Postnikov Conference

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1 Dennis Sullivan

Can you hear me in the back, Ralph? How about now? Okay.

[Yeah, can you hear me?]

I prepared a lecture as a handicap because you have to follow it. I want to get right into it, the first statement is that the basic elements of topology, homotopy theory are the Eilenberg MacLane spaces, spaces with one nonzero homotopy group.

Any space can be built up to homotopy type out of these, using the Postnikov system. I want to discuss an algebraic analogue to this. I used the word geometry, and this talk will mostly be about algebra. I hope at the end to get to one geometric application.

The first statement is that the Postnikov system allows you to take the basic spaces of algebraic topology and out of them construct any space. This will be like resolutions of an algebra.

Suppose you have an algebra A. Now the notion of algebra will vary and become quite general. Let's say this is a graded commutative algebra possibly with differential, called a dga. Is that too small for the back? Then a resolution is, you try to construct, well, a free algebra on a vector space mapping into A which is included in a free algebra on more things, and so on.

It's very easy to construct a resolution. I eventually want this to be an isomorphism on homology. So you could start with the vector space being on homology representatives. Then you need to add new generators to kill the relations to start making things injective. You get an increasing union of free algebras, and eventually the induced map will be an isomorphism.

On the other side, the Postnikov system, we have these building blocks. Let me talk about a Postnikov model. I want to think of an algebra as being like a contravariant functor that you have applied to spaces. You write it the other way around and you get spaces and maps like

$$\begin{tikzcd}
{} \arrow[d] & \\
X(\Pi_1, \Pi_2, \Pi_3) \arrow[d] & \\
X(\Pi_1, \Pi_2) \arrow[d] & X \arrow[l] \arrow[ul] \arrow[uul] \arrow[dl] \\
X(\Pi_1) &
\end{tikzcd}$$

These are all fibrations with fibers products of Eilenberg Maclane spaces. Rationally these have free graded commutative algebras as cohomology. This picture then corresponds exactly to my algebraic picture. That's the first remark.

The point of this talk, let's say, is that this analogy is quite nice if you work with rational coefficients. The question I had was, is there a type of algebra corresponding to Postnikov systems over $\mathbb{Z}$? Michael Mandell has work half-solving this problem. The maps of algebras are more general than maps between spaces. If you tweak his work you might be able to solve this problem. So now for the rest of the talk I will work with rational coefficients.

So what can you learn by going both ways with this idea? Key points:

- Starting with the idea that the Postnikov system is like a free resolution of an algebra
- Resolutions with a d which has a triangular property
- There is a notion of homotopy between dga maps from free algebras to general algebras. The notion can be defined in general but I don't know what it means for a general algebra. It includes chain homotopy but it has to respect the non-linear structure. If you have a free algebra and some other map of algebras $A \to B$ which induces an isomorphism on homology, then a map $F \to B$ has a lifting to a map $F \to A$ unique up to homotopy. This gives homotopy equivalences of free algebras. Then given two resolutions of an algebra you can show that they are homotopy equivalent. The homotopy equivalence is unique up to homotopy.

The idea is to use intuition from homotopy theory to study algebras and then extract some geometric information.

The existence is easy and uniqueness is hard but doable. Let's see. There's another point. In the analogy with spaces, everything has positive degrees and then you have an augmentation in degree zero. So you have augmented algebras, which I'll use in an important way. You have augmented algebras, like choosing a basepoint.

There's a little computation that some of you may have seen thousands of times. These are free algebras, augmented, and so then I have indecomposable generators. I can think of my algebra as made out of these and then ways of combining them two at a time, three at a time, and so on. Then d is a derivation, so it is completely determined by what it does on its generators. So $d = d_1 + d_2 + \ldots$ and you have the equation $d \circ d = 0$. Here d_i lands in the ith product of the indecomposables.

If you expand out this equation $d^2 = 0$, forgetting higher terms, you get three equations

$$d_1^2 = 0 \tag{1}$$
$$d_2 d_1 + d_1 d_2 = 0 \tag{2}$$
$$d_2^2 + [d_3, d_1] = 0 \tag{3}$$

So these say that d_1 is a differential, d_2 is a chain map $H \to H \cdot H$ and d_2^2 is zero in d_1 homology. You view this as a Lie cobracket, and $d_2^2 = 0$ is the coJacobi resolution. So this is indecomposable homology, and it already has a coLie algebra. On the algebra itself, you have a chain complex I, d_1 with a bracket given by d_2, and the Jacobi identity is homotopic to zero. So this gives an ∞ Lie algebra on (I, d_1). This is like a resolution of the Jacobi identity. There are more key points here

- an ∞ structure, true up to homotopy and higher homotopies necessary to make the whole thing contractible
- transfer of structure. So actually, this is a new part, there's not just d_2 here but there's also d_3 and d_4 and so on. These are derived from the information here. The ∞ structure can be transfered to anything else with a chain map. You get a bunch of tensors like this pulling over the invariant information. The homotopy type of an algebra is defined by the homotopy type of the resolution.

