# Moduli Space of Riemann Surfaces 

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## 1 Ravi Vakil <br> Introduction to the Moduli Space

[Tomorrow there is a reception after the first talk on the sixth floor. Without further ado we start the conference.]

I'd like to thank the organizers. There are many people here I'd like to thank.
I want to mention work that is joint with Ian Goulden and David Jackson. A want to set the stage for this work. I hope to get across the algebro-geometric point of view. For some of this material, there is stuff at http://wwww.amath.org/www/modspacecurves/

By virtue of the fact you're here is that you're interested in Riemann surfaces and how they vary in families.
$\mathscr{M}_{g}$ is the set of Riemann surfaces of genus $g$. It's a space, not really a manifold. Some prefer it to be an orbifold, others want it to be an analytic space with singularities, and some call it a stack. We want to use rational coefficients. Its dimension is $3 g-3$, with all dimensions algebraic and complex, so you might call this $6 g-6$. Then $\mathscr{M}_{g, n}$ has $n$ distinct ordered marked point. It has dimension $3 g-3+n$. This is not compact, but it has a Godgiven (Deligne-Mumford) compactification. This parameterizes certain nodal curves, like this [picture]. Here is a genus four nodal curve.

Which ones are allowed? To each nodal curve, one has a dual graph. Components become vertices labeled with their genus and marked points correspond to half-edges. Any unlabeled vertex is supposed to be genus zero. This is stable if any genus zero vertex has valence at least three, and any genus one vertex has valence at least one. Let's say connected too. Then $\mathscr{M}_{g, n} \subset \overline{\mathscr{M}}_{g, n}$ in nice ways. Then $\overline{\mathscr{M}}_{g, n}$ is compact, smooth, and stratified by topological curves or dual graphs. The codimension of a stratum is the number of edges in the dual graph.

As an example, the stratum corresponding to the generic graph is $\mathscr{M}_{g, n}$. If you call this graph $\Gamma$ then $\mathscr{M}_{\Gamma}=\mathscr{M}_{1,2} \times \mathscr{M}_{2,1} \times \mathscr{M}_{3,1}$. So each stratum looks like products of moduli spaces.

Let me write some natural maps, which being an algebraic geometer I call morphisms. If I have $\mathscr{M}_{g, n}$ I can forget a marked point. So $\mathscr{M}_{g, n} \rightarrow \mathscr{M}_{g, n-1}$. This extends to the compactification $\overline{\mathscr{M}}_{g, n} \rightarrow \overline{\mathscr{M}}_{g, \underline{n-1}}$. A second sort of map is $\overline{\mathscr{M}}_{g_{1}, n_{1}+1} \times \overline{\mathscr{M}}_{g_{2}, n_{2}+1} \rightarrow \overline{\mathscr{M}}_{g_{1}+g_{2}, n_{1}+n_{2}}$ and then ta third one $\overline{\mathscr{M}}_{g, n+2} \rightarrow \overline{\mathscr{M}}_{g+1, n}$.

Now $\mathscr{M}_{g}$ has interesting cohomology. I am going to use the Chow ring $A^{*}$ which maps $A^{*} \rightarrow H^{2 *}$. This is hopefully the same as cohomology. Let me describe some classes here Let's see what I can say first. There's the large $g$ behaviour and the fixed $g$ behaviour. Mumford's philosophy is that the topology of this space should be like the topology of the Grassmannian, $H^{*}\left(\mathscr{M}_{g}\right) \sim H^{*}($ Grass $)$. This thing on the right has beautiful topology that is easy to access and understand, say through Chern classes of line bundles or vector bundles. That's the philosophy on the left. Let me start with the part that is not relevant to my work, which is the Mumford conjecture.

Conjecture 1 Harer stability says that there are natural maps $H^{d}\left(\mathscr{M}_{g, n}\right) \rightarrow H^{d}\left(\mathscr{M}_{g+1, n}\right)$ and to $H^{d}\left(\mathscr{M}_{g, n-1}\right)$ whenever $d<\frac{g-1}{2}$. This respects the multiplication in the homology ring. So you can make sense of $H^{*}\left(\mathscr{M}_{\infty, \infty}\right)$. Mumford's conjecture is that this is generated freely by the $\kappa$ classes, which I am about to describe.

