Harvard FRG

Gabriel C. Drummond-Cole

August 25, 2009

1 D. Sullivan, Stratification of String Space

[Before you leave, go see Robbie in 325 with travel receipts. After the last talk today, we have beer and wine and stuff.]

Mumble be here. I'll start right in. Well, so, the germs or seeds of two discussions I want to make today. One is the Gauss linking numbre. Take two curves, you can use them to get a map of the torus into unit vectors, the degree of that map is the linking number. In the other discussion, the picture in yesterday's lecture by Curt, somehow, these curves are forced to cross, but not for homological reasons.

These are seeds, so then, the thing I want to talk about has three Ss in it, stratification of string space. So M^d is a smooth d-manifold, not necessarily closed or compact. Then *strat* M is smooth maps of disjoint circles into M, smooth embeddings outside a finite subset of the domain. This is a definition, and the main object. There's a little more fun than just saying it this way.

So then, I'll draw without cusps. What you see here is an even valence graph, and there's a little more structure. There is an involution that tells you when you enter how to leave. A double point, why is this a stratification? One more picture. You can think, you specify one of these combinatorial data, taking all the maps that have this data, that gives you a subset of the space of maps. Having one double point has codimension $d-2$. There are two parameters and d equations. A triple point has codimension $2d-3$. Let's say $d>1$. The three cases $d=2,3,4$ have different flavors. So the double points have codimension 0 in $d=2$. I'll pass over that one. In $d=3$ or higher, the strata have codimension at least one. I'm interested in the homological nature of this stratification, what's in the boundary of what. Let me make a table: (filled in over time)

	$d = 3$, knots or links	general d, string topolo
basic material	trivalent graphs in a circle (top dimensional things in a big complex) plus gauss map.	chains on Riemann surfa with composition
symmetry deus ex machina	collapsing two or more edges, symmetry of space	(U, ω) any chain capped with T with deformation back to stratum S^1 symmetry around small
output	enough finite type invariants	$\partial X + X \star X = 0$ This leads to a qua

A quick summary, impressionistic statement, the linking invariant with the Gauss map extends to a multi-Gauss map in $d = 3$ which leads to these invariants of knots and links that have been discussed a lot. It was kind of set up by Vassiliev. Then the other output on the right, this picture of Curt's goes with the string bracket. There is also a cooperation. Goldman and Turaev showed that this was a Lie bialgebra. The essential feature I want to talk about is dealing with codimension one boundary. That's kind of a rough outline.

Why am I talking about this here? Dealing with the codimension one boundary is related to words I hear in quantum field theory discussions, but I don't want to say those words.

Let's do $d = 3$. Then, going over to *Strat* M, this is a smooth embedding generically. So each component is a knot in the three-manifold. Let's define. I'll say knot or link. Let's just take one component for definiteness. A knot invariant is a function from the set of top dimensional stata to the real numbers. So a locally constant function. I'll suppress for now triple points, tangencies, cusps. This won't effect studying this concept. I leave in double points. So I'm naturally led to think of something called a chord diagram, so I need to know which of the four branches I go to. Then there is this interesting point about three-space, that, so, if you go to one of these double points, you can lift the strand above or push it below, and these two things can be distinguished, you have left-handedness and right-handedness. So you can extend any knot invariant I to be defined on knots with double points. You take the difference between the two. You can unresolve multiple crossings, and take differences of differences. Then there's a definition. A knot invariant is called FT if this extension is eventually zero at some level.

There are two interesting remarks about this. I don't want to get indices here. Suppose you have one of these FT knot invariants that is eventually zero. Go up to the place, the first place where it's not zero. The first nonzero piece only depends on the chord diagram of the knot.

The second interesting point (this is all Vassiliev, I think), there's a four-term relation that these invariants, any sort of antiderivative would satisfy. You unfold a triple point. Suppose that this is integrated. If I move my line over to where there's one less crossing, I suppose [unintelligible]I have one antiderivative. Moving this line around the circle, I get four different knots. So the sum of these four jumps is zero, I get back to the invariant of this knot. [confusing picture] So you can look at chord diagrams, what do they have to satisfy? And this gives some answer. You want to integrate this up. He made a spectral sequence. He formulated the question and, well, there's a whole other line of this, Goodwillie calculus, Dev Sinha, I don't know what they've achieved, using purely topological methods.

The Vassiliev theorem (Kontsevich, Bott-Taubes, Bar-Natan, other younger people, a lot of names here) is that these FT invariants, let me make a reference to yesterday's first talk. These would be filtered by the level at which they vanish. The associated grade of that is a candidate, that's our functions on chord diagrams. The filtered finite type invariants fill out. There's some fine print about framing that I'm not going to say. The finite type invariants fill out the graded pieces, and one knows this as some kind of abstract Chern-Simons theory.

