Postnikov Conference

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1 Soren Galatius Homotopy type of the cobordism category

So I'll talk about a paper which is joint with Ib Madsen, Ulrike Tillman, and Michael Weiss. It's on the arxiv as 0605429.

Let me first remind you of classical Pontrjagin Thom, that's a result that interprets smooth closed $d-1$-manifolds up to cobordism as homotopy groups of a spectrum MO or if you would like spaces better, it's $\pi_{d-1}\Omega^\infty M$ or $\pi_0\Omega^{\infty+d-1}MO$.

Let me remind you what is MO and what is the map. This is $\pi_{d-1+n}MO(n)$ where $MO(n)$ is the Thom space of the canonical bundle over $BO(n)$.

Embed M^{d-1} in $\mathbb{R}^{d-1+n}$ and it has a tubular neighborhood with a normal bundle NM which induces a map to the Grassmanian of n-planes in $\mathbb{R}^{d-1+n}$ with the normal bundle mapping to the canonical bundle U over it. Then the Pontrjagin Thom construction gives a map from $\mathbb{S}^{d-1+n}$ into the Thom space of the normal bundle into the Thom space of U, and then this can be embedded in the canonical bundle over n-planes in $\mathbb{R}^\infty$.

That's the classical construction and the main theorem of that paper is in some sense a refinement of this. We prove a homotopy equivalence where the stated isomorphism is π_0 of the equivalence.

To any category $\mathscr{C}$ there is the classifying space $B\mathscr{C}$ defined as the geometric realization of the nerve $|N\mathscr{C}|$. You start with a point for each object, an interval for each morphism, a 2-cell for each decomposable morphism, et cetera. That builds a topological space, and π_0 is the set of objects modulo equivalence generated by morphisms.

As we learned in Peter and Stephan's talks, we can think of cobordisms as morphisms in a category so if I let $\mathscr{C}$ be $\mathscr{C}_d$, where the objects are closed $d-1$ manifolds and the morphisms are cobordisms then this general result about π_0 of the classifying space says that π_0 of the classifying space is exactly the set of smooth closed M^{d-1} up to cobordsim. So "classical" Pontrjagin Thom is a calculation of $\pi_0 B\mathscr{C}_d$, and the main result of the paper is the calculation of the homotopy type of this category.

Definition 1 *Consider the Grassmanian of d-planes in $\mathbb{R}^n$. Over that I have the canonical bundle and the complement $U_{d,n^\perp}$ with fiber dimension $n-d$. Take the Thom space of that and then the n-fold loop space, and the direct limit*

$$co \lim_{n\to\infty} \Omega^n Th(U_{d,n^\perp})$$

I call $\Omega^\infty MTO(d)$. Then $\Omega^{\infty-1}MTO(d)$ is the colimit of $\Omega^{n-1}Th(U_{d,n^\perp})$.

Theorem 1 *Galatius, Madsen, Tillman, Weiss*
$B\mathscr{C} \cong \Omega^{\infty-1}MTO(d)$

There are two problems with the category, it's not small and it's not a category. Campositions are gluing and are not strictly associative. There are various ways of getting around that. There are no canonical ways of getting around it, although there are canonical ways up to homotopy. Here's one way. $\mathscr{C}_d$ has as objects pairs (a, M) where a is a real number and M is a smooth closed $d-1$-dimensional submanifold of $\mathbb{R}^\infty$. As a set it is a subset of $\mathbb{R} \times \mathscr{P}(\mathbb{R}^\infty)$.

The morphisms $mor((a_0, M_0), (a_1, M_1))$ are cobordisms $W^d \subset [a_0, a_1] \times \mathbb{R}^\infty$.

The condition, of course is that the boundary of W is $\{a_0\} \times M_0 \cup \{a_1\} \times M_1$. To have a well-defined composition I want W to be a "product near the boundary."