Postnikov systems are from the fifties, Quillen had some of this algebra in the 70s, and the ∞ version came around in the 90s. Quillen and Kontsevich are the main contributors, using ideas of Stasheff. I remember that hearing once the seed of algebra is sewn, it flourishes. I can do this to any algebras, take free resolutions of Lie algebras. If you start there you get something in terms of commutative algebras. If you know about spaces you are not surprised by the Lie structure, this is like the Lie structure on homotopy groups. We can apply this to other kinds of algebras. There's something new in the Lie case. If you have grading in negative degrees, augmented, you get something new which appears in geometry in J-holomorphic curves. We can apply this to any kind of algebra. You could do it to associative dgas, and you get something that might be called noncommutative homotopy theory. It will be interesting to work out the theory of these dgas.

Now once you have patterns you want to see if you can choose a type of algebra so that when you do this you get the homeomorphism theory of manifolds, and I have a conjecture about this. Can you choose a type of algebra so that manifolds would have this and homotopy theory of this type of algebra would be equivalent to homeomorphism type of the manifold. So rationally this would include the Pontrjagin classes, which are the main new piece of information.

There is a reasonable candidate, an algebra with commutative associative product and cocommutative coassociative coproduct and then a compatibility between multiplication and comultiplication (Frobenius compatibility). The homology of an open manifold has this structure.

Let's think about the homotopy theory of such a thing. We're stopped right away because we have to take a free resolution, and I don't know any such thing as a free algebra with a multiplication and a comultiplication. So you need a new approach. Let me indicate a couple of other, there is an interesting structure called Lie bialgebra and an interesting structure called Hopf algebra, and these have their own compatibilities between multiplication and comultiplication.

These have two generators and some quadratic relations. To have a structure corresponding to one of these on a vector space means that you can take $Hom(V^{\otimes k}, V^{\otimes \ell})$, any algebraic structure is an algebra, you give generators and relations. You don't combine them just typing them next to each other, you still get relations, you just have to glue them to one another. If you take any algebraic structure you can take a resolution, getting a free algebra, a definition with a differential, which you can call the ∞ version of your structure. You can talk about free Frobenius, which is a differential graded algebra resolving these equations. An ∞ open Frobenius algebra is something like this. You can resolve Lie Bialgebra and Hopf algebra. You can take a resolution as a differential graded algebra. Now you can use transfer of structure, how does that come about?

I'm not going to get to the geometry. This is a pure algebra talk, I guess. So let me suppose I have two complexes V and W and they're quasiisomorphic in the naive sense. I want to define a special notion of quasiisomorphism. Take free resolutions of $Hom(V, d)$ and $Hom(W, d)$. Then a special quasiisomorphism is a homotopy equivalence of these free things inducing isomorphisms on homology. If you have an open Frobenius structure on V, d, the definition of equivalent structures defines things only up to homotopy. You can lift this $F(Frob)$ action and then transport it across the homotopy equivalence and then push it down to W. I've told you how to transfer structures.

So, uh, this is how you do the homotopy theory. The other things that arose before, like Lie_∞, you have one generator and the Jacobi relation, and then the algebra down below is simpler. This idea is already in the literature for algebraic structures which only have one output.

So now I can state the conjecture, that you should be able to construct for manifolds open Frobenius structures. Hopefully when you compress these it's just to the homology. Then you get certain invariants and hopefully you see the Pontrjagin classes inside there. Starting with Frobenius you might think you'd get a Lie bialgebra structure as a result. These structures also appear in contact homology. Some people here know about that.

Now I do have time to indicate the geometric example. Let me say one more thing. The first two examples are in terms of trees, but for Hopf you have to use graphs, and the formalism of Feynmann graphs may come in, with higher loops and so on. So now let's do the key points. Let me indicate one geometric example. Take M^{2d} a symplectic manifold with J-

holomorphic curves for some complex structure compatible with this. In some, in integral second homology this picks out a set of points, a sharp cone (slices are compact), those represented by J-holomorphic curves. Then you can form a coalgebra in this cone, looking at all the lattice points, if β is in the cone then $\Delta\beta = \sum \beta' \otimes \beta''$ where $\beta' + \beta'' = \beta$.

