Postnikov Conference

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1 Veronique Godin, Higher Genus String Topology

The goal of today's talk is to say something about a theorem I proved earlier this year about extending string operations to the whole of the moduli space. Part of the problem in giving this talk is thta constructing these operations is quite technical. If I try to give details it makes things go wrong, but most of today's talk will be to explain the theorem or its consequences.

This is completely unrelated to the talk but I don't know why people think they have to bring the top board down. Let me begin with an introduction to string topology. I'll concentrate on the things I'll need. If M is a closed oriented manifold (I will always assume this) of dimension d, let LM be the free loop space $Map(S^1, M)$, piecewise smooth.

Chas and Sullivan define a product on the homology of this space

$$H_pLM \otimes H_qLM \to H_{p+q-d}LM$$

as part of a BV structure. My idea is to consider composable loops $Map(\infty, M) \overset{\rho}{\hookrightarrow} LM \times LM$ inside the product of the loop space and try to build a shriek map

$$\rho! : LM \times LM \to Thom \begin{pmatrix} ev^*TM \\ \downarrow \\ M^\infty \end{pmatrix}$$

which we can compose with the Thom isomorphism to get

$$H_*(LM \times LM) \to H_*Thom(ev^*TM) \overset{\cong}{\to} H_*(M^\infty) \to H_*LM$$

The goal is to extend this operation to operations parameterized by the homology of the mapping class groups of certain surfaces. I will have to describe which surfaces and describe the mapping class group and then I'll be able to state my theorem.

The type of surface I'm going to consider are a bit different from the simplest picture we've seen. They'll have open string and closed string boundaries, because, we'll talk about later wanting it to be a homotopy invariant.

So S will be a surface with boundary, for example [Picture]. We divide the boundary, choose two submanifolds, some incoming and some outgoing inside δS. You can have extra pieces which are called free. These we denote $\delta_{IN}, \delta_{OUT}$ and the free boundary. Some things can be closed intervals. I want to think of this as a cobordism between two things both of which have boundary.

By the way, the boundary of S at this point may be empty. We'll restrict to something else afterward. You can actually permute the extra free circles, rotate, whatever you want to do with them. The mapping class group of S, which we'll denote by $Mod^{oc}(S)$ will be $\pi_0(Diff^+(S, \delta_{IN} \cup \delta_{OUT}))$ where these are preserved pointwise. If we have two open-closed cobordisms, S_1 and S_2 and we have an an identification of the incoming of the other, then we can glue the cobordisms, compose them. This gives a group homomorphism

$$Mod^{oc}(S_1) \times Mod^{oc}(S_2) \to Mod^{oc}(S_1 \# S_2)$$

We can define a category Mod^{oc} whose objects are ordered disjoint unions of S^1 and the interval. I want every circle to have a parameterization and so on, and the morphisms between two such things,

$$mor(B_{IN}, B_{OUT}) = \bigoplus_{[S]} Mod^{oc}(S)$$

where the sum is taken over all diffeomorphism classes of open-closed cobordisms with boundary B_{IN}, B_{OUT}.

[Categorical problems]

The composition is the gluing homomorphism and we also have categories $BMod^{oc}$, $CMod^{oc}$ and $HMod^{oc}$ by taking classifying spaces, chains, and homology. All of these are symmetric monoidal categories.

Now I'm ready to define higher genus string topology. I can't get an operation for a cap, so we'll let $HMod^{oc}_{\delta IN \neq \delta}$ are those such that the incoming boundary is not everything. This won't be a category because the empty manifold will not be there. You either need the boundary of each path component of S to have some outgoing or some free boundary.

Theorem 1 *Once you restrict the category by getting rid of these morphisms, you get a functor*

$$\mathscr{A} : HMod^{oc}_{\delta_{IN} \neq \delta} \to \textit{graded Abelian groups}$$

so that $\mathscr{A}(0) = H_*LM$ *and* $\mathscr{A}$ *of an interval is* H_*M*.*

For any open closed cobordism S you get a map

$$\mathscr{A}(S) : H_*Mod^{oc}(S) \otimes H_*LM^{\otimes p} \otimes H_*M^{\otimes q} to H_*LM^{\otimes r} \otimes H_*M^{\otimes S}$$

[You can ask a question. You can't?]

