection of all these things is an ∞ groupoid, which is the one we're after.

A cocycle, a map from X to A is, a cocycle in X with values in A is a correspondence of this sort, and the coboundary is a correpsonding notion of two cell. Every [unintelligible]you have seen is of this form. In particular, observe that we can try to take the special case, the chain of inclusions

$$\begin{tikzcd} Ch(Ab) \arrow[d, hook] \\ \text{strict } \infty\text{-groupoids, or crossed complexes} \arrow[d, hook] \\ \infty\text{-groupoids} \end{tikzcd}$$

So these are chain complexes in the middle that are nonAbelian in the lowest two degrees.

Let's not do this on all of $Diff$. Suppose you have an F in sheaves on Z with values in $Ch(Ab)$, then you can associate a sheaf with values in ∞-groupoids, so call this A_F, this extension in sheaves with values in ∞-groupoids. You have this non-Abelian sheaf cohomology, which looks a little weird, is the same thing as regular sheaf cohomology, $\underbrace{H^n(X, F)}_{\text{sheaf cohomology}}$ is the same as $H(X, B^n A_F)$. So X is not an Abelian sheaf, so it's not maps from X into your thing. This lets us treat all of the nonAbelian things in [unintelligible].

The next thing is that using this setup, now you want to do something. So now what about differential cohomology. We cannot classify just bundles, but also bundles with connections.

There is a functor $P_2 : Diff \to$ smooth ∞-groupoids, which takes M to $P_2(M)$, which has objects points in M, certain classes of paths (not homotopy classes) as morphisms, and 2-morphisms classes of smooth surfaces. Then you can form the following. For A a smooth ∞ groupoid, you can, you can form A^{P_2} which is the sheaf that eats a manifold and then takes $Sh(P_2(M), A)$. These are thin homotopy classes.

Let me give the examples theorem. Let G be a Lie group, which produces a smooth ∞-groupoid BG, so then you can consider cohomology $\mathbf{H}(X, BG^{P_1})$, and this is the groupoid of G-bundles on X with connection. Let, we have again $B^2U(1)$, we can form $(B^2U(1))^{P_2}$ and compute $\mathbf{H}$ of X with coefficients in this, and it turns out this is $U(1)$ bundle gerbes over X

What is the twisted cohomology for, you get out all the physical examples that I said you might be interested in, you can interpret things you know from topological spaces. A sequence $A \to \hat{B} \to B$ of smooth ∞-groupoids is a fibration sequence means, this should be pointed, and the map should be a homotopy pullback with respect to the point:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \star \arrow[d] \\ \hat{B} \arrow[r] & B \end{tikzcd}$$

These behave as you expect so that in particular for all X, you can hom X into this diagram, form this:

$$\begin{tikzcd} \mathbf{H}(X, A) \arrow[r] \arrow[d] & \star \arrow[d] \\ \mathbf{H}(X, \hat{B}) \arrow[r] & \mathbf{H}(X, B) \end{tikzcd}$$

and so for $C \in \mathbf{H}(X, \hat{B})$, the obstruction for lifting it to an A-cocycle is $[\delta C]$.

So let me say

Definition 3 *For $A \to \hat{B} \to B$ a fibration sequence and for c a B cocycle, define c twisted A-cohomology on X as $\mathbf{H}^c(X, A)$ as the homotopy pullback of*

$$\begin{tikzcd} \mathbf{H}^c(X, A) \arrow[r] \arrow[d] & \star \arrow[d, "* \mapsto c"] \\ \mathbf{H}(X, \hat{B}) \arrow[r] & \mathbf{H}(X, B) \end{tikzcd}$$

Now you can go looking for twisted smooth cohomologies, and this produces for suitable fibration sequences the twists by magnetic higher charges, ordinary magnetic charge or [unintelligible]appearing in "nature" and one piece of the story that I didn't expect to talk about is that it allows you do describe [unintelligible]by describing non-flat differential cohomology, which looks trivial, but if you unwrap this, it's the (curvature characteristic classes)-twisted flat cohomology. This produces the twisted Bianchi identities.

Okay I'll stop here; thanks.