Now look at $S^2 \to M^{2d}$, which is acted on by SO_3, and then I let C_*, H_*, H^*, C^* be the equivariant versions with respect to this action.

If you take a couple of chains of spheres, then you can take the Cartesian product and get pairs of spheres. Then you can intersect these in places. You take the coincidence of this family and you get a circle's worth of gluings. You get an approximate map $C_* \otimes C_* \to C_*$, at least where things are transversal. This satisfies the Jacobi, so this extends to the ∞ version and we get a differential. This thing $\wedge^c C_*, d$ compresses onto $\wedge^c H_*, d$. This hasn't been computed in any example, but it's there. Now the interesting step is that each of these βs in this cone, each one you can take a moduli space of J-holomorphic curves. On the boundary of the chains, things are noncompact by splitting in a sort of inverse. If you cut these off the boundaries are described by these relations, like

$$\partial\beta \sim \sum [\beta', \beta'']$$

Then it turns out, if you take the cone coalgebra, given a coalgebra you can form the free Lie algebra on it, and the diagonal becomes a differential, so if you take this, it's generated by the free generators here so you can add this, uh, well, this is the equation, this has a differential given by this equation, this is called the bar construction, and the moduli spaces give you an approximate map of this into the chains, and the way you make it into an honest map is to go into the ∞ version of the Lie algebra, but you've got to apply this free coalgebra construction again. You use this formula to correct the mistakes in the geometry. This is like taking the classifying space of a loop space.

If you dualize this arrow, you get a free algebra on the cohomology with a derivation $\wedge(H^*, d)$ mapping to the cone coalgebra dual. This is something like a polynomial thing dualized so it's like a power series ring. It's including all the information of the J-holomorphic curves. The homology level gives the quantum cohomology ring, but this gives more information, this gives the homotopy theory. You land in the Novikov ring, which is a very interesting place. We got to the geometric application and that's all. Thank you.

2 Higher Clifford Algebras

Today I'll tell you about joint work with Douglas and Bartels. What I'm going to tell you today came from a conjecture of Stolz and Teichner, that, you can make a 2-category with objects algebras and morphism bimodules. There is a fancy version of that, where the objects are von Neumann algebras. Let $vN2$ have objects von Neumann algebras. Morphisms are bimodules, I'll tell you which, and the 2-morphisms are maps of bimodules. The interesting part is composition of morphisms. In the version where you have algebras, you take the tensor product over the algebra of the middle. In this version you have the fusion tensor product.

Then there exists an interesting (monoidal) three-category such that $Hom(1,1) = vN2$

Phrasing this thusly is a little vague. You could do it with just one object and have $Hom(1,1)$ be $vN2$. I said it had to be interesting. I decided to state it this way because this is how I thought about it.

Now I should say both what $vN2$ is and motivation.

What is a von Neumann algebra? It is a $*$-algebra over $\mathbb{C}$ (this means it has an involution $*$ such that $A^*B^* = (BA)^*$ which is complex antilinear. Also $A \hookrightarrow B(\mathscr{H})$ so it injects into bounded functions on a Hilbert space. It should be closed in a certain way, namely it's closed under "defining". This means that whenever I have a bunch of elments of A and an element of $B(\mathscr{H})$ which can be defined using elements in A then it is in A.

Unlike C^* algebras, those have an interesting representation theory, but Von Neumann algebras have a boring one. A module is a Hilbert space with such a map. The theory is not interesting because whenever I have two modules, faithful, then if I take an infinite direct sum of either side then they become isomorphic.

What does one do with these algebras? Maybe I should give an example, $L^\infty(X)$. That's a Von Neumann algebra. This can be thought of as noncommutative measure theory. With measure theory one studies L^p spaces. So a good thing to study is $L^p(X)$, in particular $L^2(X)$.

The remarkable thing which is a hard theory by Connes and [unintelligible]is that there is an analogue of these guys for noncommutative measure spaces, $L^p(A)$.

In this talk, I'm saying that given A there is a canonical Hilbert space $L^2(A)$ which is its preferred representation.