Now I wanted to give a part of the proof. These will be higher Gauss maps.

So there's a little bit of monkeying around with the four-term relation. You have two chords that come together, and there are two ways of unfolding them. The difference is two of the four terms in the four term relation. So these end up somewhere else. So you end up trying to unfold into a picture like this, unfold a four-valent vertex into something like this picture: [picture] This thing can be shortened. Unfolding, these three things are zero, they're both equal to these third thing. We'll introduce new variables, a coboundary, which will involve three-valent graphs, and the relation will be functions on some larger set which annihilate the coboundary, so some functions on cocycles. So let me just say, this is a trivalent graph, and we start adding things.

There's this really cool thing. We wanted to associate to a chord a unit tangent vector. So, adding a link to a chord, each time you do, you increase the number of edges by two. You give interior points weight three and points on the circle weight one, then the vertex weight equals the edge weight.

We have some sort of Gauss map into a product of spheres, and the invariant is like the degree. The problem is that the configuration space has boundary, and we have to deal with codimension one boundary.

The first kind of face is the collapse of two trivalent vertices connected by an edge, but the collapse of a triangle is more interesting. People who work with Feynman diagrams see this kind of picture. I want to remember the infinitessimal triangle. Let me try to keep this. This is a codimension one face. We create and compactify, create a lot of codimension one boundary. But, remember, adding these other characters to this story, then you're looking at functions on diagrams that respect the relations. These are the top cells of a space of diagrams. In the top degree, they're cocycles. It's supposed to annihilate the image of the coboundary, so it's a cycle in this complex made out of these top degree cells. It's a linear combination, a basic invariant here, it should be thought of as a homology closs in this space of diagrams.

The configuration space, take one of them, I can take care of one kind of face by saying I want a cycle, but the triangle doesn't get taken care of. For that, I need the Deus ex Machina step. If you have at least two edges collapsing, then you use the symmetry in space, you flip this over, let the edge go to here, [picture], and near this diagram there is a symmetric diagram, and there's a reversal of orientation. There's another codimension one face that cancels with this one. In the eventual output you get enough finite type invariants, degrees of Gauss maps applied according to these compactified chord diagrams.

There's a dictionary of Feynman diagrams with a cubic interaction, and the degree of the Gauss map is the integral of products of top classes of the two sphere, or more precisely, in $\mathbb{R}^3 \times \mathbb{R}^3 - \Delta$. You contract with radial vectors. On a three-manifold it will be a two-form concentrated near the diagonal. The boundary has a singular Dirac contribution near the diagonal and the Thom class, Poincaré dual to the diagonal and supported neard the diagonal. The physicists put a propagator near the edges. The ω is the kernel for d^{-1}, so then you can write down the action, and so you get $\frac{1}{3}A^3$ and then $\frac{1}{2}AdA$. So this is the Chern Simons action. What we learn is that to get invariants we have to try to collapse codimension one boundary as much as possible. When we looked at codimension one boundary, we could deal with [unintelligible]with this symmetry, and that's a pattern one can often see.

Now let's go to general d and string topology.

So we can apply *Strat M* to study string topology. We observe that this space here, where I've cut things out, has the same homotopy type as all maps. What I've thrown out has unbounded codimension. This space has the homotopy type of the product of free loop spaces of M. The definition, well, if we have two pairs of combinatorial data (Γ, α) and (Γ', α'), and a homeomorphism of the graphs, the number of circles in each case might not be the same. Check that any even valence graph can be covered by one circle.

Then I'll use the same two graphs of Curt's talk. Whenever you have this information of the circle and how it's collapsed, if you have two circles, you can take one circle and the theta graph, one involution is this, and the other involution is this.

The idea of string topology, the data indicates a map of circles into the manifold. There's a correspondence between points on different components. In fact, you can sort of think that this thing is a Riemann surface associated to this thing, and going this way, you take these two circles, and you can image an arbitrary point in this double loop space. You can cut the things and then reconnect them. You get a homology with zero area between two cycles, the same graph. The transition from two circles to one circle is effected through this graph. There are two putative operations, a product and a coproduct.

I have to finish words, I'll write in the output. Following nicely after Curt's talk, describing Thurston's theory, and the symplectic structure was developed by Wollpert. He was interested in Goldman's sister, so he told Goldman about it, and Goldman studied this bracket and extended it to other Lie groups, and eventually invented a Lie bracket on the vector space of free homotopy classes. A few years later Turaev invented the coLie algebra. This can be used to describe, at least for punctured surfaces, which conjugacy classes can be written as embedded curves.

