Moduli Space of Riemann Surfaces

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June 7, 2007

1 Penner

Recall that chc gives a decomposition $\Delta(\tilde{\Gamma})$ of F_g^s for $\hat{\Gamma} \in \mathscr{T}_g^s \times \mathbb{R}_+^s$ into ideal polygons. In a square divided into triangles abe and cde, let

$$E = \frac{a^2 + b^2 - e^2}{abe} + \frac{c^2 + d^2 - e^2}{cde}.$$

Then $abcdE = 2\sqrt{2}$ signed volume of tetrahedron. This is positive for e "on the bottom."

I apologize for the change in order, it's because of the logical dependence between my talk and Alex Bene's.

I described yesterday the convex hull construction that gave a decomposition. There was a formula, the simplicial coordinate formula. I left something out, I needed to say that this volume is a signed volume.

We have a way of assigning an invariant, and the natural thing to do is to solve the equation. Given a decomposition, let $\bar{\mathscr{C}}(\Delta)$ be the set of $\tilde{\Gamma}$ such that $\Delta(\tilde{\Gamma}) \subset \Delta$ up to homotopy.

Here's the main theorem.

Theorem 1 *We had global coordinates last time. This time it's local coordinates.*
$\bar{\mathscr{C}}(\Delta)$ is a cell parametized by nonvanishing simplicial coordinates ≥ 0 with no vanishing cycles. To each edge of the cell decomposition I assign the simplicial coordinate, and some can vanish but entire cycles cannot vanish. On the dual fat graph there is no cycle all of whose coordinates vanish

This is a hard theorem. Let me say:

- That simplicial coordinates be ≥ 0 with no vanishing cycles is already necessary, but it is also sufficient.

- This condition implies that there are unique λ-lengths realizing the putative simplicial coordinates. There really is a point in decorated Teichmueller space realizing this. This is hard, I have to solve a variational problem.
- If $\Delta' \supset \Delta$ where Δ is an ideal triangulation, the simplicial coordinates on Δ' are uniquely determined. In short, the simplicial coordinates play the role of the Strebel coordinates.

The proof is rather involved. This is the statement that there is a mapping class invariant cell decomposition. Let me state that as a corollary.

Corollary 1 *$\{\bar{\mathscr{C}}(\Delta) : \Delta$ is a decomposition into ideal polygons$\}$ is an MCG_g^s-invariant cell decomposition of $\mathscr{T}_g^s \times \Delta^{s-1}$. By scaling all the horocycles, that corresponds to scaling all the λ-lengths, and the cell decomposition is invariant under this homothety.*

I will do an example. For the once-punctured torus, the Teichmueller space is a disk, and the disk allows a fairy tesselation, and this is invariant under the mapping class group, and my decomposition is the fairy decomposition for the once-punctured torus. The torus is unimportant, it will work for any punctured thing.

All right, what I would like to do is sketch some applications of this.

I. A combinatorial presentation of MCG_g^s. Define a labeled ideal triangulation to be an ideal triangulation Δ to be an enumeration of edges up to $6g - 6 + 3s$. Define the Ptolemy groupoid to have as objects MCG orbits of labeled ideal triangulations, and the morphisms pairs of labeled ideal triangulations modulo the diagonal action. It's clear actually that the mapping class group is the set of self-morphisms of any object in this category. The interesting thing is to use the cell decomposition to give a presentation.

   Theorem 2 *The Ptolemy groupoid has presentation as follows: The generators are transpositions (i, j) for pairs of labels and labeled Whitehead moves, meaning that when you do a Whitehead move, the label carries over. The relations are those of the symmetric group plus three others.*

   - *Involutivity, doing the labeled Whitehead move twice on the same edge is the identity, $W_i \circ W_i = id$.*
   - *if i and j don't share an endpoint then W_i and W_j commute.*
   - *Pentagon relation: if they do share an endpoint, then $W_i W_j W_i W_j W_i = (ij)$.*
   - *Naturality: $\sigma \circ W_i = W_{\sigma(i)} \circ \sigma$*

   Let me indicate how these generate. Take ideal triangulations and a path between them in general position. The codimension one faces look like Whitehead moves. The relations arise from codimension two faces. Nulhomotopies can be put in general position relative to codimension two. There's more work but that's the nature of the proof.