That map has degree $\chi(H_*(S, \delta_{IN}S)dim(M))$. If you have only closed circles, this is the Euler characteristic of S but if you have incoming there's a slight change.

The first remark is that these generalize the ones that were defined a similar way. The Chas Sullivan operations, the ones Ralph and I and Voronov defined, are all contained in this setup. For example, the product corresponds to the pair of pants. You have $Mod(S) = \mathbb{Z}^3$ and $H_0 Mod(S) = \mathbb{Z}$ which contains one, which is the Chas Sullivan product.

Let me give you a couple more remarks. You have the operations generalizing the old ones, and let me say in the construction of these operations I make a "Thom Pontrjagin" (in spirit) collapse map from a classifying space

$$BMod^{oc}S \times M^{\delta_{IN}S} \to Thom(\mathscr{M}_S \to EMod^{oc}S \times_{Diff_+S} Map(S^1, M))$$

where $\mathscr{M}_S$, the virtual bundle, is some sort of twisting between a bundle over the moduli space and a bundle over M. Think of it as $H_*(S, \delta_{IN}S)$ with TM. You twist these over the space by using the right model and that's the target for the Thom collapse map. On any cell you have a difference of bundles. The bundle that you want change in dimension at every step. You patch them together. The bundles are orientable so twisting them gives that. You have a difference of bundles but I don't know a way of writing the two.

Let's move on and talk about conjectures. There's probably a way of lifting to chains, but how do you build the Thom isomorphism in a compatible way with the gluing? You might get it on spectra too. Also, there might be a way of including intervals labeled by submanifolds and associate that to paths between those.

The last thing that I would like to do is define a different version of this that will also include unparameterized circles, which should correspond to equivariant homology of the loop space.

Now I would like to talk about one more conjecture about which I have more to say.

Well, let me say. If you have, you should be able to extend these operations to include surfaces, well, intervals labeled by submanifolds $A, B \subset M$. Then your surfaces should contain this. The gluing should respect those as well. Once you build a functor you should associate to that a space of paths. You should associate to that the homology $H_*(P_{AB}M)$, paths with $\gamma_0 \in A$ and $\gamma_1 \in B$. Chas Sullivan have already defined paths in these. If you have paths in these you can look at paths that cross and compose them. This should correspond to H_0 of the flat pants. You don't get all the operations that I'm getting. If A and B are both a point you get an operation on the based loop space.

Homotopy invariance? Here is a conjecture. The isomorphism class of the functor $\mathscr{A}$ depends only on the oriented homotopy type of M. If you have a homotopy equivalence $f : M_1 \to M_2$, then $f_*[M_1] = [M_2]$ gives us a natural transformation between the functors $\mathscr{A}(M_1)$ and $\mathscr{A}(M_2)$ from $HMod^{oc}$ to graded vector spaces.

Let me tell you why that's true and then I'll finish. First of all, Cohen, Klein, and Sullivan prove that, well, all of these structures should depend on what's happening on the manifold and nothing more. It depends, seems to, on an E_∞ structure on the manifold, which you use to build a structure on the loop space.

I will look at the part of my category that has only intervals and no circles Mod^o. The theorem of Kevin Costello says that if you have a monoidal functor $\mathscr{B} : CMod^0$ to graded vector spaces, such a thing corresponds to a Frobenius algebra up to homotopy. You can complete this universally to include the circle. So you can complete this to $\tilde{\mathscr{B}}$. If $\mathscr{A}$ was what you had associated to the interval, then $\tilde{\mathscr{B}}$ of the circle is $HC_*(A, A)$. If you start with this structure on the cochains, you'd get a universal structure on the Hochschild cohomology of the cochains.

So we would like to appy this to $A = C^*M$ but the problem is using the model he has you need the trace and cotrace on the nose. In the cochain case you have this only up to homotopy. This won't give us the right structure on A, but you could probably use the ∞ version of the trace to get a similar result. In that case we would get $HC^*(C^*M, C^*M) \cong C_*LM$. Then we would have a universal structure on the homology of the loop space from the structure on the loop space, and these would determine one another in some way.

I'll stop.

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