If $a_0 \geq a_1$ then you have the identity only if $a_0 = a_1, M_0 = M_1$, otherwise no morphisms. Now composition is union of subsets, which is associative, so now it's a category. I want to consider it as a topological category, so objects and morphisms are topological spaces. Each object determines a real number and a subspace of $\mathbb{R}^\infty$. I can think of that as an embedding of M in $\mathbb{R}^\infty$ and mod out by $Diff(M)$. Then I take as objects

$$\mathbb{R} \times \amalg_{[M]} Emb(M, \mathbb{R}^\infty)/Diff(M)$$

.

Topologize using the C^∞ topology.

Let me discuss the relation to classical Pontrjagin Thom. I can include $U_{d,n^\perp}$ into the direct sum $U_{d,n^\perp} \oplus U_{d,n} = Gr_d(\mathbb{R}^n) \times \mathbb{R}^n$. Letting $n \to \infty$ I get

$$MTO(d) \to \Sigma^\infty BO(d)_+ \to MTO(d-1)$$

If you write that in a different way, you can think of it as a filtration of MO,

$$MTO(0) \to \Sigma MTO(1) \to \cdots \to \Sigma^d MTO(d)$$

It is not hard to see that the direct limit is MO. The filtration quationts are these suspension spectrums. The cofiber is $\Sigma^d BO(d)_+$. It gets higher and higher connected, so that π_{d-1} of $\Sigma^d MTO(d)$ is isomorphic to $\pi_{d-1}(MO)$, and this is precisely $\pi_0 \Omega^{\infty-1} MTO(d)$.

The $BO(d)$ is a virtual bundle and $MO(d)$ is the Thom spectrum of minus the bundle.

Let me also say about the notation, this notation was suggested by Hopkins. He explained it by, the space $O(d)$, it's the structure group of tangent bundles. You could do the same thing, if you had orientations on everything, you could take cobordism classes of oriented manifolds, or you could take spin, or et cetera. The most general statement is for any fibration over $BO(d)$.

An example of what I mean by a version, you could have as a tangential structure a map to a fixed space. Let $\mathscr{C}_d(X)$ be the category as before, but where every object is equipped with a map to X, as are the morphisms. Then the theorem says

$$B\mathscr{C}_d(X) \cong \Omega^{\infty-1}(X_+ \wedge MTO(d))$$

The right hand side is local, that is, homotopy groups satisfy excision and thus are a homology theory, where the left hand side does not look local.

So let me also relate this to, this category is essentially,

$$C_d(X) \sim RB_d(X)$$

In their theory it's kind of, a lot of that is about making stuff local in X, as far as I understand, so maybe it's surprising that taking the classifying space gives you something local.

So it's not very hard to prove this theorem so I'm spending a lot of time doing other stuff. Let me give some motivation. The original motivation was from a theorem of Madsen and Weiss in 2002,

Theorem 2

$$\mathbb{Z} \times B\Gamma_\infty^+ \cong \Omega^\infty MTSO(2)$$

Here Γ is the mapping class group. So $\Gamma_{g,1}$ is the mapping class group, in other words, it's $\pi_0 Diff_+(\Sigma_g \text{with one boundary component}, \partial)$. Here Γ_∞ is the direct limit where you just stick on a torus. B means classifying space, + means the Quillen plus construction, that this group has the same group homology as the space on the right. This states it without talking about the Quillen plus. This calculates the homotopy type of Γ_∞, which implies the Mumford conjecture, which can be stated

$$H^*(\Gamma_\infty, \mathbb{Q}) = \mathbb{Q}[\mathscr{K}_1, \mathscr{K}_2, \ldots]$$

So an equivalent statement is a map

$$\mathbb{Z} \times B\Gamma_\infty \to \Omega^\infty MTSO(2)$$

which induces an isomorphism on homology.

Before that Tillmann, in 1997, I think, proved that

$$\mathbb{Z} \times B\Gamma_\infty \to \Omega B\mathscr{C}_2^{or}$$

induces an isomorphism on homology.

If you combine those two statements. If I say this with the Quillen plus, I have three homotopy equivalent spaces $\mathbb{Z} \times B\Gamma_\infty^+$, $\Omega B\mathscr{C}_2^{or}$, and $\Omega^\infty MTSO(2)$. This theorem is of the direct homotopy equivalence of the second and third.

