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1 D. Ayala

I'm going to talk about the topology of bordism categories, and for me, $Cob_d^{\mathscr{F}}$ will be a monoidal category in Top. The objects will be $d-1$-dimensional manifolds (M,g) where $g \in \mathscr{F}(m)$, so $\mathscr{F}$ is some geometric structure like metrics or metrics of constant negative curvature, or, say pairs consisting of the complex structure and a holomorphic map to a fixed target.

This should be local data so embeddings is a non-example. This will be some model for a moduli space

$$\amalg_{[M]} Mod^{\mathscr{F}}(M)$$

My morphisms are compact d-manifolds with geometric structure (W,g) where these are closed with boundary. This is a model for the moduli space of d-dimensional $\mathscr{F}$-manifolds.

One thing we can do is look at the classifying space of this category. I'll give you an interpretation of this in a moment. I'm inclined to think of these in terms of higher category theory. So let's think that there are k-morphisms, which are maps from a $k-1$-cube into the space of morphisms. A three-morphism is a map of intervals, and composition is concatenation. I can do the path category construction on the classifying space. There's a universal arrow $C \to P|C|$. This sends an object to itself viewed as a 0 simplex in the nerve of C. It takes a morphism to the path corresponding to that morphism. This arrow $C \to P|C|$ is the groupoid completion of C. Why am I saying this? I can think of the path category as a groupoid. Every morphism has a canonical inverse. It's a functor to a groupoid universal among all such.

That said, we get maps for each d-manifold from $Mod^{\mathscr{F}}(W,g) \to \Omega_\partial |Cob_d^{\mathscr{F}}|$. Suppose we want to understand moduli spaces, well this map gives us invariants of moduli spaces. I'll call these invariants universal invariants because they don't see W.

So I might want to identify the homotopy type of the right hand side

Theorem 1

$$|Cob_d^{\mathscr{F}}| \cong \Omega^{\infty-1} \underbrace{\mathscr{F}(\mathbb{R}^d)//O(d))}_{\textit{Thom spectrum}}^{-\gamma_d}$$

Think of the right hand side as being accessible. This is coherent with Galatius-Madsen-Tillmann-Weiss. If I like this idea of having identified the homotopy type, there are two questions I'd like to address. The first question is: of what are universal invariants invariants? In other words, I'd be happy with a geometric interpretation of this classifying space. We might think the map is a good approximation of the moduli spaces:

Theorem 2 *(Madsen-Weiss)*
$Mod^{or}(W_g^2) \to \Omega|Cob_2^{or}|$ is an isomorphism in some range depending on g, which ges better as g increases.

Theorem 3 *The moduli space $\mathscr{M}_g^d(\mathbb{CP}^n)$ of genus g curves in projective space maps to $\Omega|Cob_2^{\mathbb{CP}^n}|$ is an isomorphism on homology in a range*

Theorem 4 *$Mod^{or}(W^3) \to \Omega|Cob_3^{or}|$ is zero in rational homotopy. This was also proved in rational cohomology by [unintelligible].*

So this is a non-theorem. Any three-manifold is parallelizable. This somehow, well, the right hand side, rationally this lifts to the sphere spectrum, and a three-manifold can be parallelized, it has a lot to do with this. Another is, there's a moduli space a lot of people are interested in, the Deligne-Mumford compactification, and we might hope to touch that. So question two is, is there a version $Cob_2^?$ so that $\bar{M}_g(Y)$ sits inside of the morphisms?

Okay, so this is setup for what I'm going to talk about for the rest of the talk, which is an attempt to answer the questions in terms of singular manifolds.

A very small discussion of singular manifolds.

Definition 1 *A singularity type is a stable germ $\mathbb{R}^{N+d} \to \mathbb{R}^N$ at the origin.*

It's represented by a map between Euclidean spaces of negative codimension. Two germs are the same if crossing them with $\mathbb{R}$ gives the same thing. I'll say g is local of type Σ if [unintelligible].

Here are some examples:

1. $\Sigma = \emptyset$, this is represented by $\mathbb{R}^d \to *$. Such a map is a submersion.
2. $Fold_k = \mathbb{R}^{d+1} \to \mathbb{R}^1$ given by $(x_1, \dots x_{d+1}) \mapsto x_1^2 + \dots + x_k^2 - x_{k+1}^2 - \cdots$
3. $Node = \mathbb{R}^4 \to \mathbb{R}^2$ given by $(z, w) \mapsto zw$.

