# Oberwolfach

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## 1 Alex Kahle

[There are no talks this afternoon. There will be soccer and a hike. We'll meet at 2:30 for soccer in front of the building. There is a village with a restaurant about two hours from here. We'll leave around 1:45, it's good to have a count so the secretaries can call ahead. The talks this morning are a little shorter. There's just fifteen minutes of break.]

It's hard to follow the talks that have been presented so far. The title of my talk is Superconnections and Index theory. I'll be describing the work for my thesis once upon a time. I'll divide my talk into three parts and give the third part short shrift. In the first part I'll describe superconnections. The second part is the main part and talk about what kind of index theory you can do with these. The third part is to sketch some proofs.

Superconnections were introduced to mathematics by Quillen, who gave the following

**Definition 1** *A superconnection $\nabla$ on a $\mathbb{Z}/2\mathbb{Z}$ graded vector bundle is an odd derivation on $\Omega(M,V)$. They're the right idea for a connection in the world of $\mathbb{Z}/2\mathbb{Z}$ vector bundles. If you defined an ordinary connection, you'd have a degree one derivation. Consequently, superconnections form an affine space modeled on $\Omega(M, End(V))^{odd}$. To remind you, the endomorphisms are $\mathbb{Z}/2\mathbb{Z}$ graded themselves. Every superconnection can be written $\omega_0+\nabla+\omega_2+\cdots$, where you have an endomorphism valued $0$ form, a connection, an endomorphism valued $2$-form, $3$-form, and so on.*

Quillen was thinking about representing classes in $K$-theory. A class is a guy like $f: V\to W$, and the support is [unintelligible]. The superconnection has an odd picture, an odd matrix $\left(\begin{array}{c|c} & \\ \hline f & \end{array}\right)$, and if it's unitary in the appropriate sense you get $\left(\begin{array}{c|c} & f^* \\ \hline f & \end{array}\right)$, this is the first piece. You can define the Chern character of a superconnection, you can take $ch\nabla$ which is the trace of $e^{\nabla^2}$.

The cool thing about this is that when this guy has support, he peaks near the support and falls off exponentially. This was told to us by Quillen a long time ago.

Let me talk about index theory. In order to do index theory, I'll need Dirac operators. Now $M$ will be smooth, Riemannian, and spin. Then the Dirac operator I associate to the data of a vector bundle with superconnection (unitary) over $M$ is defined by the following:

$$\mathcal{D}(\nabla) : \Gamma(\mathbb{S} \otimes V) \stackrel{V\otimes 1 \oplus 1 \otimes \nabla}{\longrightarrow} \Omega(M, \mathbb{S} \otimes V) \stackrel{C(\cdot)}{\to} \Gamma(\mathbb{S} \otimes V)$$

This will be elliptic, formally self adjoint, and as a consequence it looks like

$$\mathcal{D}(\nabla) = \left( \begin{array}{c|c} & \mathcal{D}^*(\nabla) \\ \hline \mathcal{D}(\nabla) & \end{array} \right)$$

So what's the index?

**Corollary 1** *(Atiyah Singer)*

$$ind(\mathcal{D}(\mathbf{nabla})) = \int_M \hat{A}\Omega^M ch\nabla$$

Superconnections don't tell us more about topology; every one is homotopic to a connection.

The first thing that you might want to do is prove a local index theorem. When I have a Dirac operator, I can form the heat semigroup $e^{-t\mathcal{D}(\nabla)^2}$ whose trace is the index of $\mathcal{D}(\nabla)$.

The heat semigroup is smoothing, represented by an integral kernel:

$$e^{-t\mathcal{D}(\nabla)^2}\psi(x) = \int_M p_t(x,y)\psi(y)dy$$

and rewriting this formula,

$$tr\ e^{-t\mathcal{D}(\nabla)^2}\psi(x) = \int_M tr\ p_t(x,y)dy$$

A long time ago, famous people came up with the theorem

$$\lim_{t\to 0} tr\ p_t(x,x)dvol = (2\pi i)^{\frac{dim\ X}{2}}[\hat{A}(\Omega^M)ch\nabla]_{(M)}$$

For superconnections this diverges. The proof involves stretching the manifold, which scales in time. I get a weird relationship because the individual pieces stretch differently.