You use fewer things to prove this, the statement is true in any dimension, whereas it's not clear what the first space would be like in other dimensions. Both Madsen Weiss and Tillmann uses Harer stability, which you don't have in any other dimension.

The Harer stability theorem is just that $H_k(\Gamma_{g,1})$ is independent of g in a certain range, that is for $g > 2k + 1$.

That's all I could say without talking about the proof.

The proof uses an intermediate space

$$B\mathscr{C}_d \cong D_d \cong \Omega^{\infty-1} MTO(d)$$

so it uses two steps.

Now D_d is the space of infinitely long cobordisms, a manifold equipped with a proper map to the real numbers, a d-manifold inside $\mathbb{R} \times \mathbb{R}^\infty$ so that the projection f to $\mathbb{R}$ is proper. I don't really want to, you want to move this W around in a way that a map into it $X^k \to D_d$ corresponds to a family of these things. So such a map, a family would be a $k+d$-dimensional submanifold of $X \times \mathbb{R} \times \mathbb{R}^\infty$ such that the projection to X is a submersion and the combined map to $X \times \mathbb{R}$ is proper. Maybe I'm going into a little bit too much detail.

It's similar in Peter's and Stephan's talk, instead of defining a topology I define what a family is. Then that's what D_d is. Then you can make a picture of the first homotopy equivalence.

You have your proper map f to $\mathbb{R}$, you can choose a regular value, the inverse image will give you an object in $\mathscr{C}_d$, $(a, f^{-1}(a))$. You get a morphism for a different choice between the two. The choice $(a_1, f^{-1}(a_1))$ has a morphism $f^{-1}([a_0, a_1])$ from the first object to the second. That's what you need to prove that homotopy equivalence.

I have two minutes left. Let me just say the second part. This is also, there are various ways to do it. First use classical Pontrjagin Thom theory to say what is a map into the $\Omega^{\infty-1}MTO(d)$, and then use Phillips' h-principle, and combining those two immediately gives that those two are homotopy equivalent.

I'll stop here.

2 Terilla

So I just would like to thank the organizers for the invitation and for bringing us to this nice institute. Okay, so what I'm going to state and prove a theorem and maybe I'll make one preliminary remark. If you have a differential graded, a differential BV algebra, I'm going to define this in a few minutes, you can associate to it several derived constructions, in particular two different differential graded Lie algebras, L and $L_\hbar$. The first of these is smooth and formal which means from a homotopy point of view it is uninteresting. The theorem I will state will give necessary and sufficient conditions for $L_\hbar$ to be formal. One consequence will be the structure of a weak Frobenius manifold. The point is that $L_\hbar$ is not always smooth formal. In the case that it is, you have an interaction with $\hbar$ which allows you to, the first one is uninteresting as a differential graded Lie algebra. The second one, if it is formal, then there is an interesting construction.

Definition 2 *Let $L = (V, Q, [\,,\,])$ be an odd Lie algebra. L is smooth formal if and only if there exists $\Gamma \in V \otimes SH^*$ where $H = H(V, Q)$ where*

$$\Gamma = \sum t_i x_i + t_i t_j x_{ij} + \dots$$

with $x_{i_1,\dots,i_n} \in V$ Where $\{[x_i]\}$ form a basis for H and $Q\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0$.

This equation is broken up into pieces. This implies $Q(x_i) = 0$, $Q(x_{ij}) = -\frac{1}{2}[x_i, x_j]$, so that the brackets have to vanish in homology. The t_i should be a dual basis of x_i.

If L is smooth formal, then this is the same as saying the L_∞ minimal model of L is zero for its bracket and higher brackets. Previously this was called degenerate.

This can be seen by reinterpreting this. If I view $\Gamma \in V \otimes SH^*$, this is a sequence of maps $SH \to V$. These are data $\Gamma_i : S^i H \to V$. You could say it $[x_{i_1}] \wedge \dots \wedge [x_{i_r}] \mapsto x_{i_1,\dots,i_r}$. This is Γ a map from $(H, 0) \to V$ with the L_∞ structure determined by Q and the bracket. You can write down the Γ_2 part.