3 Informal talk by David Chataur

So, um, okay, let us consider M to be a PL manifold, closed, oriented, and d is its dimension. So okay, if I consider the singular cochains $C^*(M)$ with coefficients in $\mathbb{Z}$, this is an E_∞ algebra. If you want, I will give some details about this structure, and so this E_∞ algebra structure induces on the singular cohomology a commutative structure. As we are working with a PL manifold, you have also a multiplicative structure on the singular homology, which is due to intersection. Due to Jim McClure, if you look at $C_*^{PL}(M)$, this is a partial commutative differential graded algebra and as Scott explained, when you have a partial commutative differential graded algebra, you can make it into an E_∞ algebra, which I will denote by $Rec(C_*^{PL}(M))$, for rectification, so here you have something partially defined and here something over the E_∞ operad, and the point is that $Rec(C_*^{PL}(M))$ is quasiisomorphic to $C_*^{PL}(M)$ as a partial commutative algebra. There is a sequence of quasiisomorphisms that respects the structure. So you have these two E_∞ algebras, and the point is how they are related. Okay.

Theorem 4

$$Rec(C_*^{PL}(M)) \cong C^*(M)$$

(as E_∞ algebras

Okay, so what I can explain is first, the E_∞ algebra, or one such structure on the cochains, a little bit about how this structure is constructed, and how we can compare these two things. So, first point.

[What about the partial quasiisomorphism?] This result is due to Scott. In the simplicial bar resolution of the cochains, you take the normalization, but it is still, you have a map of partial algebras and then you have a map to the rectification.

Okay. So a little bit about $C^*(M)$ as an E_∞ algebra. Should I recall the definition of an E_∞ operad. You have the operad of commutative–

[Ralph Cohen: You could ask for a rectification of the chains of an Poincaré duality space?] If you have a Poincaré duality space, you can lift to $C^*(X) \xrightarrow{\sim} C_{d-*}(X)$, and you have an E_∞ sturcture on $C^*(X)$, and then by the homotopy invariance principle you can find some $\tilde{C}$ which maps to both of these complexes quasiisomorphically, so that you get E_∞ algebra maps

$$\begin{tikzcd} C^*(X) & & C_{d-*}(X) \\ & \tilde{C} \arrow[ul, "\sim"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

[A Poincaré duality space has a Spivak normal bundle. Its Thom spectrum is a ring spectrum which has algebraic structure. Perhaps there's a Thom isomorphism.]

This result is unstable. The cochains of a space determine its homotopy type. How do you go from the stable structure to the structure on the cochains.

[There's an algebra structure on chains. I don't know if you can make it C_∞ or something.]

[A. Ranicki: You have the wrong shift.]

[R. Cohen: I think it's the right shift.]

[Ranicki: I don't think that it's this that you will get.]

[You have a Spanier Whitehead duality, so there is a relation between the chains of the spectrum and the cochains of the space.]

I don't think you recover all the information.

[R. Cohen: that's my question. At least up to homotopy, the Spanier Whitehead dual sees the cup product]

[N. Rounds: on the Thom spectrum, not the space]

[R. Cohen: But it's not obvious that it doesn't have the same information.]

I don't think so, I think you are passing to the stable world and losing information.

[Is it true that the homotopy type of the Spanier Whitehead dual of a finite complex as an E_∞ ring spectrum, determines the homotopy type of the complex?]

Can you compute the unstable homotopy of a sphere?

[Birgit Richter: Even in [unintelligible], there is a Quillen equivalence result, but the gap is not closed to extend this to the E_∞ world. This is still open. Brooke Shipley was working on it.]

[In principle, the ring spectrum structure determines the cup product structure, the coalgebra structure of the dual.]

[Ranicki: the question is a good one, but I'm sure it's not been done.]

I am too French, my culture in spectra is very low. I live in the differential graded world. So you can put an E_∞ structure in the Poincaré duality space setting, but it is not natural. It depends on the quasiisomorphism.

[N. Rounds: This structure on the chains, this transferred structure, is quasiisomorphic to the rectification?]

[I think the answer was yes]

So you have commutative algebras, and you come up with an equivalent operad, E_∞, where $E_\infty(n)$ is quasiisomorphic to $\mathbb{Z}$. This is in chain complexes. So $E_\infty(n)$ is $\mathbb{Z}[\Sigma_n]$-projective.