What you need to do is introduce a scaling on superconnections, scale the $n$-form by $|s|^{\frac{n-1}{2}}$. Then if you put this parameter in andtie it to time in this way, you do get convergence. That's the first theorem you want to prove, and Getzler proved it with stochastic techniques. You have to get at small time asymptotics, but you have a family. Things are difficult. I provided a different proof that avoids stochastic things. Hopefully I'll get to tell you how this works. Let me tell you immediately what you can do from this. Now you can consider families from this. Let me just carry on and see what happens. As we saw yesterday, we need the right notion of a geometric family. A Riemannian family (map) is a triple $(\pi, g, P)$ where

$\pi$ is a proper submersion $M \to B$, $g^{M/B}$ is a metric on $T(M/B)$ and $P : T(M \to T(M/B))$. This allows you to identify the pullback of the tangent bundle of the base with a subbundle of the normal bundle. I also want to ask so I can do index theory that the fibers be closed and spin. Then I can make a family of Dirac operators quite easily. If I have a picture like

$$\begin{array}{c} V, \nabla \\ \downarrow \\ M \\ \downarrow \\ B \end{array}$$

then I get a family of Dirac operators following my previous construction fiberwise. With [unintelligible]you can get a class of $K$-theory in the base. I'll give you a canonical $\mathbb{Z}/2\mathbb{Z}$ vector bundle over $B$ with a superconnection that represents this $K$-theory class morally.

I have for the canonical bundle $\pi_*(V)$ a fiber at $y \in B$ is $\Gamma_y(\mathbb{S}^{M/B} \otimes V)$, and finally I want to, well, [unintelligible]told us that from a connection on the top you get a superconnection on the bottom. Then I will give another superconnection and add in something that comes from the differential form part. Let me remind you that you also have different differential forms. I'll tell you what to do with the forms. If $\omega_i$ is in $\Omega^i(M, ?)$, then the pushforward of $\omega_i$, then the pushforward evaluated on $\xi_1, \ldots, \xi_i$, is $c^{M/B} z(\xi_1) \cdots z(\xi_i)\omega_i$. [Description of the meaning of this formula. I missed it.]

One more thing to say is that Riemannian maps have an action of $\mathbb{R}$, so $\pi^r$ is $\pi, rg^{M/B}, p$, and the limit as $t \to 0$ of $ch\ \pi^t\nabla$ is $(2\pi i)^{dim\ M/B}\pi_*(\hat{A}(\Omega^{M/B} ch\ \nabla)$. Let me just say that this action automatically puts in the scaling on the superconnection. You can prove an annoying formula. [unintelligible].

If you take that into account, you get exactly what the original theorem told you.

Let me just tell you what else you can do. Something that I'm interested in studying is families of Dirac operators coupled with superconnections. In wanting to study those, you run up against $\eta$-invariants. Both are defined from the spectrum of the Dirac operator. The geometry you get involve nice index theoretic quantities. With a superconnection, life is not so easy. When you want the geometric quantities, you need to take scaling into account. The Dirac operators can't be simultaneously diagonalized and your spectrum goes all over the place. So you can either do a spectral definition, rewrite an $\eta$-invariant and do the determinant bundle exactly, which is built for general operators, but you won't get nice geometric theorems. You'll get, for example, that the [unintelligible]invariant on the boundary will be a term in the asymptotic expansion of the heat kernel. It'll just be something in the middle. You won't get anything nice and recognizable. You can also do a geometric definition. Then you want to define things so that you can get at, well, the two form part should have holonomy from the $\eta$ invariant, and you want to define things so that you have a line bundle with these properties.

Look at the following situation to see how. If you have a superconnection, inside of it you have a degree 0 part. This is roughly analagous to [unintelligible]. So now I want to associate to this a line bundle whose section is the determinant of this guy, and I want the curvature of this connection to be equal to the two-form part of the Chern character of the superconnection.