It occured to me this week that a space, if you take its cohomology, you have a unit, so your product in cohomology is never zero, so this wouldn't be as interesting. That's why I'm adding the word smooth. I can justify this word in one other way. In one of the plenary talks they wanted a smooth functor, and you can create a functor of rings, which is smooth if and only if this is a smooth functor.

What about a BV algebra. This is a definition. A BV algebra is a triple $(V, \Delta, \cdot)$ where $\cdot$ is an associative unital graded commutative algebra. $\Delta^2 = 0$,$\Delta(1) = 0$, and

$$d_v(w) = \Delta(vw) - \Delta(v)w - (-1)^{|v|} v\Delta(w)$$

is a derivation of degree $|v| + 1$. The data is similar to a differential graded Lie algebra. It's not quite a derivation, but its deviation from being a derivation is a derivation. Then $[v, w] = d_v(w)$ is a bracket and $(V, \Delta, [\,,\,])$ is an odd differential graded Lie algebra.

Theorem 3 *It's not very interesting because $(V, \Delta, [\,,\,])$ is smooth formal*

You can start to prove it by construction. Choose x_i. The second condition says I have to find some primitive for $[x_i, x_j]$. I can select $x_{ij} = -\frac{1}{2}(x_i x_j)$. Then $\Delta(x_i x_j)$ is exactly the bracket since x_i and x_j are closed.

You can solve this term by term, noticing that all cohomological obstructions vanish. It's worth talking about this, Ralph Cohen was surprised by it. Let me give you a conceptually nicer way of seeing that this theorem is true.

Lemma 1 $\Delta(x) + \frac{1}{2}[x, x] = 0$ *if and only if* $\Delta(e^x) = 0$ *for* $x \in V^0$.

For a proof, note that

$$\Delta(x^n) = nx^{n-1}\Delta(x) + \binom{n}{2}x^{n-2}[x, x]$$

Then

$$\Delta(e^x) = \Delta(\sum \frac{x^n}{n!}) = \sum \Delta(x)\frac{x^{n-1}}{(n-1)!} + \frac{1}{2}[x, x]\sum \frac{x^{n-2}}{(n-2)!} = (\Delta(x) + \frac{1}{2}[x, x])e^x$$

To prove the theorem, note that $\Gamma = \log(1 + t_i x_i) = t_i x_i - \frac{1}{2}t_i t_j x_i x_j + \ldots$ Then

$$\Delta(e^\Gamma) = \Delta(e^{\log(1+t_i x_i)}) = \Delta(1 + t_i x_i) = 0$$

One would like to separate the first and second order parts of the BV operator. That leads you to

Definition 3 $(V, Q, \Delta, \cdot)$ *so that* $(V, Q, \cdot)$ *is a commutative unital differential graded associative algebra and that* $(V, \Delta, \cdot)$ *is a BV algebra and* $(V, Q + \Delta, \cdot)$ *is a BV algebra.*

From this information you can define the bracket as before. It doesn't matter whether you use Δ or $Q + \Delta$. You now have two differential graded Lie algebras, I'm going to define two. You could consider $(V, Q, [\, ,\,])$, the classical case, and the quantum, $(V[[\hbar]], Q + \hbar\Delta, [\, ,\,])$.

Definition 4 $L_\hbar$ *is smooth formal if and only if there exists a* Γ *in* $V[[\hbar]] \otimes SH^*$ *where* $H = H(V, Q)$ *and* $\Gamma = \sum t_i x_i + t_i t_j x_{ij} + \ldots$*, with the* $x_{i_1,\ldots,i_r}$ *in* $V[[\hbar]]$*. Assume* $\{[x_i^0]\}$*, the classes of the terms in* x_i *that are constant in* $\hbar$*. It's* $(Q + \hbar\Delta)\Gamma + [\Gamma, \Gamma] = 0$

Let me try to sketch a picture. I can look at $\hbar$ and t pieces. I can write Γ as living in the lattice specified by these. In order for Γ to exist, the $t = 0$ condition means that $Q + \hbar\Delta$ applied to the part linear in t, is zero, so $Q(x_i^0) = 0$, but in the $\hbar^1$ term, you get that $\Delta(x_i^0) = Q(x_i^1)$. The surprising thing which is the content of the main theorem. If you can extend the linear in t part just in the $\hbar$ direction, then all the obstructions vanish.

