Postnikov Conference

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1 Peter Teichner

Thanks for coming and giving us 2 hours to explain our stuff. Can you hear me in the back?

I can start writing my main diagrams. Let's do the first part mute. Here is my plan

1. Reminder of generalized cohomology
2. Definition of quantum field theory, part one
3. Theorems and conjectures
4. Ideas of the proof.

Let's start with the definition of a generalized cohomology.

Definition 1 *A generalized cohomology theory is a contravariant homotopy functor* E : $MAN \to gradedRINGS$ *with a long exact Meyer Vietoris sequence.*

You might be used to considering other spaces, but you can uniquely extend this functor to finite dimensional finite CW-complexes. I should have said that this is a multiplicative generalized cohomology.

The examples we're interested in, there are three main ones, H^*, ordinary cohomology, which arises (well, maybe as homology) in the 1900s with Poincaré. Then KO^*, I'll focus on real K theory, which is the 1950s, Grothiendieck, Atiyah. The last one is topological modular forms, TMF^*, in the 1990s, Hopkins, Miller and now a beautiful interpretation by Lurie. In the 1990s there was a discussion about elliptic cohomology. Every formal group law gives rise to a generalized cohomology. The standard and K theories come from the additive and multiplicative formal group laws, and then the next easiest ones come from elliptic curves. They wrote down the universal theory for this, which is not an elliptic theory but maps to all of them. It is a moduli stack. This word comes from a very important map, a topological modular form corresponds to a modular form.

A feature of these theories is the "integration map" $E^0(M) \to E^{-n}(pt) := E_n(pt)$ if M is E-orientable. For ordinary cohomology orientable is the standard notion, this is defined using Poincaré duality.

Let me give you three examples of the integration map. I am going to try erasing the European way.

Let me write down the integration maps for these, along with the orientation conditions

E^*	orientation condition	$E_n(pt)$	integration map $E \int_M$
H^*	standard orientation, $w_1 = 0$	$\mathbb{Z}$ if $n = 0$, 0 otherwise	counts points algebrai
KO^*	spin, $w_1 = w_2 = 0$	$\mathbb{Z}$ in 0 mod 4 and $\mathbb{Z}_2$ in 1 and 2 mod 8	Aarhus genus, counts s
TMF^*	string $w_1 = w_2 = 0$ and $p_1/2 \cong 0$	TMF_n	MF_n

Here MF_n is the integrable modular forms of weight $n/2$. This is a beautiful object. The canonical map $\phi : TMF_n \to MF_n$ that $\phi(TMF \int_M 1) = W(M)$, the Witten genus.

I don't want to go into what the Witten genus is, but what it counts is S^1 equivariant spinors on loop space. That doesn't make mathematical sense but the Witten genus does.

When we set out to study these things, we wanted a geometric interpretation of TMF and the Witten genus. When we set out to do that, we were interested in two dimensional conformal field theories, and then we realized we should study quantum field theories of other dimensions as well.

That was the first part of my lecture, why you should be interested in these theories. I want to start the definition now of a quantum field theory. We'll throw more and more extra things in as we need to. Let me remind you of Siegel's idea. You have two categories. Let me fix a dimension d (here 0, 1 and 2 will correspond to H,KO, and TMF).

So a d-dimensional QFT will be a a symmetric monoidal functor $RB_d \to TV$ from the Riemannian bordisms to locally convex topological vector spaces. There is a beautiful monoidal product on the right hand side given by the projective tensor product.

I want to give a little picture about how to make things Riemannian when you glue them. Let me give you a picture. A morphism in RB_d looks like a d-dimensional bordism Σ read from right to left. The objects are Riemannian d-manifolds, the open part, with a metric completion, with an embedding into the morphism. It's a germ. The surprise is that on the outcoming thing, it's a little tab stuck on the outside of the bordism. We require that it's an isometry of the open part, that it continues the Riemannian structure.

The reason we want no disjointness is that we want a category.

The last thing that I want to do in part one is to put in a manifold X and now we have a symmetric monoidal functor from a bigger category $RB_d(X)$, and now you have smooth maps from RB_d to X and the boundaries to X. To define a QFT over X you just change the domain category. To be precise you should have a d-functor. Then you should have an algebraic d-category TV_d.

Here is the main table of theorems and conjectures. I am ignoring twisting and supersymmetry. The n is twisting.

d	$QFT^n_{d\|1}(X)$
0	$\Omega^n_{cl}(X)$
1	(conjecturally) $Cliff(n)$-module bundles over ΠTX with Quillen superconnection
2	(conjecturally) a very nice NEW geometric model

We want this because the TMF is completely calculated. We don't know for sure that we get a cohomology theory for $QFT^n_{2|1}[X]$. We're very close.