Now I need to tell you about the fusion tensor product. Let's say I'm defining $H \otimes_A \bullet$. This is the unique additive functor such that $H \otimes_A L_2(A) = H$. This says that $L^2(A)$ should act like the identity for the tensor product. Since the representation theory is dull, you can take any module as a sum of such modules and this tells you what the tensor product should be.

Let me say a few words about motivation, which comes from elliptic cohomology. More specifically I'm talking about topological modular forms. TMF is something that bears a lot of resemblances to (real) K theory. Let's just say that this has been called "higher K-theory" because of those resemblances. The title of the talk reflects on this terminology. I'm looking for the objects fulfilling the role in this theory that Clifford algebras do in K-theory.

Miller and [unintelligible]made this algebraically. Stolz and Teichner have been looking for geometric cocycles that represent TMF cohomology, like the vector bundles for K-theory.

This analogy for K-theory, that this should behave like K-theory, well, let me make a diagram:

K-theory	Elliptic cohomology
$\cdot \mapsto$ vector space	$\cdot \mapsto$ von Neumann algebra
paths $\mapsto$ maps of vector spaces	paths $\mapsto$ bimodules
	surfaces$\mapsto$ maps of bimodules.

So that's just K^0. What

about K^n and Ell^n? There is this idea of Atiyah-Bott-Shapiro that you should look at $Cliff(n)$, the algebra generated by elements e_i satisfying $e_i^2 = 1$ and $e_i e_j = -e_j e_i$

Instead of bundles of vector spaces, look at bundles of Clifford moduls, and let paths correspond to maps of these.

What kind of categorical structures can we use to move over to Ell^n?

So K^0 has as its category $Hilb1$, and Ell^0 is controlled by $vN2$. Now because Clifford algebras are special types of algebras, K^n is also controlled by $vN2$. The relationship between $Hilb1$ and $vN2$ is that $Hom_{vN2}(1,1) = Hilb1$. That is the relationship between these two guys, so you want to find a three-category with $Hom_{??3}(1,1) = vN2$.

There is the following interesting feature about Clifford algebras, that inside $vN2$ we have, if I take $(Cliff(1))^{\otimes_{\mathbb{R}} 8} \cong 1$. This statement about a Clifford algebra to the eighth is a trivial object, that implies Bott periodicity for K-theory. So K^8 will be the same as K^0. This leads to the following conjecture. Whatever the higher Clifford algebras are, this TMF is periodic with period 576. So a higher Clifford algebra should have its 576th tensor product equal to 1

[Here is a fact about TMF. $TMF(pt)$ is very close to modular forms over $\mathbb{Z}$ which is like $\mathbb{Z}[c_4, c_6]$ in degrees 8 and 12, and then I adjoin $[\Delta^{-1}]$ where $\Delta = \frac{c_4^3 - c_6^2}{1728}$. After I tensor with $\mathbb{Z}[1/6]$ I get this equality, and there is complicated information that is only exactly true modulo 24^2.]

The answer is Conformal nets. It is one of the few possible mathematically rigorous frameworks for conformal field theory. There is Segal's approach, what I call a Segal CFT. This is a maximalistic approach, you put everything that you want in. There are two other approaches. One is called vertex operator algebras and the other is called confromal nets. These two approaches are minimalistic. You remember just as much as you need so you don't forget everything, but they are done with very different methods. In my work I have been using conformal nets. I check my work by asking people in vertex operator algebras what they get.

A conformal net is a Hilbert space H_0 with a positive energy representation of $\widetilde{Diff(S^1)}$ equipped with a special vector Ω (the vacuum vector) such that $\mathbb{C}\Omega = H^{PSL_2(\mathbb{R})}$. Positive energy means I only see winding in one direction. I also need an assignment from closed intervals of S^1 to Von Neumann subalgebras of $B(H_0)$. A closed interval is what you imagine it is. This should preserve the structure, the lattice structure when it's defined and the poset structure. There is also an involution, which I take as commutants (by taking all things which commute with the von Neumann algebra). The central element will act trivially on a von Neumann subalgebra, and so both sides have an honest representation of $Diff(S^1)$.

I've told you the most important axioms. There are others that are not always included. If I have two disjoint intervals, then their algebras commute. Therefore, if I look at the tensor product of these, that's a subalgebra of $B(H_0)$. That's only true for the algebraic tensor product, not the completed tensor product, because there are many possible completions. The axiom is that the completed tensor product is a subalgebra. This is analogous to being orthogonal.