Let me state this a little more coherently in a second. You have x_i as before being $x_i^0 + \hbar x_i^1 + \ldots$. You have $Kx_i = 0$ for $K = Q + \hbar\Delta$ You have (V, Q) and $(V[[\hbar]], K)$ and you have a chain map

$$(V, Q) \xleftarrow{\alpha} (V[[\hbar]], K)$$

$\alpha(y) = y_0$. If you have all of these conditions, you can extend something Q-closed to being K-closed, then you can create a section β of α

$$(V,Q) \underset{\beta}{\overset{\alpha}{\leftrightarrows}} (V[[\hbar]],K)$$

with $\alpha\beta = id_V$

Theorem 4 *$L_\hbar$ is smooth formal if and only if there exists $\beta : (V,Q) \to (V[[\hbar]],K)$ such that $\alpha\beta = id_V$*

Rather than proving this, let me make some closing remarks. The next page in my notes is to compare and contrast this with the version of the d-$\bar{d}$ lemma that applies here. Kontsevich-[unintelligible]proved that if you have two differentials that commute and behave in a certain way, then you get $L_\hbar$ to be smooth formal. The $Q - \Delta$ lemma implies that there exists this map β which can be taken as the identity. You can choose elements that are Δ-closed, and you don't even need to choose $\hbar$ terms. The second thing is that if you have any Γ which is a solution, you can push the solutiononto just the homology of H and you get a Frobenius structure (with no metric) on the homology, with $\hbar$s. If you follow the theorem, if you use β, in the proof, going to the universal solution, you can push Γ to the homology of the manifold with no $\hbar$s in the Frobenius manifold structure. This is what physicists call special coordinates.

I'm saying the proof of this theorem reveals a corollary. There is a Frobenius manifold structure on the homology of (V,Q) which can be constructed with Γ or with Γ_β. The one constructed with Γ_β has no $\hbar$s in its homology. The construction of Manin in his book on Frobenius manifolds says that if you have $Q - \Delta$ you can do this but you don't need that information.

3 Zenalian

So let me start, I am going to do an algebraic construction on certain objects. Let me set up which objects I am going to talk about. I need a smooth oriented compact manifold M and over this I need o bundle of $\mathbb{Z}_2$-graded algebras, so I can take sections of this bundle $\mathscr{A} = \Gamma(A)$, and over that I consider an odd operator Q which is a derivation, a differential, first order differential operator, and elliptic.

That means that the associated complex of symbols is exact. I'll tell you why I need this property and you can take that as what you need for this definition. So on top of this I need a trace map $\mathscr{A} \to \mathbb{C}$ with the two properties $tr(ab) = tr(ba)$ and $tr(Qa) = 0$. I need a map $*$ which squares to $\pm Id$, and I want $\langle a,b\rangle = tr(a, *b)$ is Hermitian.

The first part is an elliptic space, the trace makes it Calabi Yau, and then the last part makes it Hermitian. I don't know any examples where if you have the first two, you don't have the third one, but I don't think it logically follows.

For example, take $A = \wedge T^*M$ and then $A = \Omega M$ and $Q = d$. Let $tr(a) = \int_M a$; let M be Riemannian and then $*$ is the Hodge star.

Another example that may be more interesting is to start with a compact complex manifold with a Hermitian metric, and using the Riemmanian metric that comes from that, it's $\mathbb{C}$-linear. When you look at the top exterior algebra of the holomorphic cotangent, you have a section of it, nowhere zero, also called a holomorphic volume element. This is Calabi-Yau. Let the A to be the exterior algebra $\wedge$, furthermore, let E be a holomorphic bundle over M. Then I can take $A = \wedge \bar{T} \otimes E$ with $Q = \bar{\partial}$ with $\mathscr{A} = \Omega(M, EndE)$ The trace is $tra = \int_M a \wedge vol$. This a contains all the $d\bar{z}$s. The volume forms have the dz part.

