Harvard FRG

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August 26, 2009

1 D. Nadler

[This afternoon we have scheduled discussion sections where I wanted to talk about the goals of this FRG. Today there's an interesting talk about the Kervaire invariant at MIT. There's a talk at 3:00, and some of the people here are going. There's going to be a smackdown at 2:15 because Dennis forbids it. I suggest we start our discussion at 1:30. Now Jacob will introduce David.]

Thanks for coming back. I hope you found something in the first two talks to entertain yourselves. Now you know D-modules and they can be a black box. I'll talk about quantum field theory.

Let me recall, to X a smooth scheme or Artin stack we were able to associate the derived category of D-modules, so $D(D_X - mod)$. I want to say, you don't need to think about this in any particular way. You had three ways. I can happily say that these are D_X-modules if you understand that these are the derived version. One way to think about them was as a linear PDE. Smooth functions satisfying an equation give a D_X module. The second perspective was geometric: a quantization of $\mathscr{O}_{T^*X}$-modules. The last was topological, I gave a couple of ways to think about this, the regular holonomy things [unintelligible]constructible sheaves. You could have just thought about this topological category, which is the same thing as Lagrangian branes, objects in the Fukaya category of T^*X, and then there is a correspondence to S^1 equivariant sheaves on the loop space. Maybe the easiest access point is to focus on constructible sheaves and proceed from there. The solutions look regular at ∞.

Fix an algebraic variety. A constructible sheaf is something so that along each algebraic piece it's locally constant. It looks like a vector bundle with a flat connection. You can think that the Grothiendieck group is the vector space of constructible functions.

The output for today, I want to be able to write down my favorite scheme or stack and then pass to the category of D-modules on it. Today, I'd like to list the categories of D-modules that are the lifesblood of representation theory and show how they're pieces of a topological field theory.

Let me start with the first theorem of representation theory. Our first category, here's a list of categories, is

1. D-modules on the flag variety G/B. If G is SL_2 then G/B is $\mathbb{P}^1$. Now Beilinson-Bernstein tells you that there is an equivalence between the categories, well, I need more terminology, $U\mathfrak{g}$ is the enveloping algebra of the Lie algebra. Fix G as a Lie group. The enveloping algebra has a center. It has an Abelian base. For example, the center will have to act by a character. For the moment, let's fix that our representations of the center act trivially. So I'll write U_0 for $U\mathfrak{g}/Z\mathfrak{g}$, and we'll look at U_0-modules. This is not finite dimensional representations, and it's not trivial at all, it's very interesting but sort of orthogonal to that part of the theory. So Beilinson-Bernstein tells you that U_0-modules are an equivalent category to D-modules over G/B. The functors are $Hom_D(D,\ )$ from D-modules to U_0-modules. In the other direction it's localization, $D\otimes_{U_0}$. You write down the most naive things and the miracle is that this is an equivalence.

   If you're a representation theorist, well, you can now be a geometer. We've spread out representations with respect to all of the Borels. You can ask for the fiber of this representation funtor, and the fiber will be [unintelligible].

Variations

1 There are twisted versions, if you like finite dimensional versions you're unhappy now but where I've divided out by the trivial character, you can divide out by a different character, and you get a twisted version.

2 The equivariant version exists and is very nice. Suppose you fix a subgroup K of G. Consider (U_0, K)-modules. You insist that K acts as an algebraic subgroup should. The Lie algebra of K acts in the way you expect, but it's integrable to K. So for example $K = unip(B)$, another is, K is fixed points of an involution. If you study modules like this, that's equivalent to studying representations of the real group.

The Beilinson-Bernstein says this is equivalent to D-modules on the stack $K\backslash G/B$.

If K acts with finitely many orbits, then all D-modules on this stack are regular holonomy.

Example 1 *If $G = SL_2$ and $K = N$, then we have these orbits, a point and the rest. We classified D-modules whose singular support was contained in the zero section and the tangent bundle at this point. These are K-equivariant. So you already know that $(U_0\mathfrak{sl}_2, N)$-modules has five indecomposables.*

I'll tell you the five constructible sheaves: they're the constant sheaf, the constant sheaf on the point, cochains on the complement that are allowed to touch the point, and cochains that are relative to the point. The fifth are cochains on the complement which are relative to half of the point. Every other thing is a sum of these. But you know the representation theory now.