You generally won't land in the right stratum, but [unintelligible]. The stratum are nice, you need to intersect with them, you need to know about the boundary, but you can use the Thom class and the deformation of the Thom class to the actual stratum. I'll just write in the rest of the table.

This compactification, well, Deligne Mumford comes up from maps Σ to a closed symplectic manifold. But if you have an open manifold with two ends, then you can only, well, surfaces run through from one end to the other, the pieces stretch out in both directions. So the one you use for this type of symplectic manifold should be this one.

2 K. Poirier, String diagrams and compactification of moduli space

This compactification is not the Deligne Mumford one and is not the one that Dennis talked about, although that's closer. So I'll define X which is homotopy equivalent to $\mathscr{M}$, and I'll take $SD \subset X$ and both of these sit naturally inside their own compactifications $\overline{SD}$ and $\bar{X}$. I'll state in part three that $\overline{SD} \subset \bar{X}$ is a homotopy equivalence. If there's time, I'll discuss the relation with compactified string topology. This discussion is sort of a precursor to the things Dennis said. Most string topology operations are organized by these open spaces here, and we'd like to extend to the compactified spaces and get more infolmation.

In part one, I'll go through in some detail, because in part two, X and $\bar{X}$ will be very similar. For the whole discussion, we'll fix g, k, ℓ, where g is at least zero and k and ℓ at least one.

Definition 1 *A pre-string diagram of type g, k, and ℓ is a graph with extra structure. So there are k input circles. On each circle there is one distinguished vertex. Part of the extra structure is, on each input circle, an angle structure. All I mean is that for any two points I can measure the angle between them. We might choose to let the distinguished vertex be 0. To continue constructing, I've got other parts of the graph. I can add corollas whose external vertices are on the input circles. At each vertex there should be a cyclic order of the half-edges incident on it. You can thicken all the edges, and then the cyclic order tells you how to attach bands to a disk at the vertices. The cyclic order has to satisfy the condition that the ribbon surface is an orientable surface of genus g with $k + \ell$ boundary components. k of these should be isotopic to the k circles we started with.*

Remark 1 *The set of pre-string diagrams of type (g, k, ℓ) form a space. Each component corresponds to a graph of the same combinatorial type, meaning the data of the underlying graph without the angles.*

What are the parameters of a component. I can add this corolla vertex anywhere on this circle, so I have one dimension. In the closure of a component I could have coinciding points. If I have multiple points on a circle, that gives a simplex, because the angles add to 2π. So the closure of a component is a product of k simplices with possible identification on the boundary.

That describes the space of pre-string diagrams of type g, k, ℓ. I'm going to give an equivalence relation and then the quotient will be $\overline{SD}$.

Let me start with a pre-string diagram and produce a metric space. I'll start by giving the construction and then finish by drawing a picture. The construction does not depend on the marked point.

The first thing I'll do is assign a length to each input circle. I'll give length one to each input circle.

Since I have the angle structure, I know the distances. All the edges of the input circle now have length. I can form the ribbon surface, and then augment it, and now I'll draw in pictures. I have a disk for each vertex and a band for each edge. I have another band coming out like this. Here's the ribbon surface associated to this graph. Here's how I'll augment this. So we have the cyclic order of the edges coming into this vertex. In between each two edges, I'll stick in an edge of half-infinite length. On the other side the same thing. Here's my augmented ribbon surface, and now I'll stick things onto it to make a metric space. So I'll start by sticking on a half infinite cylinder of radius one. I'll put rectangular strips for each edge in as well. The whole input circle has length one, and so I'll take a rectangular strip in of the appropriate length (given by the length of the input circle. I'll mark a point at a particular height, arbitrary but the construction will be independent of choice. Beneath this, I glue along one of the half-infinite edges. The two points go to these two points here. The pink and orange are very close. I also have the same thing on the other side with purple and green edges. I've produced a surface. Above it looks like a cylinder and below it looks like two cylinders. I also have the original graph I started with living on this space. This is generically the construction that takes you to a metric space and in this case it's a Riemann surface. For example, if these two points come together, what happens if the points are really close? Now the edge is really short, so the strip that I glue on is really narrow. So I get another pair of pants with one leg really narrow.

When the points coincide, the construction will give you, I want to sew in a strip with zero length. The one leg will be infinitely thin. It's a cylinder with an interval attached. Generically we get a Riemann surface but not always. It might be a singular space.

The singularities might be a cylinder with an interval attached, but you might also get nodes which are double and even higher multiplicity points.