II. Integration over moduli space, instead of integrating over a fundamental domain, integrate over one cell in each orbit. This requires enumerating triangulations, and happily there are matrix models from higher energy physics that let you do this in closed form.

III. Torelli Johnson Morita theory. This is what I'll describe this afternoon.

IV. Cluster algebras

V. quantum Teichmueller theory

Here's what I'd like to spend the rest of the time discussing, the following question. Given a cell $\bar{C}(\Delta)$, to which stable curves is it asymptotic? What are the asymptotics of the cell decomposition? The answer depends on a very basic fact that I'm about to state

Lemma 1 *Fix a triangulation Δ and suppose the simplicial coordinates are non-negative with no vanishing cycles. Then there are corresponding λ-lengths and these corresponding λ-lengths satisfy the strict triangle inequalities on each triangle complementary to Δ.*

The proof is a calculation, but it illuminates. Take the usual notation, and suppose that $c + d \leq e$. Squaring both sides we conclude that $c^2 + d^2 - e^2 \leq 2cd$. The coordinates are nonnegative, so $0 \leq E \leq \frac{a^2+b^2-e^2}{abe} - \frac{2}{e} = \frac{(a-b)^2-e^2}{abe}$. This is nonnegative, so it fails there, and you get a cycle of triangles of failure. This obviously telescopes, cancelling like terms, and 0 is greater than a sum of λ-lengths, but they cannot vanish in a cycle.

Let me put a sentence up. There exists a cycle of failures, which telescopes, absurd.

This is the guts, this silly little lemma, is the guts of the solution to the asymptotics. There are a few definitions and then I'll state the theorem.

Suppose G is a fat graph with set E of edges. Given $A \subset E$ there is a corresponding subgraph which inherits a fattening. So you get a possibly disconnected fat graph G_A. Then here is a key definition. Say that $A \subset E$ is recurrent if G_A has no univalent vertices. This is equivalent to saying, for each edge of G_A there is a closed edge path in G_A with no reversals.

A slightly wordier part of the definition. Following Fulton-MacPherson, let me define a screen on G to be $\mathscr{A} \subset \mathscr{P}(E)$ satisfying certain conditions:

- $E \in \mathscr{A}$
- If $A \in \mathscr{A}$, then A is recurrent.
- if $A, B \in \mathscr{A}$ then $A \cap B \neq \emptyset$ implies $A \subset B$ or $B \subset A$.
- for all $A \in \mathscr{A}$, the set $\cup\{B \in \mathscr{A} | B \subsetneq A\} \subsetneq A$.

Each $A \in \mathscr{A}$ other than E has an immediate predecessor A', and we have two fatgraphs G_A and $G_{A'}$, which have corresponding skinny surfaces. Consider the surface associated with G_A, that is a subsurface of $F(G_{A'})$. The first of these has a relative boundary in $F(G_{A'})$. Consider this, and let $\delta_{\mathscr{A}} A$ be the relative boundary of $F(G_A)$ in $F(G_{A'})$. Let $\delta \mathscr{A} = \cup_{A \in \mathscr{A} - \{E\}} \delta_{\mathscr{A}} A$.

Theorem 3 *(Joint with McShane)*
For any fat graph G, a cell $\mathscr{C}(G)$ is asymptotic to a stable curve with pinched curves K if and only if $K = \delta\mathscr{A}$ for some screen $\mathscr{A}$.

Let me give an example.

[pictures.]

Let me make a few remarks about the proof. I've talked about how a one-parameter family gives rise to a screen. The first step is to convince you that the relative boundaries are short. The procedure is really cool. There's a weakening of the triangle inequality invariant under suitable Whitehead moves. I can change the graph as long as there's only one left turn. This is like an upper triangular matrix almost, very easy to compute. This weakening is what you need to calculate the trace to find out that it goes to 2.