Let me make one final comment. When $K = N$ (or B), then the N-orbits are Schubert cells, and the studiy of $D(N\backslash G/B)$-modules is the beginning of Kazhdan-Lusztig theory. There are two natural bases. There's the basis you really want to understand, the irreducibles, but the easiest ones to think about are induced ones. So there are the two bases, and you want to know the transition matrix. So you want to know how to turn the irreducible representations into reduced ones (or vice versa?)

2. The second category are character sheaves. Let me try to give you a sense of what they are. Study the adjoint quotient G/G. Now we want to study D-modules on G/G. So character sheaves will be some subcategory. Let's call these character sheaves. The first and most important condition is that we will proscribe where the characters can be supported in the cotangent bundle. Recall that T^*G is $G \times \mathfrak{g}^*$. We can use the Killing form to identify this with $G \times \mathfrak{g}$. I have my favorite conical object in $\mathfrak{g}$, I have the nilpotent cone of eigenvalue zeros:$\mathcal{N} \subset \mathfrak{g}$. So these character sheaves are G-equivariant and have nilpotent singular support. It's an exercise to check that all of them are holonomic. We're dealing not with algebra but topology at this point. This is the fiber of the eigenvalue map so it's Lagrangian.

   Generically, nilpotent singular support, this will be a local system, and generically I want this to have trivial monodromy. There's a huge open stratum of regular, semisimple elements. I want this to be roughly trivial on this open piece with distinct eigenvalues.

   Let me give you an example that it's nice to keep in mind. My favorite example is the Springer D-module. We have G and the favorite space over it, $\tilde{G} = \{g, B : g \in B\}$ which maps to G/B. For SL_2, above a semisimple point you have two points, above the nilpotents you have one point, and above the cone point you have $G/B = \mathbb{P}^1$.

   So you can take $\mu_* \mathscr{O}_{\tilde{G}}$ as an element of this character sheaf Ch_G. I think I'm almost, are there questions? We so far have two categories of interest. There's a category that interpolates in a way explained by field theory, that's the Hecke category. H_G is D-modules on $B\backslash G/B$. This is a monoidal category. You can rewrite $B\backslash G/B$ as $BB \times_{BG} BB$ where B means both classifying space and Borel.

   I want to point out two structures that are important. Any time you have $D(K\backslash G/B)$, you naturally have an action of H_G. You should believe $BK \times_{BG} BB$, and of course the Hecke category will act on this.

   The second thing will be that you have a natural correspondence from G/G by the adjoint to the G/B adjoint action, which maps to $B\backslash G/B$. Lusztig correspondence constructs character sheaves.

   Let me state a theorem joint with David Ben-Zvi. Now I presume everyone knows what a two dimensional topological field theory is. There exists a categorified extended two dimensional TFT, so this means that a point has a 2-category, so the shadow of a three dimensional TFT. To a point it gives the category of H_G-modules. So H_G is a dualizable Calabi-Yau algebra in stable categories. We'd like to study smooth functions, so we categorify them. To S^1 it assigns Ch_G. I'll quit, but hopefully I've conveyed that these constructions are organized by a format you like very much, related to Jacob's work, et cetera.

   Suppose we are classical representation theorists, then we want to study smooth functions and integral operators. There is a relation with one dimensional field theory, but now we want to, well [Dennis question]

What is representation theory? It's the study of classifying spaces. Big spaces are too hard, so we'll study classifying spaces. We study maps into them, that's gauge spaces. All of the things that arose in representation theory, pretend we want to topological field theory from the start, you unwind it and find that we are doing representation theory. It's hard, it's a miracle that representation theory exists, they didn't know they were doing this.

There aren't very many functions on a classifying space. So I need to pass to a richer linearization. One should continue up the ladder. Let me quit. I'm around.