I'll use the construction to define the equivalence relation on the space of graphs. We'll say two string diagrams are equivalent if they produce the same metric space, so if there is an isometry of their metric spaces that preserves the decomposition according to the graph, including the marked point on the input circle. You actually end up mixing combinatorial types of prestring diagrams. For example, these two pictures have the same metric spaces: [picture]

The surface that these build is genus zero with three inputs and one output. We'll take the equivalence classes as string diagrams.

Definition 2 *An equivalence class is called a string diagram. We'll let the space* $\overline{SD}$ *be the space of string diagrams. We'll let* SD *be the space of string diagrams whose constructions produce a surface. That occurs if* Γ *without its input circles is a forest.*

Remark 2 *The space $\overline{SD}$ is a cell complex of dimension $2|x|$. Inside, SD sits as a non-proper subcell complex.*

One more thing to note, if your g, k, and ℓ, if you have 0, k, and 1, then $SD = \overline{SD}$. Why is that? You started with k circles. You wanted a surface of this type, [picture]

Part two will be almost exactly the same. We'll start with

Definition 3 *A pre-string diagram with levels is a pre-string diagram with extra structure. I begin with k input circles with a distinguished point and an angle structure. My prestring diagram will be level 1. Then I'll keep going. Then I can attach corollas at the next level, saying, this one is attached at level two. Here's level three. This remembers the levels of each corolla, and there is at least one corolla at each level. I'll also remember spacing parameters between the levels less than one. If you have a spacing parameter equal to zero, I want to let the corollas attached at the corresponding levels to be in the same level, and this involves renumbering.*

I have an analagous construction to produce a metric space. We already know how to do this. I don't want to just forget the levels and apply the construction of part one. I won't go through the whole example. We marked arbitrarily but at a fixed distance. Now you use the spacing parameters to tell you how to mark the further points. So your strips have multiple points. Generically this produces a Riemann surface but we can have singularities, intervals as cylinders shrink, multiple points, and smooth surfaces of lower genus, and now you can attach intervals at both endpoints, and even corollas. It's still the same as in part one, the construction defines an equivalence relation, and:

Definition 4 *An equivalence class is a string diagram with levels. That is our space $\bar{X}$. The space X are those string diagrams who give surfaces. This gives a map $X \to \mathscr{M}$.*

Let me state a theorem.

Corollary 1 *(to theorem of Bödigheimer)*
Let $\mathscr{M}$ be the moduli space of surfaces of genus g, with $k + \ell$ punctures divided into two sets, with inputs decorated with a tangent direction, that's a choice in a circle, and outputs decorated by a positive number with sum 1. The statement of the theorem is that there exists a subspace $\mathscr{M}_1 \subset \mathscr{M}$ which is homotopy equivalent and a map $\mathscr{M}_1 \to X$ which is a homeomorphism.

Let me say something about the proof. This lets you construct a harmonic function on the Riemann surface. You construct a graph by looking at the combinatorics. The spacing parameters can be unbounded.

I should have said before, $\bar{X}$ is a finite cell complex of dimension $3|\chi| - 1$. Then $X \subset \bar{X}$ as a nonproper cell complex again.

You can take those diagrams producing surfaces and put in the faces that aren't there to pass to $\bar{X}$.

There is a lemma in one of Bödigheimer's papers that uses this argument, so

Theorem 1 *(Bödigheimer, Poirier)*
$\overline{SD} \overset{i}{\hookrightarrow} \bar{X}$ *is a homotopy equivalence.*

So Γ a stirng diagram gives a string diagram with one level. If $\Gamma \in \bar{X}$, just respace to give all spacing parameters zero. [some pictures]

Why can't you do this without the bars? I'll show you. If $\Gamma \in X$, then $r(\Gamma)$ need not be in SD. Take a single circle with two two-corollas with the same endpoints.

I'll stop here.

[Can you say something about part 4?]

I didn't say how these would act on the loop space of a manifold. I clam that $\delta\bar{X}$ has two types of components: small outputs and spacing parameter equal to 1. So you can cut it at one, and look at it as something that is viewed as a composition. So if you had a way of associating operations to this space, you can write one kind of boundary as a composition and eliminate the other type of boundary. Then you can write, in quotes,

$$\partial\varphi = \varphi * \varphi$$

3 Nadler, continuance

Thanks, we did half of part one last time. I was inspired by Curt and Dennis' lectures, Curt because of his advocacy for spontaneity, but also Dennis because I'd like to emphasize the role of loop spacs in D-modules. I'd still like to spend some more time on background material. Let me remind you where we were and then pause for questions. Last time, X was a smooth scheme or smooth Artin stack. One way to talk about D modules was analytic, but as a linear PDE. I could turn a PDE into a D-module.