Let me put down theorem two. The partition function "evaluate on tori" gives a map $Z : QFT^{-n}_{2|1}(*) \to MF_n$.

The left hand side is defined, we just don't have a connecting map. This associates something to every surface. You get something like numbers on each torus. This is a function on the moduli space of tori. This function is holomorphic and $SL_2(\mathbb{Z})$ invariant. This is one reason we thing this theory could be TMF because we know we have the map to modular forms in TMF.

Let me go through the $d = 0$ case carefully with my last few minutes. I'm just going to use what I explained. We have the functor $RB_0(X) \to TV$. The objects are -1 manifolds mapping to X, but there is only one such manifold, the empty set. We're going to make differential forms out of the empty set. So we better get the unit in the monoidal category of vector spaces. The functor is symmetric monoidal, so we should look at a point mapping to X. The point is closed. For every point in X we get a map to $\mathbb{C}$. So $QFT_0(X)$ is all functions from X to $\mathbb{C}$.

We want this to be a smooth functor which will land us in $C^\infty(X)$. But we don't want functions on X, we want differential forms, so $C^\infty(\Pi TX)$. Then ΠTX is $Map(\mathbb{R}^{0|1}, X)$. There is a supergroup acting here, the automorphisms of the superpoint. This group is $\mathbb{R}^{0|1} \times \mathbb{R}^\times$, translations and dilations, and the infinitessimal generators of this action act on the functions, on the differential forms they are the de Rham differential and the grading. The last sentence I'm saying is that in the morphisms of the Riemannian bordism is dividing out by isometries. This should be translation invariant, so you translate out and only get the closed forms.

Thanks a lot.

2 Stolz
Supersymmetric quantum field theories and generalized cohomologies II

Thanks very much to the organizers. i wanted to continue with the second part of the talk.

I want to start out by talking about the second part of the definition of quantum field theory. I want to define what is a quantum field theory of dimension d over a manifold X. That's a symmetric monoidal functor Q from a bordism category equipped with maps to a space X, $RB_d(X)$ to the category of locally convex topological vector spaces

I want to make this precise by putting a superscript fam to stand for families. Enlarge the categories substantially by thinking of smooth families of the things we had before.

Let's look at objects in these categories. Let's look in the range. A family of locally convex vector spaces is just a smooth vector bundle over a manifold S. The fiber should have the old structure, locally convex topological vector space.

A morphism in this category is a map of bases covered by a map of vector bundles

$$\begin{tikzcd} V \arrow[r, "\hat{g}"] \arrow[d] & V' \arrow[d] \\ S \arrow[r, "g"] & S' \end{tikzcd}$$

Now we want families of $Y \xrightarrow{f} X$ in the domain category, and we replace this with smooth bundles of Riemannian closed $d-1$-manifolds. That's a formal definition, almost, and we want to put in two requirements on this functor.

The first is that you have forgetful functors $RB_d^{fam}(X) \to Man$ by forgetting the bundle, just remember the base S. You have another from TV^{fam} and you want the functor to commute

$$\begin{tikzcd} RB_d^{fam}(X) \arrow[rr, "Q"] \arrow[dr] & & TV^{fam} \arrow[dl] \\ & Man & \end{tikzcd}$$

You also want Q to be compatible with pullbacks.

So imagine you had a smooth functor. If you had a smooth functor you could produce a functor that would give a family out of a family of Riemannian bordisms, so that's what we define smooth as.

This, you can look at more general versions. If you had pullback here, you can define pullback in general by writing down properties that characterize it. You can say this is a Groethendieck fibration, meaning you have pullbacks. You can define a functor that takes pullbacks of the forgetful functor to pullbacks and commutes with the forgetful functor, you call that a smooth functor.

Now we want to define supersymmetric field theories.

Let me write down the definition. A supersymmetric quantum field theory over X of dimension $d|1$ is again a symmetric monoidal functor

$$Q : RB_d^{fam}(X) \to TV^{fam}$$

but I want to do certain replacements to what we had before. The fibers before were d dimensional manifolds with Riemannian metrics. Now we replace those with supermanifolds of dimension $d|1$. This d is the ordinary dimension and the 1 is the fermionic dimension, I'll explain in a minute, and they have a super-Riemannian structure. Doing this change, I replace d with $d|1$ in my notation. The second replacement, in the world of supermanifolds, is to be systematic and allow supermanifolds as bases of my families.

Then you denote this by

$$Q : SRB_d^{fam}(X) \to STV^{fam}$$

with S for super.

So that's the definition. Let me check that I didn't forget anything. Now we need to define supermanifolds. That's the second part of my talk.