A collection J of singularity types is proper if

- All are of finite codimension in $\operatorname{co\,lim}_N Map_0(\mathbb{R}^{N+d}, \mathbb{R}^N)$
- For $f \in \Sigma \in J$ minimal, then, well, $f : \mathbb{R}^{N+d} \to \mathbb{R}^N$, and $[germ_x f] \in J$ for all $x \in \mathbb{R}^{N+d}$.

Now I want some kind of moduli space of singular manifolds. The picture of the double cone is a singular manifold, but I should realize this as the fiber over zero of a map $\mathbb{R}^4 \to \mathbb{R}^2$. So a singular manifold is the data of how it resolves. I have to capture somehow resolution of singularities. So to do that I'll use sheaf stuff.

Let Man_∂ be a category with objects smooth manifolds with corners and morphisms are embeddings sending k-dimensional faces to k-dimensional faces. So the simplices and their face maps are in this category.

Now consider a sheaf B_N^J from $Man_\partial^{op} \to Set$. This will send $X^r \to E^{r+d} \subset \underbrace{X_e^r}_{\text{extension}} \times \mathbb{R}^N$ such that the projection $E \to X_e^r$, the class of this projection is in J and so that the map to $X_e^r \times \mathbb{R}^1$ is proper. Also the restricted map on faces should also be in K.

Bear with me for a moment and consider another sheaf on the same category $Man_\partial^{op} \xrightarrow{\bar{B}_N^S} Set$. Let X^r go to $E \in B_N^J(X)$ and $a \in \mathbb{R}^1$ so that $E \to \mathbb{R}^1$ is fiberwise transverse to a.

Overline just means it's equipped with this a, and $\bar{B}_N^J$ is naturally a sheaf of posets were $(E, a) \leq (E', a')$ if $E = E'$ and $a \leq a'$.

A poset is a category in a canonical way.

Definition 2 *A cobordism category of J-singular manifolds*

$$Cob_d^J = \underbrace{\lim}\, \textit{homotopy colimit}_N |\bar{B}_N(\Delta)|$$

So I think of this as being long in one direction, and there are a and a', it should be a cobordism between these two slices.

Theorem 5 $|Cob_d^J| \cong |B_\infty^J(\Delta^\cdot)| \cong \Omega^{\infty-1} MTJ$ *Here MTJ is a spectrum with the $N+d$ term equal to $Map(\mathbb{R}^{N+d} \to \mathbb{R}^N)$ which are J-singular, modulo $O(N)$. Then I pull back γ_N. The point is, it's something fairly easy that you can get your hands on.*

So we're going to take this theorem and try to classify maps from $|Cob_d^J|$.

in words, I don't think you can have a Madsen Weiss theorem. You can split into irreducible pieces of low genus which will always be in the boundary. I can see Deligne Mumford space as a component of the morphism space of this boundary. It is a component. We get a map in particular from Deligne Mumford into this realization. One way that this shows up is looking at coarse Deligne Mumford.

So, in the spirit of trying to understand these maps, the classifying space comes with a filtration as we remove singularity types. Suppose J' is proper. Then the stratum $strat(\Sigma) \hookrightarrow \lim_N Map_0^J(\mathbb{R}^{N+d}, \mathbb{R}^N)//O(N)$, and $Strat(\Sigma) \cong B\ Aut\Sigma$ which are automorphisms of $\mathbb{R}^{N+d}, \mathbb{R}^N$, orthogonal transformations, which preserve the singularity. If Σ is a node, then $Aut(\Sigma) = \mathbb{Z}/2 \wr S^1$

This group comes with maps into $O(N+d)$ and $O(n)$, whcih I will call $-\nu$ and γ. I was going to do some examples but I'll jump to the node case. Say J consists of the empty and nodal singularities. Then there's a functor $Cob_2^{\rightarrow} Cob_2^J$, which regards a smooth manifold as a nodal one with no nodes. This gives me on the level of classifying spaces a map $MTO(2) \to MTJ$. In light of what I said as the stratum identification, and I forgot to say, with the inclusion of the normal bundle, you have this mapping to $B\ Aut(Node)^{\gamma+\nu}$ and this is a cofiber sequence of spectra.

With any cofiber sequence, I can recognize the middle term as the cone of $MTO(2)$ being mapped in by a desuspended version of $B\ Aut$.

So a map $MTJ \to \Sigma^{-1}(C)$ is equivalent to a map $MTO(2) \to C$, and a nulhomotopy of the map $\Sigma^{-1}B\ Aut(Node)^{\gamma+\nu} \to MTO(2) \to C$.

