# Postnikov Conference 

Gabriel C. Drummond-Cole

July 3, 2007

## 1 Craig Westerland

Thank you. I'd like to thank the organizers. Before I forget to say it, most of this work is joint with K. Gruher and a lot of it builds on work she did with Paolo. It's about the string topology of the classifying spaces of compact Lie groups. So $G$ will be a compact Lie group and $X$ will be a $G$-space. I can form the translation groupoid $[X / G]$. This will be a category and the objects will be $X$ with morphisms $G \times X$. The source of $(g, x)$ is $x$ and the target is $g(x)$. You compose things by multiplying in the group. So $(h, g(x)) \circ(g, x)=(h g, x)$.

Take the classifying space of this category, and it will be $B[X / G]=E G \times_{G} X$. I'll call these stacks, just to intimidate you. I'll focus on two of these, $[* / G]$ and $I([* / G])$ which is $[G / G]$ where the action is conjugation. When I take the classifying spaces of these guys I get $B[* / G]=B G$ and $B[G / G]=G \times_{G} E G \cong L B G=\operatorname{Map}\left(S^{1}, B G\right)$.

This is where it starts becoming string topology. This will be about producing algebraic structure on these gadgets. Here is the Ur-idea that these things are built on. "A Frobenius object" in the category of correspondences of groupoids.

If I look in $[G / G]$ I want a multiplication $[G / G] \times[G / G] \rightarrow[G / G]$. What I can do is cook up a map the wrong way $\pi:[G \times G / G] \rightarrow[G / G] \times[G / G]$ where the action on the left is diagonal. This is just factoring out by another $G$. There is also another map $\mu$ to $[G / G]$. If I could invert either arrow, I'd get either an algebra or a coalgebra.

We are going to construct operations using umkehr maps. If I can turn $\mu$ around, then $\pi \circ \mu$ ! is a coproduct (FHT). If I turn $\pi$ around then $\mu \circ \pi_{!}$is like a Chas Sullivan product.

What happens if I apply $B$ to this diagram? $B$ of a product is the product of the $B$ s so I get

$$
L B G \times L B G<\leftarrow_{\pi}^{<}(G \times G) \times_{G} E G \underset{\mu}{\longrightarrow} B G
$$

The middle term fibers over $B G$ with fiber $G \times G$.
Let's try to make $\mu_{!}$This was $\mu: G \times G \rightarrow G$. This is a principal $G$ bundle. So I get a transfer map $\tau: S^{a d} \wedge G_{+} \rightarrow G \times G$. This is a topological version of the thing you get in homology,
moving from the fiber to the base. Take the orbit of a cycle. The $\mu$ was $G$-equivariant and so is $\tau$, and that leads to

$$
\mu_{!}:\left(S^{a d} \wedge G_{+}\right) \times_{G} E G \rightarrow(G \times G) \times_{G} E G
$$

Collectively I get a map $L B G^{a d} \rightarrow L B G \times L B G$. I get rid of the $a d$ by desuspending by two copies of $a d$. So I get $L B G^{-a d} \rightarrow L B G^{-a d} \times L B G^{-a d}$.

Let me try to turn $\pi$ around now. There are a number of different ways of doing that. There are two ways to do this. We could do a transfer again. You're quotienting out by an extra copy of $G$. That produces something I don't really understand. It looks like it's zero most of the time. Let's do something that is more along the line of intersection theory. This is Gruher-Salvatore. The $L B G \times{ }_{B G} L B G$ can be thought of as a subspace of $L B G$. But $B G$ is a very large space. So the map into this is infinite codimension. So I have to replace the map with finite dimensional approximations.

What's the fact that this relies on? That we have finite dimensional approximations. Part of being a compact Lie group is being linear, so there is a finite dimensional faithful representation $V$ of $G$. If I define $E G_{n}$ to be the set of injective linear maps $V \rightarrow \mathbb{C}^{n}$ we get a Stieffel manifold. What do we know about it? When it's nonempty, it's a finite dimensional manifold with a free $G$ action. The quotient $E G_{n} / G$ we will call $B G_{n}$, and that's a finite dimensional manifold.

Now $E G_{n}$ sits inside $E G_{n+1}$, because I can stick $\mathbb{C}^{n}$ inside $\mathbb{C}^{n+1}$ and theses become increasingly connected. So $\lim E G_{n}=E G$ and $\lim B G_{n}=B G$. I want to do intersection here on these guys and then assemble them together.