If I think about this for a while, I'll see how to build this, using the transgression, which is the trace of something, which can be made sense of in an infinite dimensional setting as well. Maybe I should just stop.

[I have a philosophical question. One way I know what to say what a connection is, is an endomorphism valued one-form. This is a generalization, what about holonomy, what is the generalization of holonomy?]

They don't generalize [unintelligible]as much as they do derivation. [unintelligible]has written a thesis where he defines parallel transport for superconnections. The physics is telling you something else. Parallel transport is not obvious.

[You get, it's not invariant under reparameterization, but it is under the adiabatic limit.]

## 2 Scott Wilson, Categorical Algebra and Mapping Spaces

I realize that the title is a bad title, so I won't write it on the board. Let me give an outline of what I want to talk about. So three things. The first thing I want to talk about is a language for some elementary topology, algebraic topology. I want to introduce it for my second topic, an application to generalizations of Hochschild complexes, which is in the spirit of several things that have gone on so far. In these generalizations I want to give some examples for invariants on mapping spaces, generalizing things like Chern characters, and also relations to known constructions related to deformations of the Laplacian on a manifold. That's the outline.

Let me start with something that we can either take as a definition or a lemma:

**Definition 2** *A differential graded algebra is something living on the category of chain complexes with chain maps and the tensor product of chain complexes. There's another category, of finite sets with maps of sets and disjoint union of finite sets. What's the idea? In finite sets I want to include the empty set and sets with one and two elements and so on. So for example, the empty set will be sent to the unit, the one element set will be sent to a chain complex, and then $k$ will be sent to $A^{\otimes k}$. On morphisms, a morphism from a set with two elements to a set with one element, that's a product.*

This is really an equivalence of categories. I want to use this language to extend this by changing one word.

**Definition 3** *A partial dga, a partial algebra, is a lax monoidal functor from finite sets to chain complexes. It assigns to $j$ a space $A(j)$. It respects the monoidal structure up to* *natural transformation. There exists a natural equivalence from $A(j \sqcup k) \to A(j) \otimes A(k)$. This is a natural transformation that is also a quasiisomorphism. This should satisfy coherence properties, roughly symmetry and coassociativity.*

I'm using this, [discussion about lax versus strict], well you can use coalgebras or any operad. Change the morphisms of finite sets to work with a different operad. If you note that the category of based finite sets is a module over the category of finite sets, you can generalize to modules, comodules, et cetera.

Now you may ask, what is the meaning of this definition. One way to think about it is that partial algebras can be functorially replaced by $E_\infty$ algebras. (this a theorem due to the speaker). There's a proof that's elementary using the bar construction. Let me give an example, and then I'll move on to the second part. The example is going to be very elementary, although it doesn't pop up in elementary topology books. Let $X$ be a space, and if you have a function from a set with $j$ elements to a set with $k$ elments, I can take $X^j = Map(j, X) \leftarrow Map(k, X) = X^k$. Let me linearize this and get $C_*(X^j) \leftarrow C_*(X^k)$ or $C^*(X^j) \to C^*(X^k)$. This doesn't work for every set of chains or cochains you put here. There's no cellular diagonal map. If you take piecewise linear chains and cochains, you'll get the Kunneth map of the Cartesian product of chains and its dual. It's a little tricky to take the dual, but in the PL setting you can work with chains and make things nice.

This gives a partial coalgebra on $C_*(X)$ and a partial algebra on the cochains. This partial algebra, secretly an $E_\infty$ algebra (by Mandell) determines the integral homotopy type if $X$ is simply connected or nilpotent. It's a small package for that structure.

Let's go on to two, which is drawn like —. Let $Y$ be a space and $A$ a partial algebra. By a space I mean a simplicial set, a functor from $\Delta$ to finite sets, and a partial algebra is a functor to chain complexes. This is a simplicial complex in $Ch$ so it has a total complex $CH^Y(A)$, which is a generalization of the Hochschild complex, defined for any partial algebra $A$.