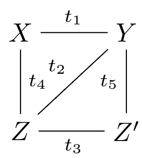
The star operator is the Riemannian star but then when you get the result, you divide out the dz parts. I have these two examples. What can you do with this definition? One nice thing is that you can look at $Q^\dagger$, and take $H = [Q^\dagger, Q]$. This is an elliptic second order self-adjoint positive operator. These objects have kernels for their heat operator. So what I really need, this is where the conditions come in. I want to give meaning to the operator e^{-tH}. Those are sufficient to give $e^{-tH}\alpha(x) = \int_M X(x,y,t)\alpha(y)dy$.

Here $K \in \mathscr{A} \otimes \mathscr{A} \otimes C^\infty(\mathbb{R}^+)$.

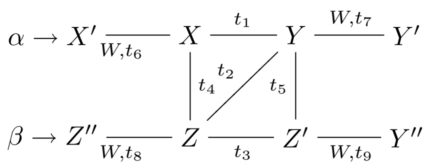
So to get a little bit of feel for what this is, you have differential forms, this is the tensor product of two copies of differential forms, so forms on $M \times M \times \mathbb{R}^+$. For fixed t the diagonal is a class and this form is the dual to it. Two different choices would give you cohomologous objects. So $K(x,y,t_1) - K(x,y,t_2) = Q_x + Q_y(\int_{t_1}^{t_2} Q_x^\dagger K(x,y,t)dt$ I call L by $Q_x^t X(x,y,t)$. I am using this first example; they are both forms on $M \times M \times \mathbb{R}$. If I add Ldt , well $W = K + Ldt$ really lives, if you want it in the general case it lives in $\mathscr{A} \otimes \mathscr{A} \otimes \Omega\mathbb{R}^+$.

Now, why am I discussing this? This comes out of work of Kevin Costello, and what is to follow is joint work with Kevin and Thomas Tradler.

What can you do with this? Given a Calabi-Yau elliptic space, you get a differential form in the moduli space of Riemann surfaces. I'm going to use the moduli space of metric graphs. You can take this ribbon graph [picture] and put some lengths



Then start for each edge with, say, $W(x,y,t_1), W(y,z',t_5), \cdots$ and for each vertex take tr_x. You need to bring them in the right order with the cyclic ordering. The well-definedness comes from the commutative property of the trace.



The graph might have edges that are loose, and you can do something even better. You can declare that some of these are inputs and others as outputs. You can put elements of $\mathscr{A}$ into this. Then the variables at a vertex get absorbed with trace and the elements of $\mathscr{A}$. At the other end, the W has two variables, and you get an output, you get a map $A^{\otimes 2} \to \Omega(\mathscr{M}) \otimes A^{\otimes 2}$. The semigroup property of a self-adjoint operator proves that these things glue together correctly.

Let me be precise about what it means to put a form on the moduli space of graphs. I can get a graph of a different combinatorial type by collapsing things. There is some nonsmoothness that is happening. You have to be a little careful about what type of differential forms I'm talking about. These are examples of stratified spaces. I am going to write, X is a stratified space if it is a union of X_α, a disjoint union such that each X_α is a smooth manifold. There are weaker notions, but this will be quite sufficient. You want the closure of X_α in X to be a smooth manifold with corners. You want $\bar{X}_\alpha - X_\alpha$ to be $\sqcup X_\beta$ and you want $\bar{X}_\beta \subset \bar{X}_\alpha$ to be a submanifold with corners.

What do you mean by a differential form on X? If ω is a k-form on X, then for every α you have $\omega_\alpha \in \Omega(\bar{X}_\alpha)$. If $\bar{X}_\beta \subset \bar{X}_\alpha$ then $\omega_\alpha|_{\bar{X}_\beta} = \omega_\beta$.

So now I'll define $\Omega_k X = \bigoplus \Omega^{\dim X_\alpha - k}_{\text{compact support}}$

So a 1-cycle would be a 1-cycle on the 2-strata and a 0 form on a 1-strata.

Now $\partial\sigma = d\sigma + \sum \sigma|_{X_\beta}$ where β ranges over those such that $\bar{X}_\beta \subset \bar{X}_\alpha$ with codimension one.