Now I have to convince you that there's no other short curves. It lives in some A and isn't a boundary component. Therefore it turns left and right at least once, and then all of these are comparable, and the cross-ratio is near one.

I have to convince you of the other implication, that all screens arise. Define the depth of a screen element to be the number of steps down, and the depth of an element of E to be its maximum screen depth. Let $\lambda_t(e) = t^{depth(e)}$. The maximum depth is not uniquely achieved at the edges near a vertex. You then have E strictly positive. A little more work is required for high codimension cells. I'm sorry I didn't get to tell you more about the proof.

This is a piece of a larger project to give a cell decomposition. Screens form a partially ordered set. Contract trees or put more subgraphs into the screen to give a partial ordering, and take the geometric realization.

The quotient of this poset realization by a finer equivalence than MCG, by symmetries of the graph as well. The strong version is, homeomorphic to a blowup of Deligne Mumford.

2 Penner II

Thank you again, I'm sorry to be speaking twice. I thought I'd start with Torelli-Johnson-Morita theory. Everything is joint with Morita.

[Slide] Let F_g be a closed genus g surface with basepoint $*$. Let π be $\pi_1(F_g, *)$ and $H = H_1(F_g, *, \mathbb{Z})$.

There is the lower central series $\Gamma_0 = \pi, \Gamma_{k+1} = [\Gamma_k, \pi]$. The kth nilpotent quotient is $N_k = \pi/\Gamma_k$. There is a central extension $0 \to \Gamma_k/\Gamma_{k+1} \to N_{k+1} \to N_k \to 1$.

Then $M_{g,*} = Homeo_+(F_{g,*})/Iso\ rel\ *$ acts on N_k. The kth Torelli group is $M_{g,*}[k] = \ker\ M_{g,*} \to Aut\ N_k$. So $M_{g,*} \supset M_{g,*}[1] \supset \cdots$ and in particular

$$1 \to M_{g,*}[1] \to M_{g,*} \to Sp_{2g}\mathbb{Z} \to 1.$$

Then $\varphi \in M_{g,*}[k]$ and $\gamma \in N_{k+1} \to \varphi(\gamma)\gamma^{-1} \in \ker N_{k+1} \to N_k$ so the central extension plus work gives the kth Johnson homomorphism $\tau_k : M_{g,*}[k] \to Hom(N_{k+1}, \Gamma_k/\Gamma_{k+1})$ where $\ker \tau_k = M_{g,*}[k+1]$.

Facts are (Johnson) for $g \geq 3$, $M_{g,*}[1]$ and $M_g[1]$ are finitely generated by explicit "torus BP maps" and $M_{g,*}[2]$ and $M_g[2]$ are generated by Dehn twists on separating curves. We know $M_2[1]$ is an infinitely generated free group (Mess) and $M_{g,*}[k]$ and $M_g[k]$ are not finitely generated for higher k (Biss Farb)

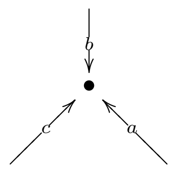
I've told you about an $M_{g,*}$-invariant cell decomposition of $\mathscr{T}_{g,*}$. This decomposition is invariant under $M_{g,*}[k]$ so we get $T_{g,*}[k+1] \to T_{g,*}[k] \to \dots$ where this $T_{g,*}[k] = \mathscr{T}_{g,*}/M_{g,*}[k]$, where this is a manifold Eilenberg MacLane space.

This is a nice picture, all of these are coherently triangulated by the triangulation I told you about this morning.

The kth Torelli groupoid is the fundamental path groupoid discretized by this triangulation.

Cells in these spaces are indexed by fat graphs with extra structure, namely, and to be concrete let me concentrate on the first Torelli group and space and so on, let me tell you the additional structure required.