So now I have told you all of the axioms. I should give you some examples. Whenever you have a compact Lie group and a [unintelligible]then you get a conformal field theory. I'll give you the conformal net. Let me tell you a level for a loop group. If I have a compact Lie group G with trivial center for simplicity, then I can look at the loops LG which has an interesting central extension, universal, denoted $\widetilde{LG}$. The center will be an S^1. A level is a character for the center of $\tilde{LG}$.

$$\begin{tikzcd} 0 \arrow[r] & S^1 \arrow[r] \arrow[d] & \widetilde{LG} \arrow[r] & LG \arrow[r] & 1 \\ & S^1 & & & \end{tikzcd}$$

So given a group and a level there is a unique Hilbert space H_0 such that $\mathbb{C}\Omega = H^{PSL_2(\mathbb{R})}$, which will be the Hilbert space of my net. Now I consider, start with an interval, take the variant L_IG, where you take $\gamma \in LG$ supported in I. That's a subgroup, so I can restrict my representation and define $A(I)$ to be the von Neumann algebra generated by $\widetilde{L_IG}$. It is nontrivial to check the axioms, especially the involution one. It is nontrivial to see that it is the full complement.

Okay, let's see. Maybe I should give another example, namely the one that will come into play here. This is the free Fermion, which I will call higher Clifford. How is it defined?

Consider the Hilbert space $L^2(S^1)$. You need a vector space and an inner product to make a Clifford algebra. So now we can take $Cliff(L^2(S^1))$ and take H_0 to be the "basic" representation of this Clifford algebra. Let me tell you a little bit more. It's constructed as a Fock space. You take the exterior algebra on $L^2(S^1)_+$ and then make a Hilbert space. Then you look at particular subalgebras corresponding to intervals. You get a subalgebra $Cliff(L^2(I))$. I can define $A(I)$ to be the von Neumann algebra generated by this Clifford algebra.

I should maybe close with a few remarks. I haven't told you anything about morphisms, and that's too bad, but one can still ask, this particular conformal net, what about the order of it? One encouraging fact is that this has an invertible object and the order of the invertible object is divisible by 24.

3 Ono, Floer cohomology and symplectic fixed points

Let me apologize that I sent the wrong abstract. I wanted to talk about the Floer theory and symplectic fixed points, but I sent the wrong abstract.

First of all I want to remind you of some notation. In my talk, (M, ω) denotes a closed symplectic manifold, and I denote by $Symp(M, \omega)$ all symplectomorphisms, φ a diffeomorphism so that $\varphi^*\omega = \omega$. If you consider the identity component of this group, this contains the Hamiltonian diffeomorphism group $Ham(M, \omega)$. Being in the identity component means there is a symplectic isotopy to the identity. What is $Ham(M, \omega)$?

Well, φ_t, a family of symplectomorphisms, gives a family of symplectic vector fields X_t with $\mathscr{L}_{X_t}\omega = 0$ and $i(X_t)(\omega)$ is a closed one-form.

φ is Hamiltonian if there is a path to the identity such that $i(X_t)(\omega)$ is exact.

[oops! I was too tired!]

4 Tradler

Well, I guess, thank you to the organizers for inviting me. Basically I'm going to talk about some of thet things I have thought about in the last several years, so string operations in an algebraic setting. I should probably say that A lot of this is in collaboration with Zenalian. I want to refer to his talk later this week where he constructs some kind of CFT.

The first thing I want to talk about is like elementary, which is the BV algebra on Hochschild cohomology. You start with an associative algebra with unit, graded, and then there is the Hochschild complex you can define $CH^\cdot(A, A) = \prod_n \geq 0 Hom(A^{\otimes n}, A)$. More generally, if M is a bimodule over A, then you can build $CH^\cdot(A, M) = \prod Hom(A^{\otimes n}, M)$. There is a differential that many of you probably know,

$$(\delta f)(a_1, \ldots, a_{n+1}) = a_1 f(a_2, \ldots, a_{n+1}) \pm \sum f(\ldots a_i a_{i+1} \ldots a_{n+1}) \pm f(a_1, \ldots, a_n)a_{n+1}$$

So $\delta^2 = 0$ and you have the homology $HH(A, M)$. So you have a BV structure here, starting with the cup product

$$f \cup g(a_1, \ldots, a_{n+m}) = f(a_1, \ldots, a_n)g(a_{n+1}, \ldots, a_{n+m})$$

and $\cup$ respects δ so it descends to homology.