So how can we build an action of an E_∞ operad on the cochains. Here you will have ones that are non-cofibrant.

So a typical example of an operad is an endomorphism operad of a chain complex, so you can build the following gadget. You can take $D(n)$ to be the natural transformations between $C_*(\ )^{\otimes n}$ and $C^*(\ )$. So you have n to one maps. You have an action $D(n) \otimes C^*(X)^{\otimes n} \to C^*(X)$. You have evaluation maps, $\phi \otimes c_1 \otimes \cdots \otimes c_n \mapsto \phi(c_1 \otimes \cdots \otimes c_n)$. You also have a morphism of this operad to Com, evaluation of the natural transformation on the cochains of aa point. You can reformulate the theorem of acyclic models by saying that this operadic map is a quasiisomorphism. For example, if you take the cup product, you have the cup product. The two operations $a \cup b$ and $b \cup a$ are the same in Com, and you have a natural homotopy between these, and you can continue and hope that this is a quasiisomorphism. So this acts on cochains, and the action of the symmetric group on this very big operad is not free. What you can also notice is, you have the diagram of operads

$$As \to D \to Com$$

where $As(n)$ is the regular representation of Σ_n, and you send e_{Σ_n} to the cup product in a certain order.

[Ralph: when you take the orbit space as opposed to the homotopy orbit space, you don't get the same thing, and the Steenrod squares detect the difference.]

So if you want to replace, you can lift any E_∞ resolution of Com to $D \twoheadrightarrow Com$. This is not explicit at all.

$$\begin{tikzcd} D \arrow[r, "\sim"] & Com \\ & E_\infty \arrow[u] \arrow[ul, dotted] \end{tikzcd}$$

So there are versions by McClure Smith and Berger-Frsese. For instance, you can take for $E(n)$ the standard bar resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Sigma_n]$-module. You get an operad, and you get an action of this operad on the singular cochains of any space.

Is that enough detail on that?

The point is that this is an E_∞ operad by definition but it is not cofibrant.

What about chains?

So the PL chain complex is obtained by taking lim over the triangulations of M of $C_*^T(M)$ so for any triangulation you have a simplicial complex. You have a category of triangulations given by refinements, and you take the limit. You have a map from $C_*^T(M)$ to $C_*^{T'}(M)$ if T' is a refinement of T. So this limit is $C_*^{PL}(M)$ and this computes the homology of M.

What is the advantage of working with this chain complex? The advantage is that you can define the Gysin map at the chain level. If you have $f : N \to M$ which is a PL map between two PL manifolds, one can define a morphism $f_!$ from a chain complex $C_*^f(M)$ which sits inside $C_*(M)$ and maps to C_{*+n-m}. These are generated by chains in general position. You look at the support of the chain and the pullback is in general position when it has the good dimension, and when the boundary of the support has the good dimension.

[What about multiples of simplices?]

Forget the multiples, look at the support of the chain.

So the support of $C_q^f(M)$ is in general position is a condition on $dim(f^{-1}(supp(C)) \leq q+n-m$ and $dim(f^{-1}(supp\, \partial C)) \leq q+n-m-1$. You need this condition to define your chain complex.

So what do you do for the intersection product? You look at the diagonal embedding $\Delta : M \to M \times M$. At the chain level you have $C_*^{\Delta}(M \times M) \overset{\Delta_!}{C}_{*-d}(M)$, and you have

$$\begin{tikzcd}
G(2) \arrow[rr] \arrow[d, hook, "\sim\text{[McClure}"'] & & C_*^{\Delta}(M \times M) \arrow[r, "\Delta_!"] \arrow[d, "\sim\text{[McClure}"'] & C_{*-d}^{PL} \\
C_*^{PL}(M) \otimes C_*^{PL}(M) \arrow[rr, "\epsilon"'] & & C_*^{PL}(M \times M) &
\end{tikzcd}$$

Here $G(2)$ is the pullback. So now you can look at

$$\begin{tikzcd}
G(n) \arrow[r, hook, "\sim"] \arrow[dr] & C_*^{PL}(M)^{\otimes n} \\
& C_*^{PL}(M)
\end{tikzcd}$$

Where $G(1)$ is $C_*^{PL}(M)$.