This classifies not field theories but field theories landing in groupoids as this thing I spelled out. So geometrically (then I'll finish), by, when we take the infinite loop space, we can geometrically interpret, the answer is, a map $|Cob_2^J| \to \Omega^\infty C$ is the data of a section $D^2 \to \Sigma^\infty C|\alpha(S^1) = Z_0([unintelligible])\}//Aut(node)$ over $B\ Aut(node)$.

2 Orlando

Let me tell a story about sitting in on Raoul Bott's geometry class. Someone asked him, "don't you need the axiom of choice for that?" He said, you must be a logician, turned around, and went on writing. I might do that, "you must be a mathematician."

Mike asked me to talk about why you need p_1 vanishing for elliptic cohomology. I'm going to try to talk about that, and today I'm going to try to tell a little bit about path integrals, and how the second Stiefel-Whitney class has to vanish. Mapping a torus into a manifold in field theory is what gives you p_1 vanishing.

The first thing that I want to warn you is that when you talk about topological quantum field theory, I won't talk about that at all. I'll talk about supersymmetric models. In a topological quantum field theory, there are no time evolutions. For us, where things are is going to matter. We'll have geometry, not topology, but you will be able to have topological information.

So we should have an operator Q with $Q^2 = 0$. So in a supersymmetric theory, you have

$Q^2 = H$, where H is the Hamiltonian and that acts as time evolution. So for instance if $Q = d + \iota_x$ then $Q^2 = \mathcal{L}_x$, the Lie derivative in the direction of x.

We use the following terminology. There's always time, so we write things like $n + 1$, where n is spacial and 1 is time. So quantum mechanics is $0 + 1$ and quantum field theory is all of the other cases. For example, for $p_1 = 0$, this comes out of path integrals three different ways, the modular properties, anomalies, and renormalization. They seem very different but it's interesting to see how they come together.

When you read a physics paper, you'll see some physics terminology. So I'll talk about the Dirac operator and Dirac-Ramond operator. In colloquial speech, let me translate what a physicist means by a Dirac operator.

You have a map $S_+ \to S_-$ in spinors, that's what mathematicians call Dirac, and physicists call the chiral Dirac operator. Physicists call the matrix

$$\not{D} = \begin{pmatrix} 0 & D^* \\ D & D \end{pmatrix}$$

Then we have γ_5, it's standard notation to call it the block diagonal matrix with I and $-I$. This is also sometimes called $(-1)^F$.

So now I want to make a historical comment about this which goes back originally to basically the heat kernel proof of the index theorem. That's the observation that $Tr\ e^{-tD^*D} - Tr\ e^{-tDD^*}$. This is indepenet of t and is the index of D. We think of Schrodinger evolution, $\frac{\partial\psi}{\partial t} = -Ht$.

A formal solution is $e^{-tH}\psi(0)$. So if you know what the heat kernel is, you can evolve it in time.

The question that you can ask is, is there a physical system whose Hamiltonian allows you to solve this problem? I would like to describe this.

The last thing I wnated to do is write this way, this is the $Tr\ (-1)^F e^{-t\not{D}^2}$.

So let X be a map $(1|1) \to M^{2n}$, a map of a supermanifold to M^{2n}, so let $X^\mu(t, \theta) = x^\mu(t) + \theta\psi^\mu(t)$. I am leaving out coefficients, so for me $1 = 2 = \pi = i$ because it will take a long time.

Now what is the supersymmetry transformation $SUSY$? That takes $\theta \mapsto \theta + \epsilon$ and $t \mapsto t + \theta\epsilon$. Then $\delta x = \epsilon\psi$ and $\delta\psi = \epsilon\dot{x}$. Then $SUSY^2$ has $\delta\ x = \dot{x}\epsilon_1\epsilon_2$ and $\delta\psi = \psi\epsilon_1\epsilon_2$. If you have a symmetry, it can be written in terms of a vector field. These are written in terms of operators. They will generate the Dirac operator and the Hamiltonian.

As an exercise $dt - \theta d\theta$, maybe, there's an invariant form, something like this, maybe, its square is a metric. In higher dimensions there's a gamma [unintelligible]here.

Let me just say one thing about the Dirac operator. I just want to make a story about the Dirac operator. Look at the tangent space to M at a point. I can move in a variety of directions. Because I have a basis, I have Clifford multiplication. Then $\gamma_\mu = C(\frac{\partial}{\partial x^\mu})$, and so you have the symbol part of the Dirac operator and you want to try to complete it.