Let me give you two examples of stratified spaces. One is the moduli space of metrized ribbon graphs. Of this I want to take differential forms as well as de Rham chains. Another example is Δ^n, the usual simplex. I want to consider chains on that. I'm going to work with these. As it turns out, and I'm going to talk about this, I'll consider two types of graphs. The types I considered before I call $\Gamma(k, \ell)$ and the closed ones I'll call $G(k, \ell)$. In the $G(k, \ell)$ the inputs and outputs are obtained by thickening the graphs.

So I want to define what $Hoch(\mathscr{A})$ is now. Our definition is slightly different than the usual one, but equivalent. It will be

$$\bigoplus \underbrace{A \otimes \cdots \otimes A}_{n+1} \otimes \Omega_.(\Delta^n)$$

This tells you where to put the things on the circle. The relation on this will be

$$a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_n \otimes c = a_0 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes I_{i*} c$$

Now the boundary operator is

$$\partial = \partial_A \otimes id + id \otimes \partial_{\Omega_.()}$$

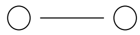
This equivalence should be cyclic too.

Now I will consider

$$\Omega_{.}G(m,n) \otimes Hoch(\mathscr{A})^{\otimes m} \to Hoch(\mathscr{A})^{\otimes n}$$

The Γ were complicated graphs with m input edges and n output edges. The outputs in $G(m,n)$ are circles connected by graphs, and the other boundaries are inputs.

Let me draw a simple one, $G(1,2)$, one input and two outputs



I also have to mark the outputs and the inputs. Let me take an element over here. The element I have on the left could maybe be $a_0 \otimes \cdots \otimes a_n \otimes c$ and let me take $\sigma \in G(1,2)$. Then I will take $\Omega_{.}G(1,2)$ and tensor it with where c lives, in $\Omega_{.}(\Delta^n)$. So this can be thought of as $\Omega_{.}(G(1,2) \times \Delta^n)$. You have this $G(1,2)$ which has one input, and then if the length here is five, I could stretch it to a sequence of numbers between zero and five, and I could go around and put marks at specified lengths. The combinatorics will change as the ts vary. I have a current on Δ^n and I want to substratify to respect the combinatorics as the points vary. Now $G(1,2) \times \Delta^n$ has the product stratification $S \times S$ but I will pass to a better one, S'', the combinatorics of this being fixed. Now the sum of chains on this strata, well, I can work on one stratum. On one stratum I can start labeling my [picture]. Some of the labels will land on the output circles. Later I will collect those. I get the beginning of an element of Hochschild, but I need the appropriate chains here. On the core, I have, if I cut the circles out, a form of those Γs with some lengths zero.

Then I apply Kevin's machinery and get as output a bunch of As which I collect. In addition I get a differential form on the moduli space of this piece. Now I know how to collect the elements, $a_{i_1}, \ldots, a_{i_k}$ and $a_{j_1}, \ldots, a_{j_\ell}$. I haven't used σ, but it's a chain in the moduli space of G-type graphs. This is a union of a circle and some graphs, and the moduli is the product of the respective moduli spaces. You use the projection, I get ta map from the G to $\Gamma \times \Delta^L \times \Delta^K$. I push this forward and get a Γ piece, a chain. Integrate that and get a number. Then you get the chains you needed from Δ^L and Δ^K to make Hochschild elements. Let me tell you, take three more minutes. Everyone knows in the simply connected case if you put differential forms in Hochschild you get something interesting. Let me discuss something less familiar. There is an abstract theorem in category theory that says, choosing a bundle appropriately, the Hochschild complex calculates the deformations of the complex structure, the extended deformation complex, [unintelligible]. Then this theorem gives you an action of the chains on the moduli space on the deformation complex. Kevin, in a categorical way, has shown that, these are called the B-models of all genera. The Frobenius part from Kontsevich only reflects genus zero. If I can get my hand of the bundle E explicitly, this would be a way to get at the B-model of all genera. So the categorical version of this is in Kevin. It would be interesting to relate this to what Thomas talked about, chains on the moduli space of ribbon graphs relative to directed ribbon graphs.