To any D-module you can assign $Hom_{D_X}(M, \mathscr{F})$ and this is solutions to this PDE in $\mathscr{F}$. There was also the geometric picture, where D-modules are quantizations of their classical limits living on T^*X. I would draw X and its cotangent bundle, and I either gave you a classical limit to quantize or [unintelligible]. I could ask for a D-module whose colimit lies on a subvariety. Now I'm going to try to explain a topological perspective on D-modules. But first, any questions?

You could ask for a module over $\mathscr{O}_{T^*X}$ or D_X. The Rees algebra interpolates between these, where at $\hbar = 0$ you get one and at $\hbar = 1$ you get the other. I just wanted for you to have a picture. To D-modules I have a subvariety, well, [unintelligible]

I'll build to joint work with Ben-Zvi by first recalling a Koszul dual perspective on D-modules. I'm happy to discuss more details with anyone who wants it. We have modules supported on T^*X. Suppose for a moment that these are supported near the zero section, well, seen by the zero section. I have a restriction functor, I want ones that are seen by the restriction functor. What data tells me how to put this module back on the entire cotangent bundle. We can also put the shifted tangent bundle, so $Spec\ \Omega_X^*$. Ths doesn't make sense in a classic perspective, but what I mean is a module over this sheaf.

If i just include sheaves seen by the zero section, and include this, you can move back and forth. There is a functor from the derived category of D-modules over X to the derived category of Ω_{dR}^*X, where this now has the de Rham differential, so is nonformal.

Let me write down the functor, send M to its solutions, $Hom_{D_X}(M, \mathscr{O}_X)$. The de-Rham complex acts on this for the usual descent picture. Everything that $\mathscr{O}$ sees as a module will be a module over some X over $\mathscr{O}$.

Let me say how this works for a couple of examples. What does D_X go to? It goes to $\mathscr{O}_X$.

Once you quotient out by crazy D-modules, this becomes an equivalence. So for example on coherent D-modules it's an equivalence.

Why am I advertising this as as step toward a topological picture, but the de Rham complex is our favorite topological object.

Let's do an example first. For $X = \mathbb{A}^2$, we have $D_X \to \mathscr{O}_X$, and what gets killed here? ∂_x and ∂_y, and so the next thing is $D_X\xi_x \oplus D_X\xi_y$. Then the new kernel is $\partial_y\xi_x - \partial_x\xi_y$. So we need $D_X\eta$. So then we get $\mathscr{O}_X \to \mathscr{O}_X dx \oplus \mathscr{O}_X dy \to \mathscr{O}_X dxdy$.

[Some fusses.]

Let me remind you of the Riemann-Hilbert correspondence. A holonomic D-module, when we thought of singular support, it could be Lagrangian but no smaller. A holonomic D_X-module is one whose singular support $SiS(M)$ is Lagrangian. Something like D_X is not holonomic because its singular support is too big. We wrote one down like the exponential. I want to restrict as well to what are called regular D-module, which have simple poles, not deep ones.

Now looking at the derived category of regular holonomic D-modules, this is equivalent to complex constructible sheaves on X. Hopefully I've gotten rid of everything bad. This is the complex variety as a complex space where you also know the complex subsets.

Just to remind, this is also equivalent to the Fukaya category of the cotangent bundle. So great. What I'd like to do is forget D-modules and just study deRham modules, but I'll just study the derived category of $(\Omega_{dR,X})$-modules.

I'd like to get loop spaces into the picture. The goal here is to realize this category $D(\Omega_{dR,X}^* - mod)$.

So I want to imagine S^1 to your favorite scheme, be it $\mathbb{P}^1$ or BG or whatever. So in derived algebraic geometry you can have stacks, and study maps from a stack to something like $\mathbb{P}^n$. Let me give you a two second context. I want to study maps from S^1 to an algebraic variety. The B-module is studying functions on this. We want to have a format to do this. What is algebraic geometry about? It's the study of functors from rings to sets. An algebraic variety is a fuctor of points. This satisfies you until you try to study moduli spaces with isomorphisms. So then you study groupoids. There are maps comparing sets and groupoids. Once one likes this, you can go on to arbitrary spaces. So to sets is the theory of schemes, to groupoids of stacks. For BG this is good enough but $B(BG)$ you'd need higher stacks. If you want topological rings to spaces, that's derived stacks. These are differential graded algebras that are commutative and concentrated in negative degree.

What's the basic example of why we add such a thing to our game? The theory of derived stacks is to properly take subobjects. So take X, Y, and Z affine schemes, and I want to calculate $X \times_Z Y$, and you would get $Spec\mathscr{O}_X \otimes_{\mathscr{O}_Z} \mathscr{O}_Y$, and then you might want to derive the tensor product. Now we can take subs without losing functions on higher intersections.