Definition 2 *A supermanifold M of dimension $p|q$ is a pair*

$$\underbrace{M_{red}}_{\text{a topological space}}, \qquad \underbrace{\mathscr{O}_M}_{\text{sheaf of commutative } \mathbb{Z}_2\text{-graded algebras over } \mathbb{C}}$$

which is locally isomorphic to $\mathbb{R}^{p|q} = (\mathbb{R}^p, \mathscr{O}^{p|q})$ where $\mathscr{O}^{p|q}(U) = C^\infty(U) \otimes \wedge[\theta_1, \ldots, \theta_q]$. The θs here are to be thought of as odd variables.

You are supposed to think of it as giving the spaces of functions on the supermanifold. If I have a supermanifold M by smooth functions I mean global sections of the structure sheaf.

Let me do some examples. The space M_{red} becomes a smooth manifold from the requirements on the supermanifold.

Suppose that you have a vector bundle E^q over X^p a manifold. Then you can produce a supermanifold out of it called ΠE which is $(X, \wedge E^*)$. This gives you a sheaf of commutative $\mathbb{C}$-algebras. Locally it looks like the standard model, so this is a supermanifold of dimension $p|q$. I am confusing bundles and the corresponding sheaf of sections.

This is a very good example of a supermanifold because every supermanifold of dimension $p|q$ is isomorphic to one of these. There are a lot more morphisms in the super-world and that's why you would rather work there.

Take ΠTX. I wanted to remark, the Π you should think of as parity reversing. You reverse the parity of your vector bundle.

So you should get a supermanifold of dimension $p|p$. The global sections are the symmetric algebra on the dual T^*M, so the $C^\infty(\Pi TX) = \Omega^*(X)$.

Now we want to do everything we normally do over manifolds over supermanifolds. For example we should do vector bundles. These are projective modules over the ring of functions and you can export that to the world of supermanifolds. A vector field is a derivation of the ring of functions. That can be brought to the language of supermanifolds too. It is a graded derivation of the ring of functions.

For example, think about $\mathbb{R}^{1|1}$. The functions on it are the functions on $\mathbb{R}$ tensored with $\wedge[\theta]$. So you can take, for example ∂_t and ∂_θ or linear combinations, like $D = \partial_\theta - i\theta\partial_t$. Let me discuss parities. ∂_t is even whereas ∂_θ is odd and D is also odd. Here's something that you can't do in ordinary manifolds. If you square an odd vector field, you get $D^2 = -i\partial_t$. You don't have that on ordinary manifolds.

I want to define a Riemannian structure on a supermanifold. This is piecemeal, doing it for one d at a time. We can do it for $d = 1$,$d = 2$, maybe $d = 4$. Let me jot down the definition for $d = 1$ and then motivate it.

A superRiemannian structure on $Y^{1|1}$ is an odd vector field D which locally looks like the D that was defined before. I want to be able to pick local coordinates to make this vector field look like that D.

Here are motivations. One is physical. What do physicists do with a Riemannian metric? They write down action functionals, energy functionals with it.

This structure is what a physicist would need to do this. The other motivation is that looking at the underlying reduced manifold of dimension one, the superRiemannian structure induces a Riemannian structure on the underlying manifold.

You can do the same thing for supermanifolds of dimension $2|1$ but it looks completely different. I could discuss that later.

Now I would like to look at supersymmetric quantum field theories in detail in dimensions 0, 1, and 2.

Okay. Let me briefly discuss the case $d = 0$. What is a 0-dimensional field theory over X? Well, it's a functor $SRB_{0|1}^{fam}(X) \to STV^{fam}$.

We need a bundle with $0|1$ dimensional fibers over S and then a map into X as a morphism in the domain category. Then the functor Q has a bundle homomorphism from the trivial bundle to itself, so a smooth function of the parameter space S. Let's specialize to $S = \Pi TX$. Think of this as the supermanifold of maps $\mathbb{R}^{0|1} \to X$, which gives a canonical map from this $\times\mathbb{R}^{0|1}$ to X which is evalution.

Let's see whether $\omega \in C^\infty(S) = C^\infty(\Pi TX) = \Omega^*(X)$. Argue a little and you get a closed form and work a little more and it's a bijection. That's the $d = 0$ story. Let me do the $d = 1$ story, starting with the normal story before doing the supersymmetric story.

Take Q a one-dimensional quantum field theory over X. That's a functor $RB_1^{fam}(X) \to TV^{fam}$. An object, you could take the single point sitting inside $\mathbb{R}_{\geq 0}$, and you need a collar to be precise, so I guess I want to do the easy case where there is no X involved. This could be parameterized over a point. Call this object pt. Now I want endomorphisms of this. For any positive real number I see an endomorphism of this object so I can apply the functor to get endomorphisms of $Q(pt)$, continuous operators on this locally convex vector space. If I have a positive number t I think of the interval of length t as a bordism of a point to a point. Then composing we get a representation of the $R_{>0}$ semigroup. I can differentiate that and I get Lie algebras.