Theorem 5 *(McClure) $\{G(n)\}_{n \geq 1}$ is a partial commutative algebra. So $G(n) \overset{\sim}{\hookrightarrow} G(1)^{\otimes n}$ is a quasiisomorphism.*

How can you compare this to singular cochains? The idea is to introduce a third chain complex that mixes the PL chain complex together with cochains. So how do we do that? I will begin with, let X be a topological space. I leave the world of PL spaces for a moment. You can define $C_*^{bi}(X)$ for bivariant singular chain complex to be $C_i^{bi}(X)$ is generated by couples σ, c where $\sigma : \Delta_p \to X$ and $c \in C^q(\Delta_p)$ (here $i = p - q$) together with the relation that $\lambda(\sigma, c) + \mu(\sigma, c') = (\sigma, \lambda c + \mu c')$. The boundary is boundary, coboundary, and restriction on the boundary.

So $d = d_g$ (restrict to the boundary) plus d_a (coboundary), a geometric part and an algebraic part. You have morphism to the chain complex. You have a map $C_i(X) \to C_i^{bi}(X)$, where you take σ to $(\sigma, 1)$. The other way you use the cap product, which goes $C_i^{bi}(X) \to C_i(X)$, where $\sigma, c \mapsto (-1)^{pq}(c \cap \sigma)$. You filter this by the geometric dimension, and if you do that, you compute the homology of a p-simplex so you get a quasiisomorphism. Is it okay?

In the same spirit, one can define $C_*^{PL,bi}(M)$ and for a PL manifold you have the following diagram

$$\begin{tikzcd} & C^*(M) \arrow[d] \\ C_*^{PL}(M) \arrow[r] & C_*^{PL,bi}(M) \end{tikzcd}$$

Now you have a nice model for the cap product. You can choose a fundamental class, which is well defined in the PL complex. You can take a cochain a and send it to $([M], a)$. I use the notation $[M]$ for the fundamental class. You must restrict this to every chain of the triangulation. This is a quasiisomorphism. You send a chain σ again to $(\sigma, 1)$. On this side, what will be the partial product? If you take, if you want to do the product of the fundamental class by itself it is in general position, so you will take the intersection product, and on cochains you do the cap product. The intersection product is well-defined. Let me give the idea of what are the algebraic structures. On chains you have something which is partial commutative. On the bivariant PL chains you have partial version of $com \otimes E_\infty$. On cochains you have E_∞. Why the tensor? You have a commutative part and an E_∞ part, and so $com \times E_\infty$ is an operad. So you have maps

$$\begin{tikzcd} Com \otimes E_\infty \arrow[r] \arrow[dr] & Com \otimes Com \arrow[r] & Com \\ & E_\infty & \end{tikzcd}$$

You have a strict E_∞ algebra, and this is partial $(comm \otimes E_\infty)$, and so the idea is, how can you define the $G(2)^{bi}$? You take your couple (a, b), and $G(2)^{bi}$ you take $(\sigma, c) \otimes (\sigma', c')$ and geometrically you take $\sigma \cap \sigma'$, triangulate this, so it can be written as the sum of simplices $\sum \sigma''$ where these are contained in $\sigma \cap \sigma'$ which maps to $\sigma \times \sigma'$, and you push $c \times c'$, you take, oh yes, sorry, sorry, you have $c \cup c'$ that you can pull back, so you have $\sigma'', \iota^*(c \cup c')$. So it is a little subtle, you have to go back to the definition of the Gysin map and play with support. You have this diagram of partial algebras, and the first theorem is that each $G(n)$ into $G(n)^{bi}$ and also a map from $C^*(M)^{\otimes n}$, and the theorem is

Theorem 6 *these maps are quasiisomorphisms of partial com $\otimes$ E_∞ algebras*

$$\begin{tikzcd} & \{C^* M^{\otimes n} \arrow[d] \\ G(n) \arrow[r] & G^{bi}(n) \end{tikzcd}$$

[The bivariant complex is the pair subdivision.]