There is a Δ operator on $CH^\cdot(A, A^*)$ where the bimodule structure is $a_1 a^* a_2(a) = a^*(a_2 a a_1)$. The Δ, if $y : A^{\otimes n} \to A^*$ is

$$\Delta f(a_1, \ldots, a_{n-1})(a_n) = \sum f(a_j, \ldots, a_n, a_1, \ldots, a_{j-1})(1)$$

So Δ respects δ and $\Delta^2 \cong 0$. Let F be a bimodule map $A \to A^*$, and you can say $\langle , \rangle : A \otimes A \to k$ or you can say $F^{-1} : A^* \to A$ or $\langle , \rangle : A^* \otimes A^* \to k$ which can transfer Δ to $CH^\cdot(A, A)$. Then the Δ operator is

$$\langle (\Delta f)(a_1, \ldots, a_{n-1}), a_n \rangle = \sum_{j=1}^{n} \langle f(a_j, \ldots, a_{j-1}), 1 \rangle$$

Now you can compare Δ with $\cup$ and $\Delta(f \cup g) \pm \Delta f \cup g \pm f \cup \Delta g \pm \{f, g\} \cong 0$, where $\{ , \}$ is the Gerstenhaber bracket. So on $HH(A, A) \cong HH(A, A^*)$ you get a BV algebra.

That's the first thing I wanted to get to. This is not the general case because in general you don't get the relations strictly. If X is a formal Poincaré duality space (K ahler manifolds or spheres or Lie groups) and $A = C^{\cdot}(X) \cong H^{\cdot}X$ and we get $HH^{\cdot}(A, A)$ is a BV algebra. There is also the connection with string topology, that $HH^{\cdot}(A, A^*) \cong H_{.}(LX)$ which gives a BV algebra on the free loop space of X. This is rationally the same as the one from string topology but there are some subtleties with torsion.

I want to make a graphical representation of this. If I look at $CH^{\cdot}(A, A^*)$ this is $\prod Hom(A^{\otimes n}, A^*)$ then you get in there elements of $(A^*)^{\otimes n} \otimes A^*$. You think of these as sitting on a circle, and it applies a coproduct to each of these terms.

The Δ inserts a new last element in all possible ways and you evaluate it on 1

The cup product glues together along last elements, and construct a new last element with these two by means of $F \circ \times \circ (F^{-1} \otimes F^{-1})$.

You have more general such things. You can glue along any pair of things, how do I read off the new thing? I decide where to start, reading a word off by looking at the description looking at the graph, and you can get operations like $CH(A, A^*)^{\otimes 3} \to CH(A, A^*)^{\otimes 2}$. If you let these vary a little bit, you get

Theorem 1 *Chains on the ribbon graphs with inputs, outputs, and and markings on Hochschild cochains of A over A^*.*

[Much discussion]

If X is a formal Poincaré duality space then chains on ribbon graphs act on $CH^{\cdot}(A, A^*)$ where $A = C^{\cdot}X$

How can we generalize to a homotopy version of this? I will relax having a strict bimodule map. The two main ingredients we have are the product $A \otimes A \to A$ and the unit $1 \in A$, and then the coinner product in $A \otimes A$. I can be infinite dimensional or whatever, I just use an inner product like this. These are the two things I want to generalize. Once I do that I will get a homotopy version.

So if I resolve the multiplication I get A_∞. To resolve the coinner product, I get $A \otimes (A^*)^{\otimes k} \otimes A \otimes (A^*)^{\otimes \ell}$. Then the appropriate relation is that the sum of applying the A_∞ structure all around this zero, so I would say $D(F) = 0$.

So the graphs change now in the following way. We get a direction on these graphs. We get the following action. Let's say I have three inputs with data on where to start. So graphs with either two outgoing or one outgoing act on this. There is a very nice construction of this for any manifold, for any Poincaré duality space, there exist D and F which are homotopy coinner products and you get local maps of that form. Therefore if $A = C^{\cdot}X$ then there is an action of these new directed ribbon graphs, chains of these on $CH^{\cdot}(C^{\cdot}X, C_{.}X)$. I want to say one last thing. There is an equivariant version of this. If you don't assume you have markings here, then you can instead define an equivariant version on the cyclic Hochschild complex of A.

There is an action of chains of directed ribbon graphs without markings on the cyclic Hochschild chain complex of A but it turns out that you can define a gluing between two cyclic complexes, the analogue of the Lie bracket, this satisfies Jacobi up to homotopy. You can do a further generalization. Put all of these together, with one generator for each cyclic ordering of inputs and outputs, and this is V_∞. Ribbon graphs for all types resolves biLie.