This is a free Lie algebra in one generator, and so the whole homomorphism is specified by ∂_t whose image is A, the infinitessimal generator. Out of a supersymmetric one dimensional field theory we can produce this operator, or we can use the operator to get homomorphisms and create the functor. All the other Riemannian intervals can be constructed by gluing together. If you want to do a quantum field theory, it's just picking a vector space and an infinitessimal generator. I'm cheating a little.

I want to do the super story. I look at $1|1$ dimensional QFT. This is a functor

$$SRB_{1|1}^{fam} \to STV^{fam}$$

Again I want a particular object and a particular group of endomorphisms. Look at $\{0\} \times \mathbb{R}^{0|1}$ sitting inside $\mathbb{R}^{1|1}_{\geq 0}$. This I will call a superpoint. I can generalize this way of looking at intervals, being careful of the geometric structures. I have this collar, but this needs a super Riemannian structure, which I get from the standard one on $\mathbb{R}^{1|1}$ by restriction.

Then I can ask what are the automorphisms of $\mathbb{R}^{1|1}$ equipped with this structure. It is

$$Aut(\mathbb{R}^{1|1}, D) \cong \mathbb{R}^{1|1}$$

with the group structure

$$(t_1, \theta_1) \times (t_2, \theta_2) = t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2$$

The algebra on functions on $\mathbb{R}^{1|1}$ is generated by θ So think of these as being generators on the left, then we know where to send t and θ, to make this group law honest.

Now let's generalize what we did for the point. You can produce a homomorphism $\mathbb{R}^{1|1}_{+} \to End(spt) \to End(Q(pt))$ which will be a smooth representation of a supergroup, which you can differentiate to get $Lie(\mathbb{R}_{+}^{1frm[o]--}) = \langle \partial_t, Q \rangle$ So this is determined just by the image of Q, supersymmetric field theory.

Being supersymmetric means that the original [unintelligible]is the square of this odd D.

Now I'm ready to go to $d = 2$.

[Too fatigued.]

3 Michael Sullivan

We're looking at Legendrian contact homology and string topology. Let me just present a brief overview of symplectic geometry, J-holomorphic curves, and so on. One could argue that symplectic geometry was born in 86 when Gromov proved his theorem for the compactification of the moduli space of J-holomorphic curves in a symplectic manifold. With this proof that the moduli space could be compactified in a nice way, that enabled people to use holomorphic curves in a number of ways. I can call this symplectic rigidity. It should be called geometry, not topology. Now there's a parallel field known as contact geometry. I'll define this, it doesn't look the same, but there are a great many similar results. The thing about contact geometry is that it's odd dimensional, so there are no holomorphic curves. Because of that, contact rigidity results were lagging by about ten years, so starting in the 90s, only then were people able to, notable Hoffer, to start using holomorphic curves in a symplectic manifold related to a contact manifold. More recently, in this decade, these holomorphic curves started addressing problems in differential topology, low dimensional, and there are a number of ways in which these started helping answer problems, most notably Heegaard Floer. So that theory based on holomorphic curves in a symplectic manifold was able to recover the Alexander polynomial. There are a lot of attempts to go from contact geometry to differential topology. I learned from Eliashberg the idea that a smooth submanifold in $\mathbb{R}^n$ gives you a Legendrian submanifold in a particular contact manifold known as the unit cotangent bundle. So this contact manifold and the Legendrian manifold have ways to study them. This translates to differential topology. The one of most interest is the case of a knot in $\mathbb{R}^3$.

The final, there are these Floer theories for holomorphic curves, there are contact homologies, and the contact homology for a Legendrian submanifold in a contact manifold also recovers the Alexander polynomial, but my interest is the knot invariant constructed from open string topology. There are a lot of technical details required to nail this thing down.

Okay. That will be the first half of the talk, just kidding, that will be the talk. Let's start with some definitions of contact manifolds, contact manifolds and Legendrian curves, and contact homolgy. A contact manifold is a pair M, ξ where M is an odd dimensional manifold, and it's not closed but we'll say orientable, dimension $2n+1$, and ξ is a $2n$-dimensional distribution sitting in the tangent bundle which is "maximally nonintegrable." In this talk it suffices to say that you can define it in the cases I care about as there existing a one-form α such that $\ker \alpha = \xi$ and so that $\xi \times (d\xi)^n$ is never 0. Then α is called a contactt form.

In my example, well, I might be sloppy and omit the distribution in notation, but in $\mathbb{R}^3$ take $dz - ydx$.

All contact manifolds look like this in the neighborhood of a point.

[I'm too hot and my seat is in the back.]