There's a way to say algebraically, and this is joint with Tradler and Zenalian, and versions of this go back to [unintelligible], Hochschild, Tradler, Zenalian, Ginot. For $A = \Omega(X)$, the differential forms on a space, then $CH^Y(A)$ computes the cohomology of $X^Y$ if $X$ is sufficiently connected. For higher dimensional simplicial complexes, Tradler, Zenalian and Ginot have shown that there is an iterated integral that does [unintelligible]for higher connected spaces.

There's another example due to Jim McClure where the partial algebra is the intersection product on a manifold. This is the same $E_\infty$ structure, which I was told at this conference by David [unintelligible].

Let $A$ be a strict algebra, and $Y$ a circle. Then $CH^{S^1}$ is the Hochschild complex, made up of $\prod_{n \geq 0} A \otimes A^{\otimes}$, and the differential is two terms, one of which is the internal $d$ and the other of which multiplies adjacent terms. The algebraic structure corresponds to the cup product on the mapping space. The partial algebra is well understood for the Hochschild complex, it's the shuffle product on the Hochschild complex. You multiply the first elements and then shuffle all the other terms. It's commutative and associative, which implies there is an exponential map. I want to do a quick calculation. Let's calculate, I can exponentiate an element, so I want to exponentiate $1\otimes x$, and so this is $1+1\otimes x+1\otimes x\otimes x+\cdots$. Let me continue this calculation. Let me call the differential $D$. Then $D(e^{1\otimes x})$ is $(1\otimes(dx+x\cdot x))\cdot e^{1\otimes x}$. So if $dx+x\cdot x$ is zero then $De^{1\otimes x}=0$. This reminds us of curvature of connections. For example, if the algebra is matrices of froms, then $dA+A\cdot A$ computes the curvature, and this analogy can be taken further.

We saw in the first talk this morning, we saw a map $K(M)\to\Omega(M)$, an one can consider $CH^{S^1}(A)\cong\Omega(M^{S^1})$ and so:

$$\begin{tikzcd} & CH^{S^1}(A) \arrow[d] \\ K(M) \arrow[ur, "\text{Getzler, Jones, [unintelligible], also Bismut}"] \arrow[r] & \Omega(M) \end{tikzcd}$$

There's even an equivariant version. [Some description of this argument]

Now the algebraic machinery is set up for other mapping spaces, so you can consider maps of the torus, and we've shown that there's a closed class above extending a closed class on the loop space. We're working to generalize to other groups and mapping spaces.

Let me do an example, for $Y=I$. This Hochschild complex, $CH^I(A)$ is the two sided bar construction. I mentioned that we can talk about partial modules. I didn't write it out, bet these can be defined in terms of partial modules. Let me put in symbols. I picture the modules over the vertices at either end of the interval, and $CH^I(A,M,N)$ is $\prod_{n\geq 0}M\otimes A^{\otimes n}\otimes N$.

I'll let $A$ be the forms on a Riemannian manifold with differential $d$ and wedge product. $M$ will be differential forms with $d$ and the usual module structure. $N$ will be differential forms with $d^*$ and the dual module structure. You might say, what's so interesting, you're just transporting by an isomorphism. Miraculously, you see that you have all of this structure from differential forms. In this example, one can compute the following. Let $D$ be the differential on the Hochschild complex $CH^I$ for the usual algebra and take both to be the standard module structures. I let $D^*$ be the differential for $A$, $M$, and $N$ described earlier. Then you transport one, these don't commute, but they have a commutator, $\Delta=[D,D^*]$. I'll end with one calculation. I want to calculate $\Delta x\cdot e^{s(1\otimes x\otimes 1)}\cdot x$. This turns out to be

$$xe^{1\otimes sx\otimes 1}(\Delta_{usual}x+sd*(x\wedge *x)+s*(x\wedge *dx)+s^2*(x\wedge *x)\cdot x)$$

This is the term from Witten's deformations of the Laplacian. If $x$ is in the kernel of $d^*$, then the nonlinear term corresponds to an interesting equation in PDEs, namely

$$\Delta x+dp+\frac{1}{2}d*(x\wedge *x)+\underbrace{*(x\wedge *dx)}_{x\times curl x}$$

So this is the Navier Stokes equation.