Now the loop space is the mapping stack $S^1 \to X$, which can be written as the fiber product of X with itself with respect to the product of two diagonal maps.

This kind of thing is why you see Hochschild homology apparing in the B-model, because this is some kind of loop space.

Example 1 *If X is affine, $X = Spec\ \mathscr{O}_X$, Then $\mathcal{L}_X = Spec\ \Omega_X^{-*}$. Let's say everything's smooth. This looks like the de Rham algebra.*

The second example is, if we take BG, then $\mathcal{L}BG = G/G$, the quotient acting on itself by quantization.

Let me give you one more, well, maybe later I'll get to flag varieties.

Now if we wanted we could try to do string topology. Here, it may look funny to think of this as the loop space, but that's perfectly reasonable. The one and two dimensional part of the B-model is understanding this. But I want to categorify and talk about sheaves on this loop space. Let's consider quasicoherent sheaves on the loop space. So this does some string topology but misses intricacies, this has more duality than you had before. I'll focus on the S^1 action of loop rotation. Let me state some results. First proposition, G/G is very big. The first thing I want to do is restrict attention to small loops, to define $\hat{L}X$ to be the formal completion along constant loops. I just want a little formal neighborhood of constant loops. So $\hat{L}x = \hat{T}X[-1]$. For X affine these are the same, but for BG they are not.

I'll tell you what the S^1 rotation is. Globally it's interesting, locally it's just the de Rham differential.

Theorem 2
$$QCoh(\hat{\mathcal{L}}X)^{S^1} = D(\Omega_X^{-*}[d] - mod)$$

This is not yet the de Rham complex. You know when you're on the first or second date and you realize you really shouldn't be telling that story, maybe I shouldn't have told this story. If I actually want the de Rham algebar with its differential, then I need to see that this lives over S^1, and you need to look at the periodic objects.

What is the upshot? If we up to being the de Rham algebra in negative degrees, not positive degrees, well we can forget and just study S^1-equivariant sheaves on $\mathcal{L}_X$. Let me pause, and now that Dennis is awake, now there's a question?

[That's the model of the loop space?]

Yes.

[Peter: Can you modify that and put in more points? Is that equivalent]

Yes. I think I'm going to, well, I want to say one further thing.

[Is the loop space morally finite dimensional or infinite dimensional?]

Functions on it, that's the only interesting thing. To head toward representation theory, this gives interesting things, not just differential forms.

I'm just going to say a final variation and quit for the day. Next time I'll talk about topological field theory. I want to say, for experts, how to see twisted D-modules from this picture. I haven't told you what twisted D-modules are. We can take sections of any line bundle. So these are modules over rings that locally look like D. They're roughly parameterized by line bundles. How do I see twisted D-modules from this setup? We've been studying S^1-equivariant sheaves, but we could consider the free group on S^1, which I think some would call the loop space of the suspension, with kernel ΩS^3. This is a rotation of the Hopf fibration. I rotated the diagram. Now we can study $Free(S^1)$-equivariant sheaves on $\mathcal{L}_X$, which should be the same as quasicoherent sheaves on X with connection. Now you can ask for the connection to be given with curvature, which will have to do with how you treat ΩS^3.

If one wants to study twisted D-modules, you're studying something to do with the Hopf fibration. Thanks for your patience.

[So, can I think of this, this connection of loop spaces and D-modules. A point in the loop space is point and an automorphism. For a scheme, in derived algebraic geometry, I think of an automoryhism as a vector field. So then the S^1 action, that's to give a point and an automorphism.]

Yeah, so, I think of the S^1 action as being a vector field. The de Rham differential is the difference, it's, if all the objects are locally constant, then all you see is the monodromy, which is de Rham d.

[So the S^1-equivariant sheaves are the modules.]

[unintelligible]

[Can you say a little more about the loops on S^3?]

So to give an equivariant object, I have to give you a function on the loop space, there is a generator u in degree -2, so I need a function of degree -2 with values in the automorphism. So roughly it will be a second cohomology class. It can be in H^2 of 0-form,s H^1 of 1, or H^0 of 2-forms. You could ask for a $1,1$ form, you can't have [unintelligible].

4 Nathaniel Rounds

I believe the title of the FRG connects the algebraic topology of closed manifolds to quantum field theory and string theory. I don't know how this relates to quantum field theory and string theory, but I hear those people say locality a lot. My motivating questions is, what algebraic structure is there on chains and cochains of a manifold. I'll start out by talking about things we know, and that's going to lead me to want to develop an idea of based chain complexes, and I'll use them to define a notion of locality, and then use that idea to state a theorem about manifolds. That's my plan. Let me say, in case I forget later, that when I say space I mean simply connected space, homotopy type of a finite complex. Whenever I say manifold, I mean closed oriented connected topological manifold.