So $L = \frac{1}{2}(\frac{dx}{dt})^2 + \frac{1}{2}\frac{d\psi}{dt}\psi$ and the map $L \to L + \frac{d}{dt}\mathcal{S}$.

So we have $\psi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \dot{x}^\mu$ and $\pi_m u = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\mu} = \psi^\mu$, and you have the Poisson bracket $[x^\mu, \phi_\nu] = \delta^\mu_\nu$, which is the same thing as $\{\psi^\mu, \pi_\nu\}$. So here $Q = \psi^\mu \phi_\mu$.

Quantization, we have this Poisson algebra and you want to represent it in terms of operators. You want to represent it so that $\frac{1}{i\hbar}[\hat{x}^\mu, \phi_\nu] = \delta^\nu_\mu$, and $\frac{1}{\hbar}\widehat{\{\psi^\mu, \psi^\nu\}} = \delta^{\nu\mu}$ So in $L^2(\mathbb{E}^{2n})$ has $\hat{\phi}_\mu = -i\frac{\partial}{\partial x^n}$. I have one nice representation, with $\hat{\psi}^\mu$ as the Dirac gamma matrix γ^μ. Then $\hat{Q} = \gamma^\mu \frac{\partial}{\partial x^\mu}$. Then this is $L^2(\mathbb{E}^{2n}) \otimes \mathbb{C}^{2n}$.

What if I curve the manifold. What you get is the Dirac operator on that manifold, and the quantization ar the square integrable sections.

I can rewrite L as $p\dot{x} - \frac{1}{2}p^2 + \psi\dot{\psi}$. Writing this way, you see that we already have an odd symplectic manifold type variable.

We try to turn this algebra, this Poisson algebra, to find a Hilbert space on which it operates. We find a representation of $\hat{p}$ as the derivative with respect to x_μ and on the other side we have the Clifford algebra.

On a manifold these things just become L^2 sections of the spin bundle.

γ_μ is just the matrix for the operator $\hat{\psi}^\mu$.

One last thing, the operator $hatQ^2 = p^2$. Dirac asked how to take the square root of the Hamiltonian.

Let's talk about path integrals. What I want to do is, so, the idea of path integrals is due to Dirac. There's a great paper where he asks how Lagrangians enter the Hamiltonian view. It's all there except for computation. Feynman sat down and did the computation.

Let's talk about the path integral. Let's ask the following question. Assume I start at $|x_I\rangle$, and ask, what's the probability amplitude that I'll end up at x_F? So what is $\langle x_F | e^{-TH} | x_I \rangle$? Well, it's $\int_{\mathscr{M}} [\mathcal{D}x] e^{-\int_0^T L}$ where $\mathscr{M} = \{x : (0, T] \to M$ such that $x(0) = x_I$ and $x(T) = x_\Gamma$. In this one dimensional case the right hand side has rigorous mathematical meaning.

So earlier we had a trace. You want to take the diagonal part of the matrix for the trace. So the trace of e^{-tH} is the integral over periodic paths of

$$\int_{per.} [\mathcal{D}x] e^{-\int_0^T} Ldt$$

I'll move to the SUSY case.

So $L = \frac{1}{2}||\frac{dx}{|}|^2 + \langle \psi, \frac{D}{dt}\psi \rangle$. We can evaluate

$$\int [\mathcal{D}x][\mathcal{D}\psi] e^{-\int_0^T Ldt}$$

Have to stop. Too fast.

3 Kevin Costello, Bosonic string theory

[Our next speaker is Kevin Costello.]

My title is overly ambitious. I'm going to talk about, first, some general approaches to QFT, and I'll be very handwavy and broad. What I really want to talk about is how to apply this to a particular example which is called bosonic string theory. We'll see some fun Lie algebra homology calculations. I'll start with the general idea. This approach is kind of analagous to the deformation quantization approach to quantum mechanics. There will be algebraic structures encoding quantum and classical field theory, and quantization is turning a classical field theory into a quantum field theory. The key theorem is that one can do this. There is a cohomological problem about quantization. I should start by telling you what these structures are. The structure that I claim encodes quantum field theory is a factorization algebra.