I define $\operatorname{Ad}\left(E G_{n}\right)$ to be $E G_{n} \times{ }_{G} G$.
Start forming a diagram.


Theorem 1 Gruher-Salvatore
$\tilde{\mu} \circ \tilde{\Delta}_{!}$makes $A d\left(E G_{n}\right)^{-T B G_{n}}$ into a ring spectrum

Part of the Pontrjagin-Thom collapses turns things a round. We can form a pro-ring spectrum $L B G^{-T B G}$.

How do you compute these rings? You can do this as Hochschild cohomology of the cochains on the manifold. There are ring isomorphisms
1.

$$
H_{*}^{\text {pro }}\left(L B G^{-T B G}\right)=\lim _{\leftarrow} H_{*}\left(A d\left(E G_{n}\right)^{-T B G_{n}}\right) \cong H H^{*}\left(C^{*}(B G), C^{*}(B G)\right)
$$

2. 

$$
H^{*}\left(L B G^{a d}\right) \cong H H^{*}\left(C_{*} G, C_{*} G\right)
$$

These are not terribly hard to prove. I'll sketch the idea.

1. À la Cohen Jones, give a cosimplicial model for $\operatorname{Ad}\left(E G_{n}\right)^{-T B G_{n}}$ with

$$
H_{*}\left(A d\left(E G_{n}\right)^{-T B G_{n}}\right) \cong H H^{*}\left(C^{*}(B G), C^{*}\left(B G_{n}\right)\right)
$$

Assemble into the theorem using limits.
2. from Jones, $H H_{*}\left(C_{*} G, C_{*} G\right) \cong H_{*}(L B G)$. Then using dualization and Poincaré duality you get this switched over to Hochschild cohomology.

The ring spectrum and the pro-ring spectrum are intimately related

## Theorem 2 Gruher

$L B G^{-T B G}$ is Spanier Whitehead dual to $L B G^{-a d}$ and the duality takes the product to the coproduct

This duality should not be thought of between spectra and spectra but spectra and prospectra. This is multiplicative. Since they're dual, the homology of one should be the cohomology of the other, so the two guys in the theorem should be isomorphic.

Now I want to interpret the same statement using Kozsul duality. The $B$ in $B G$ stands for bar, and so the algebra $C^{*} B G$ is Kozsul dual to $C_{*} G$. What does that mean?
$C^{*}(B G)=B^{*}\left(C_{*} G\right)=\operatorname{Tot}\left(k \rightarrow \operatorname{Hom}\left(C_{*} G^{\otimes k}, F\right)\right)$ with the duality $B^{*} B^{*} \cong i d$. The following statement is known, I don't know how well,

Proposition 1 For some dga $A$ there is a ring isomorphism

$$
H H^{*}(A, A) \cong H H^{*}\left(B^{*} A, B^{*} A\right)
$$

It is because of this that $B^{*} B^{*} \cong i d$. The moral is that

$$
H H^{*}(A, M)=\text { RHom }_{A \otimes A^{o p}}(A, M)
$$


Now when we go back to the statement that I have the two spectra as dual, the homology of one is the cohomology of the other. We just made a connection between them with Kozsul
duality. Before I jumped into it, this was a ring isomorphism by functoriality. On one side it was the Uneda composition. Composing and then applying $B^{*}$ I get a composition.

Anyway the upshot is that Kozsul duality carries the FHT type product on $H^{*}\left(L B G^{-a d}\right) \equiv$ $H H^{*}\left(C_{*} G, C_{*} G\right)$ to the Gruher or Chas-Sullivan product $H_{*}\left(L B G^{-T B G}\right) \cong H H^{*}\left(C^{*} B G, C^{*} B G\right)$.

There's a little bit more to say, it's not worth jumping into with only a few minutes left. Thanks for listening.
[Question]
How do I construct $L B G^{-a d}$ ? I have $G^{-a d}=G^{-T G}$ which $G$ acts on. So I can form $G^{-T G} \wedge_{G} E G=L B G^{-a d}$. This is a homotopy orbit spectrum. Desuspending by the tangent bundle gives the Spanier Whitehead dual. So this gives $(D G) \wedge_{E} E G$.