Let me start by reminding you of some of the things we know.

Theorem 3 *(Quillen, Sullivan)*
The C_∞ coalgebra structure on rational chains of a space determine the space up to rational homotopy equivalence.

If you want to know a space up to rational homotopy groups, it suffices to give this structure on rational chains.

Theorem 4 *(Mandell)*
The E_∞ structure on cochains of a space determines the space up to homotopy.

You take the cup product and resolve those operations on cochains. If you just want to study homotopy types, this is enough. What if you have a manifold, is there anything else to say?

Definition 5 *A Poincaré duality space X is a space with a cycle $[X] \in C_n(X)$ such that $[X] \cap \bullet : C^{n-*}(X) \to C_*(X)$ is a quasiisomorphism of cochain modules.*

As most of you probably know, manifolds are Poincaré duality spaces. Here is a new property not contained in chains and cochains alone. One knows from surgery theory that not every PD-space has the homotopy type of a manifold. It might be nice to know, is there some way to enrich or add to this structure so that you know you have a manifold and not just a Poincaré duality space?

So the question is, can the duality map from cochains to chains of a manifold be enriched detect more than homotopy type.

So first I want to tell you some things that you might take as negative evidence. Here are two results I heard in Oberwolfach from David Chataur, which give negative answers. I'll use a few words, but ask me if you don't like these statements:

1. The E_∞ algebra structure on PL-chains of a PL-manifold (constructed by Jim McClure and Scott Wilson) is equivalent to the E_∞ algebra structure on singular cochains.
2. If X is a PD-space then the PD-map extends uniquely to an equivalence of E_∞ cochain modules.

I hear that Mike and Jacob's work implies something about the Frobenius structure, [unintelligible], but I can't back it up. Let me mention one more theorem:

Theorem 5 *(McCrory) If X is a PD space which is a pseudomanifold, then X is a homology manifold if and only if the inverse of the duality map from cochains to chains is represented by a cocycle $U : C_* \otimes C_* \to \mathbb{Z}$ whose class is in the image of $H^*(X \times X, X \times \backslash\Delta) \to H^*(X \times X)$.*

Being a homology manifold is not a homotopy invariant. A homotopy can destroy it. This is property that is not a homotopy invariant that you can hope to include in algebra.

So now I want to develop some algebra for this. Hopefully this suggests that there's some juice in local Poincaré duality. I will describe what a based chain complex is pedantically so that I'm clear:

Definition 6 *A based chain complex (or bcc, because I'm lazy) (C, B, ∂) is an $\mathbb{N}$-graded finite set $B = \amalg B_k$, a chain complex C_k which is a free module on B_k, and a differential of degree -1.*

So, for example, take cells in a cellular complex.

Remark 3 *B has a partial order where $b \le b$ and $b \le y$ if b appears in the formula for the boundary of y, and extend this transitively. I will make a definition, let $\Delta(b)$ be the minimal subcomplex containing all $x \le b$. If I had a based chain complex with simplices.*

Here's something that you can do with such objects: given (C, B, ∂), there is a new complex P, which is the pair complex, which will be the protagonist of our story. I'll start with a picture. Imagine the chain complex was just a simplicial complex. The new complex will have as a basis all pairs in the old thing. It has basis

$$B_k = \{(y, x), y, x \in By \ge x, ||y| - |x| = k\}$$

where $d_p(y, x) = (\partial y, x) \pm (y, \delta x)$ with illegal pairs are zero. Here the coboundary δ, says that $\langle \delta x, z\rangle = \langle x, \partial z\rangle$. So the pairs should give you a one dimensional generator for each codimension one face. For each generator x, you get a one-dimensional thing, and two dimensional things, for this picture, as vertices inside top cells.

Let me say that P is a differential graded coalgebra by

$$(y,x) \mapsto \sum_y \geq z \geq x(y,z) \otimes (z,x)$$

. This is literally the diagonal.

This isn't very strong or useful, so I'll need axioms.