Definition 3 *A factorization algebra on a manifold M is the data (this should be the space-time of the theory) of*

1. *for each open set a cochain complex. The structure is sheafy. Just like with a sheaf, there will be maps between open sets.*
2. *If $U_1, \ldots, U_n$ are disjoint open subsets of V, then there is a map $\mathscr{F}(U_1) \otimes \cdots \mathscr{F}(U_n) \to \mathscr{F}(V)$. In particular if $n = 1$ there's a map, so it's a cosheaf, but there's this multiplicative structure.*
3. *There are evident compatibility conditions you want this to satisfy. If I have small open subsets, I can map to middle size subsets, and then to large ones, or I can go directly to the large ones. [Picture] I should have associativity.*
4. *This is sheafy. Kevin calls this blob homology, Jacob calls it topological chiral homology. $\mathscr{F}(V)$ is built from sums of products of elements of $\mathscr{F}(U)$ where U is arbitrarily small. This is rough. Slightly more precise, it is the colimit of $\mathscr{F}(U1) \otimes \cdots \mathscr{F}(U_n)$.*

You might ask why this has to do with quantum field theory, so the idea is that this vector space $\mathscr{F}(U)$ is supposed to be the set of observations one can make on the open subset of spacetime U. The key thing we would like to know is what the correlations between various bservables are.

There's no algebra, this higher dimensional structure is the replacement for the algebra structure. In good cases, if our manifold is compact, $H(\mathscr{F}(M)) = \mathbb{R}[\hbar]$ so we get these maps $\mathscr{F}(U_1) \otimes \cdots \otimes \mathscr{F}(U_n) \to \mathbb{R}[[\hbar]]$ and these are the correlation functions.

If we start with an isolated classical solution, we get this; if it is not isolated, then we have to do some second [unintelligible].

I should mention how this will relate to the previous talk. This purports to encode much of quantum field theory. Let me say, for ordinary quantum mechanics, $M = \mathbb{R}$, we have $\mathscr{F}(0, \epsilon)$ is the algebra of observables for quantum mechanics. This was some kind of bundle, combination of Clifford and Heisenberg algebras, so $\mathscr{F}(I_1) \otimes \mathscr{F}(I_2) \to \mathscr{F}(I_3)$ is just the product in this algebra.

We wouldn't expect to have such a simple structure in higher dimensions, because time isn't discrete like this.

[How do you put in that observables which are spacelike separated have to commute?]

I don't understand what happens in the Minkowski signature.

[Argument from Alvarez.]

I wanted to try to explain how to construct these gadgets using obstruction theory.

We'd like to start by saying what happens at the classical level. Suppose we have some classical field theory. Let for the first example, fields be smooth functions and the action be something standard, like

$$S(\phi) = \int \phi \Delta \phi + \Phi^{\Delta}$$

The second example has M a surface Σ and fileds are (J, φ) a pair of a complex structure on Σ and φ a map to $\mathbb{R}^d$ modulo the gauge group of diffeomorphisms of Σ. The action is

$$S(J, \varphi) = \int_{\Sigma} \langle \partial \varphi, \bar{\partial} \varphi \rangle_{\mathbb{R}}^{d}$$

I want to explain what structure classical observables satisfy and what structure they have in these examples.

For classical field theory, we find a factorization algebra where $\mathscr{F}(U)$ is also a commutative algebra and all maps respect this structure. This is a classical factorization algebra. (A classical field theory needs a Lagrangian and fields, which can be made precise).

If we have an action, let $EL(U)$ be the derived space of solutions to the Euler Lagrange equation. These must be taken to be infinitessimally close to a fixed one.

[Jacob: how big is this space?]

The quadratic term is elliptic, so if U is compact, it will be finite dimensional. Now $\mathscr{F}(U)$ will be $\mathscr{O}(EL(U))$, which is a commutative differential algebra.

The Euler Lagrange equation is the critical points of an action and [unintelligible]is a critical points of a one-form. Let me say what it looks like to take the derived critical locus. What do these look like?

In finite dimensions, for X finite dimensional and $\mathscr{F} : X \to \mathbb{R}$ a function, then $\mathscr{O}(Crit\ f)$ looks like

$$\cdots \to \wedge^2 TX \overset{\vee df}{\to} TX \overset{\vee df}{\to} \mathscr{O}_X$$

which is $\wedge TX, \vee df$. So $x \in \pi_0 Crit\ f \subset X$ then the tangent space $T_x\ Crit\ f = T_x X \to T^{\cdot}_x X$ via the Hessian of f.