To construct $L B G^{-T B G}$ is quite a bit of work. This thing is $(G)^{n G}=F\left(E G_{+}, \Sigma^{\infty} G_{+}\right)^{G}$. This isn't entirely true. It's also not apparently a pro-object. How do I make it into one? I really want to take $F\left(E G_{n+}, \Sigma^{\infty} G_{+}\right)^{G}$. This collection is homotopy equivalent to $L B G^{-T B G}$. One is a homotopy orbit spectrum and the other a homotopy fixed point spectrum. The duality of orbits and fixed points and $G$ with $D G$ jibes perfectly with the duality before.
[Question]
You want $L B G^{-T B G}=T H H^{*}(D B G)$. Oh, that's interesting, here's a question. Is $L B G^{-T B G} \cong$ $T H H^{*}(G)$ ? I don't know, I hadn't thought about that.

## 2 Scott Wilson <br> Homotopy Frobenius algebras and forms on a manifold

First I want to say thank you for everyone who made it possible for me to be here. This is joint work with Dennis Sullivan. Here is my outline:

1. I will recall the definition of an open Frobenius algebra
2. I will define/construct a free resolution of the open Frobenius algebra structure
3. I will give a construction realizing this on the forms of a smooth manifold.

Let me recall the definition of an open Frobenius algebra. You begin with a graded vector space $A$ with a differential $d$ and a graded commutative associative algebra, a graded cocommutative coassociative coalgebra, and the coalgebra map is a map of $A$-bimodules. $d$ is a differential for the multiplication and a codifferential for the comultiplication.

I'm going to describe this in terms of trees. We can describe the algebra by a tree with two inputs, keeping track of which is which. The coalgebra I can keep track of with a tree with
two outputs. I have the Frobenius relation, in tree form.
Two consequences, well, let me give an example, the homology of a manifold $H .(M)$ where you take the intersection pairing and the diagonal. Also, if a manifold is closed, has Poincaré duality, it's true on the cohomology of the closed manifold. These are examples. The point of the talk is, if this is the structure on the homology, then what is the structure on a chain or cochain complex that computes this homology. We know it's not reasonable to expect the exact same structure; there will be higher homotopies which are invariants of the space.

Remark.

1. The relations imply there exists a unique map $A^{\otimes k} \rightarrow A^{\otimes \ell}$ for any tree with $k$ inputs and $\ell$ outputs.
2. The dual of a finite dimensional open Frobenius algebra is an open Frobenius algebra. When one dualizes, the algebra and coalgebra relations turn around. The relations read left to right are the same as the relations read right to left.
Another question is whether a particular Frobenius algebra is isomorphic to its dual. The structure conceptually is dual whether or not a particular one is isomorphic to its dual.

It's a theorem that an open Frobenius algebra with a unit and a counit is finite dimensional. You can't do that on the differential forms of a manifold.

Okay, that's what I wanted to say for recalling the definition.
Now I want to construct a free resolution of this open Frobenius algebra.
For those that are familiar, let me start with an example. $A_{\infty}$ is a resolution of associative. Let's do the following. I'm going to construct a cubical cell complex, and every cube will be labeled by a tree colored black and white. These will fit together to form a dioperad. I'll try not to use that word. These will contract to a point and be free on a certain set of generators. The algebraic structure will be free on those generators.

For $k, \ell \geq 1 \mathrm{I}$ want to consider abstract trees with valence at least three, $k$ labeled inputs and $\ell$ labeled outputs, with internal edges having length between 0 and 1 . If a tree has an internal edge of length zero I identify this with a tree with that edge contracted. External edges have length $1 / 2$.

Every internal edge has some length between 0 and 1 . This space, the space of all such trees forms a cubical cell complex. Let me draw a picture that labels each cell of the cube. Then we'll see the identification. Let me draw a cube and label subcubes of it with certain trees. An $n$-cube is labeled by what I'll call a black and white tree. Every internal edge will be either black or white. On the board, more chalk means white, so that's white.

Let me say what that means. An $n$-cube will be labeled by a black and white tree with $n$ white edges. The white edges vary in length between 0 and 1 . The black edges have fixed length 1 . That's a realization of the space.

For instance, take three to one, $k$ to $\ell$. This is a cubical cell complex for $k=3, \ell=1$ This says that all the associators are homotopic, this space contracts to a point.

Let me just say that the differential can be easily understood in terms of black and white trees. Take the sum over all white edges of crushing the edge to zero, contracting it to a point, or painting it black.