Definition 7 *A regular based chain complex satisfies two axioms:*

1. *for each $b \in B$ the closure should be contractible, by the map $\Delta(b) \to \mathbb{Z}$ via sending zero-degree basis elements to 1 and every other basis element to 0, and extending linearly. This should be a quasiisomorphism.*
2. *$C \to P$ via $b \mapsto \sum_{|v|=0}(b,v)$ is a chain map by the first condition, and this should be a quasiisomorphism as well.*

Example 2 *Something that is not regular is a degree one basis element b with boundary the sum of three degree zero elements. In the boundary of a regular cell complex, the boundary of a degree one basis element must be the sum of two degree zero basis elements, one with coefficient one and one with coefficient negative one.*

There's a cone construction, given a regular based chain complexes to spaces, which I'll build inductively. X_0 will be a disjoint union of points, one for each $b \in B_0$. Then I will start building new strata by attaching a cone on everything less than it. In X_{k+1}, atach a cone to ∂b for each $b \in B_{k+1}$. This won't quite be a cell complex. What is true is you have a homology sphere, although it may not be a PL-sphere. Chains on the space are equivalent to chains on your original based chain complex.

All right, so maybe I won't say anything else about that. Now I want to define a category of B-local objects. So part three is the B-local category. Fix a regular based chain complex (C, B, ∂). I'll tell you a category of things spread out overe the basis.

Definition 8 *A B-local structure on the module D ($\mathbb{Z}$-module) is a decomposition $D = \sum_B D(b)$. This is a direct sum of pieces, one over each b. A B-local map $f : D \to D'$ satisfies*

$$f(D(b)) \subset \sum_{y \geq b} D'(y)$$

A B-local chain complex is a chain complex in this category.

Let me tell you the motivating example, it's the pair subdivision. P is B-local. So $P(b)$ is generated by pairs (y,b), and D is certainly the direct sum of these. You're just filtering by the smaller element of the pair. [Picture]. It's the dual cone on your basis element. If you only remember the degree, that gives you a filtration, and that's the Ziemann[sic] spectral sequence.

Remark 4 *If D is B-local, then $D^{-*} = Hom(D, Z)$ need not be. Now since we don't have an ordinary duality functor.*

Definition 9 *So instead, T is a functor from B-local complexes to one another. If you do this and forget, you'll get something chain equivalent to the Hom dual, and you get something equivalent to the identity by doing this twice.*

$$TD(b) = \sum_{y \geq b} Hom(D(y), \mathbb{Z})$$

(shifted down by the dimension of b)

So take a bunch of Hom duals. This has a differential which imprecisely is the dual of the old differential $T(\partial_0) +$ "δ''

What is TP? It's cochains on the triples, meaning that it is a chain complex local over B with a generator for each triple.

Proposition 1 *Let $\Sigma^n TP$ denote the n-fold suspension of TP, which is a B-local chain complex, and let $\Sigma^n P^{-*}$ denote the Hom dual. Given a cycle $\mu \in C_n$, the original based chain complex, there is a commutative diagram of maps*

$$\begin{tikzcd}
\Sigma^n TP \arrow[r, "\varphi_\mu"] & P \\
\Sigma^n P^{-*} \arrow[u, "\epsilon"] \arrow[ur, "s(\mu)\cap"'] &
\end{tikzcd}$$

Here ϵ is the subdivision quasiisomorphism, and $s(\mu)\cap$ means subdivide and then $\cap$. So $\cap$ is now B-local.

Let me make a definition.

Definition 10 *A Poincaré duality regular based chain complex is (C, B, ∂, μ) where (C, B, ∂) is a regular based chain complex so that φ_μ is a quasi-isomorphism.*

This has some inverse. Will that map be B-local? Not necessarily, as you can deduce from McCrory's theorem, take something that is not a homology manifold. Let me state a theorem, but unfortunately, before that I need one more definition.

Definition 11 *A PD-regular based chain complex of dimension n is pseudomanifoldish if*

1. *for each $b \in B_{n-1}$, $\langle dy, b \rangle \neq 0$ for exactly two y.*
2. *for all b, $C \to P(b)$ sending $\mu \to (\mu, b)$ induces an isomorphism on $H_n(C) \to H_n(P(b))$.*

Theorem 6 *(Ranicki, Rounds)*

1. *closed topological manifolds in the homotopy type of the space associated via cone construction to a regular PD pseudomanifoldish based chain complex (C, B, ∂, μ), of dimension $n \geq 5$ are in one to one correspondence with B-local cobordisms from (P, μ) to (P', μ'), where μ' has a B-local inverse.*
2. *B local cobordism class of (P', μ') is topologically invariant.*

Definition 12 *A B-local cobordism between (P, μ) and (P', μ') is $P \oplus P' \to D$ and $\theta : \Sigma^{n+1}TD \to D$ where this data is B-local, such that $d_{Hom}\theta = \mu - \mu'$, and $\Sigma^{n+1}TD \to cone(f)$ given by $\theta \oplus (\mu - \mu')$ is a quasiisomorphism.*