So what is the derived critical locus in infinite dimensions? The easiest thing to grasp is what the tangent space looks like. In bosonic string theory, fix $(J, \varphi = 0)$. The tangent space $T_{(J,0)} EL(\Sigma)$ is, well,

$$\begin{array}{cccc} \Omega^{0,0}T\Sigma & \xrightarrow{\bar\partial} \Omega^{0,1}T\Sigma & \Omega^{1,0}T^{\cdot}\Sigma & \xrightarrow{\bar\partial} \Omega^{1,1}T^{\cdot}\Sigma \\ & \Omega^0(\Sigma, \mathbb{R}^d) & \xrightarrow{\partial\bar\partial} \Omega^2(\Sigma, \mathbb{R}^d) & \\ -1 & 0 & 1 & 2 \end{array}$$

The physicists call the top parts ghosts and their antifields.

What is the quantization procedure? In the finite dimensional situation, we've taken these polyvectorfields, which have the bracket. This means that functions on the derived critical scheme have a bracket of degree 1 in the classical setting. This is also true in a homotopical sense in the infinite dimensional setting. This means that $\mathscr{F}^{cl}(U)$ should be not just a commutative algebra, but a commutative algebra with this Poisson bracket, what Jacob would like to call a P_0 algebra.

A quantization of a P_0 algebra A is a differential $\hat{d}$ on $A[\hbar]$ so that $\hat{d}^2 = 0$, $|\hat{d}| = 1$ which restricts to the ordinary differential $\hat{d} = d \mod \hbar$, and the relation that we want to move to the chain complex world, $\hat{d}(ab) - a\hat{d}x - (\hat{d}a)b = \hbar\{a, b\}$. We try to find quantizations over every subset in a coherent fashion, so a $\hat{d}$ on every U which are compatible with the factorization structures. So the map $\mathscr{F}(U_1) \otimes \cdots \mathscr{F}(U_n)[[\hbar]] \to \mathscr{F}(V)[[\hbar]]$ is a cochain map.

The theorem is that you can quantize if the obstruction groups vanish.

Theorem 6 *There is a deformation or obstruction complex Def^{cc} for deforming classical theory. If we have a quantization modulo $\hbar^n$, then there's an obstruction $O_n \in H^1(Def^{cl})$. If $O_n = 0$, then quantizations to order $n+1$ is a torsor for $H^0(Def^{cl})$. Topologists would say that we should turn this guy into a simplicial set, and it's a torsor for the simplicial Abelian group. So $H^1(Def^{cl})$ is automorphisms, and so on.*

The way I'm writing it, this is just a complex. I only understand the linearization. If you had a complete description of the moduli space of quantum field theories, you'd know the L_∞ structure here.

So how do we compute this? I left the example on the board for a purpose. To describe a moduli problem, to give an all order description, I need to know that the tangent complex is a Lie algebra. We'll have a sheaf of Lie algebras on the space on which we're doing quantum field theory. Let's discuss the case $T_(J, 0)(EL(\Sigma))$. This is a differential garded Lie algebra, in fact a sheaf of such. So what's the Lie bracket? Vector fields are a Lie algebra and act on everything else. Another piece comes from, if we deform our complex structure, we change our $\bar{\partial}$. There's a pretty evident Lie algebra structure here. So let, call this complex E. It's a vector bundle on Σ, but the bracket is not a bundle map, but a differential operator. This means I can take jets of sections, and this is a D_Σ Lie algebra. I can form a new one at a point made of germs of sections, and it always has a flat connection. When we take jets, it's perfectly fine. What's the punchline? What is the deformation complex? The deformation complex, we take cochains, Lie algebra cochains,

$$Def = \underbrace{\omega_\Sigma}_{\text{top forms}} \otimes_{D_\Sigma} \underbrace{C^e_{red}(J(E))}_{\text{Lie algebra cochains}}$$

Theorem 7 *In bosonic string theory,*

1. $H^1(Def) = \mathbb{R}^2$
2. $H^{\leq 0}(Def) = 0$
3. *Only the first obstruction O_1 can exist for a symmetry reason, and $O_1 = 0$ if and only if $d = 26$.*

This should preserve the $\mathscr{C}^*$ action that I didn't tell you about.

When you have this quantization, you get from uniqueness the flat connection, and it's the square root of the determinant of the complex, so that's determinant of one complex times the Pfaffian of the other one, and so you can calculate this.

One thing I wanted to say briefly, the key point is that the obstructions can only lie in the piece of the Lie algebra $\Omega^{0,0} \to \Omega^{0,1}$ by symmetry. So obstructions really come from Lie algebra cochains of jets of holomorphic vector fields. Locally, it's just the cohomology of $C^\cdot(\mathscr{C}[[z]]\partial_z)$, and we know that Gelfond Fuks, H^3 of this thing is $\mathbb{R}^2$. I'll stop there.