Now let me talk about gluing trees. For gluing trees, I want to glue single edges. There's something going on in the middle, and I glue single edges. When I glue them I get a new tree with inputs and outputs and an induced labeling, and the glued edge has length 1 . This respects the boundary operator, which takes a white edge, crushing it to a point or making it a black edge. You can perform these operations on a white edge in one or the other factor, that's the derivation property.

I'm describing the operations of an algebraic structure abstractly. Let me list further properties. A pure white tree is a tree with no black internal edges.

1. Under composition, black and white trees are free under gluing on the pure white trees. When you compose things you make a black edge. You regard that as a composition. Cut along all black edges and you have a disjoint union of pure white trees whose composition is the original tree.
2. For each $k$ and $\ell$ the space of trees and associated cell complex is contractible. The homology is thus concentrated in degree zero. We have finite length edges between zero and one, and contract down to the corolla with no internal edges by homotoping all the internal edges to zero.
3. $d$ is triangular or minimal. This means that if you take $d$ of a generator, a pure white tree, you get a collection of terms all of which involve pure white trees earlier in the partial ordering. There will be a nonlinear term and a linear term. The nonlinear term will be a gluing of pure white trees with fewer white edges and the linear term will have fewer white edges.

By my first remark, saying what the operations were in an open Frobenius algebra, I get a free resolution of the open Frobenius algebra structure. I need a map from these things to the operations of the Frobenius structure. That property helps make inductive constructions.

So this brings me to the third part. I want to realize this structure on the differential forms of a closed oriented manifold. To every tree we will assign an operation on forms compatible with the composition and $d$. That will be an action of this structure on forms. The first part we know very well. The one output part we know from our first course of geometry of manifolds. The differential forms on a manifold form a dga. I want to talk about the coalgebra structure. So let me say the coalgebra. I am going to give a map $\Omega(M) \rightarrow \Omega\left(M^{2}\right)$. There's a lot of interesting stuff here. Take the diagonal sitting inside $M \times M$. Then there are two projection maps $\pi_{1}$ and $\pi_{2}$. One of the great properties of being a manifold is that in any regular neighborhood of a cycle is a dual cocycle. If I choose a tubular neighborhood $U$ of the diagonal (which contracts to the diagonal), then I'll put the dual cocycle, an $n$-form,
and I'll call it a Thom form on $M \times M$ with support on a neighborhood of the diagonal. De Rham made a nice construction of this, regarding the diagonal as a current and diffusing it to make it a form. I have this form, and again, it's not unique, but all of the choices are cohomologous. This is a closed form, $T$ is a cocycle.

The fact that you have a degree $n$ form which is closed will give you a degree $n$ chain map by multiplication. If I have $\omega$ on $M$ I could look at $\pi_{1}^{*} \omega \wedge T$ or $\pi_{2}^{*} \omega \wedge T$. These two operations are homotopic. The reason is from the properties I've listed here. If you restrict the projection maps, well, $U$ is contractible to the diagonal where the two maps are the same map.

I will use this sort of construction over and over again. You might say there are some choices there. So up to homotopy I've taken care of the ambiguity. So maybe if I were more of a homotopy theorist I would build up more homotopies. In this abstract algebraic thing I'm trying to construct I'm going to get a cocommutative coproduct. I can make one of these skew-symmetric things once and for all, symmetrize and proceed. Similarly, one can define all of the one to many corollas by pulling back, taking a Thom form, and pull back a form along all the factors and wedge with the higher Thom form with support near the diagonal.

So I just, I guess that's my last board. Let's consider coassociativity. Let me draw a picture first. This has something to do with the original thing that involves compositions.

## [Picture]

The idea is that if we compute in two different ways, in the interest of time I won't identify every number with a factor, one will be a first coproduct on the third factor and another factor. Computing both of these will not literally agree, but agree near the diagonal and are supported near the diagonal. I can integrate out the contracting homotopy to the diagonal.

I used the same arguments to form the homotopy. Everything is built up out of chain maps that agree on the literal diagonal and is supported nearby. Inductively I can look at a pure white tree, which labels a cell, which has some boundary. I have the operation defined on the boundary, and each of these is made up of something, Each homomorphism agrees when restricted to the diagonal. I can cross with the interval and build an operation with each $t$. As $t$ goes to one I'm contracting to the diagonal.
[Is the wedge product a special case?]
That sits inside of this.
[I want to make sure I understand. You are freeing up the coassociative part only.]
I am also resolving the Frobenius relation.
[Grading?]
The degree of a cell is the dimension of it, which goes down. I would flip the complex to get something that respected degree.