It's rather simple, define a homology marking on a trivalent fat graph G to be a function $\mu : \{\text{oriented edges of } G\} \to H$ so that $\mu(-e) = -\mu(e)$ and so that

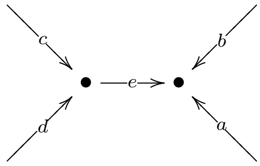


has $\mu(a)+\mu(b)+\mu(c) = 0$. If you have a fat graph embedded in the surface, it gets a canonical homology marking. If G is the spine of $F - \{*\}$ there is a canonical homology marking. Dual to an edge is an arc from $*$ to $*$, and I use the orientation of the surface to tell me the direction. Cells are indexed by trivalent fat graphs with homology markings.

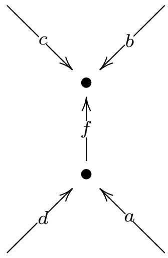
I'm cheating a little because not every homology marking arises.

I hope I've given you some sense of how cells in this tower are named. Instead of H I use N_k with the appropriate changes of group operation and inverse.

What is the effect of a Whitehead move on a homology marking? A Whitehead move acts on homology as follows:



Where $e = c + d = -a - b$, this moves to



and $f = a + d = -b - c$.

Here's the theorem, grand in its scope and boring because it's so easy.

Theorem 4 $M_{g,*}[1]$ *admits the following presentation:*
The generators are sequences of Whitehead moves beginning and ending on the same $M_{g,*}$*-orbit and preserving the homology marking.*
The relations are involutivity, commutativity and the pentagon, just as this morning.

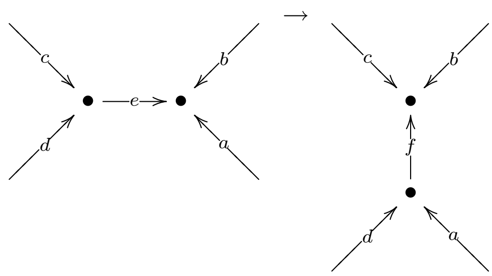
The proof is just by the same argument as this morning, general position.

Remark.

1. Likewise for presentations of $M_{g,*}[k]$.
2. This is a brutally inefficient set of generators so as to get a nice relation set. They enumerate with repeats $M_{g,*}[1]$.
3. A student of Farb, Andy Putnam, gives a more efficient presentation of $M_{g,*}[1]$ with nicer generators but crappy relations. The techniques don't extend to $M_{g,*}[n]$.
4. Using Johnson's work you can give richer finite presentations of the first two Torelli groupoids, the fundamental path groupoids of $M_{g,*}[1]$ and $M_{g,*}[2]$.
5. You can also use these techniques to give finite presentations of the level n Torelli groups.

I hope you see, this is a nice theorem because these are open problems, but it's a little deflating because they are easy. Let me move on to the first Johnson homomorphism τ_1. I alluded to this. It's convenient to consider the dual to the cell decomposition, $\hat{G}$. Let me jump right in and so a Whitehead move is actually an oriented 1-simplex.

Define a 1-cochain, which, when you see



Then j of this Whitehead move is $a \wedge b \wedge c = c \wedge d \wedge a \in \wedge^3 H$.

Theorem 5 *$j \in \mathbb{Z}^1(\hat{G}, \wedge^3 H)$ is an $M_{g,*}$-invariant cocycle. The associated group homomorphism $[j] \in H^1(\hat{G}/M_{g,*}[1], \wedge^3 H) = Hom(M_{g,*}[1] \wedge^3 H) = 6\tau_1$.*

You want to calculate the Johnson homomorphism, choose a sequence of Whitehead moves, kick off one of these wedge factors each time you do a Whitehead move, sum them up, and BAM! That's the Johnson homomorphism.

How about the proof. You have to check that the value is invariant under the choice of Whitehead paths. You sit down that j of the relations vanish, you sit and calculate them, so this is well-defined, so it is a cocycle and gives a homomorphism. Now step two is to calculate the value on a BP map. Let me just allude to this, it's good to show an example on the level of fat graphs, now is when you really have to pay the piper. BP-maps are made up of a bunch of Whitehead moves. Here's the BP map:

[slide]