This was proved by Madsen, Weiss, and later work by Galatius. This also proves the stronger Madsen's conjecture.

I'd like to discuss what is happening at a finite level. Let me define some Chern classes on bundles on the moduli space of curves. So let me cook up some vector bundles. Over $\mathscr{M}_{g}$ is $\mathscr{C}_{g}$, the universal curve. This fits into a stack, which admits a line bundle of differentials, the dual of the space of tangent vectors, the relative differentials $\mathscr{C}_{g} / \mathscr{M}_{g}$. Then take the Chern class $\psi=c_{1}(\mathbb{L})$. Then we can push down $\pi_{*} \psi^{a+1}=\kappa_{a} \in A^{a}\left(\mathscr{M}_{g}\right)$ or $H^{2 a}\left(\mathscr{M}_{g}\right)$.

Every curve has a $g$-dimensional vector space of differentials (that is one way of defining genus) so we can define the Hodge bundle $\mathbb{E}$ over $\mathscr{M}_{g}$ whose fibers are differentials on the corresponding curve. Then $\lambda_{i}=c_{i}(\mathbb{E}) \in A^{i}\left(\mathscr{M}_{g}\right)$ for $i=1, \ldots, g$.

Let me define the tautological subring. It's a subring because any class you can think of geometrically is contained in it. It's very well behaved.

Definition $1 R^{*}\left(\mathscr{M}_{g}\right)$ is the subring generated by the $\kappa$ classes.

Any interesting class you can think of lies in this ring. The $\lambda$ classes, for example, are in this ring.

$$
\sum \lambda_{i} t^{i}=e^{\sum_{i=1}^{\infty} \frac{B_{2 i} \kappa_{2 i-1}}{2 i(2 i-1)} t^{2 i-1}}
$$

Now let me state another conjecture, dating from twelve or fifteen years ago.

Conjecture 2 Faber
This ring $R^{*}\left(\mathscr{M}_{g}\right)$ is like the cohomology of a complex projective manifold of dimension $g-2$.

Let me divide this into different pieces.

Theorem 1 Vanishing part (1997)
$R^{i}\left(\mathscr{M}_{g}\right) \cong 0$ for $i>g-2$ and $R^{g-2}\left(\mathscr{M}_{g}\right) \cong \mathbb{Q}$.

## Conjecture 3 Perfect pairing

The map $R^{i}\left(\mathscr{M}_{g}\right) \times R^{g-2-i}\left(\mathscr{M}_{g}\right) \rightarrow R^{g-2}\left(\mathscr{M}_{g}\right) \cong \mathbb{Q}$ is a perfect pairing.

## Conjecture 4 Intersection number conjecture

Let me write this down and then say a few things in words.

$$
\frac{(2 g-3+n)!(2 g-1)!!}{(2 g-1)!\prod_{j=1}^{n}\left(2 d_{j}+1\right)!!} \kappa_{g-2}=\sum_{\sigma \in S_{n}} \kappa_{\sigma} .
$$

For any $d_{1}+\ldots+d_{n}=g-2$ if $\sigma=(234)(1)(5)$ then $\kappa_{\sigma}=\kappa_{d_{2}+d_{3}+d_{4}} \kappa_{d_{1}} \kappa_{d_{5}}$.
Getzler and [unintelligible]showed that this is a consequence of the physical Virasoro conjecture. This is the unique solution to the recursion. Givental has proven this for $\mathbb{P}^{2}$. This is really a theorem. The combinatorial information is not really known yet.

I'd like to restate this in a way that is tractable to methods that are out there. This restatement is due to Faber as well. I'll need to introduce more classes and move to the compactification. I'd now like to talk about the tautological ring on $\overline{\mathscr{M}}_{g, n}$. As soon as we have marked points, there is another kind of class that we have. In the universal curve, you have $n$ distinct sections. You have the cotangent line at every point on the section. So define $\mathbb{L}_{i}$ which is the cotangent line at point $i$. Then I define $\psi_{i}=c_{1}\left(\mathbb{L}_{1}\right) \in H^{2}\left(\mathscr{M}_{g, n}\right)$.

## Definition 2

$$
\left\{R^{*}\left(\overline{\mathscr{M}}_{g, n}\right)\right\}
$$

for all $g$ and $n$ is the smallest set of subgroups containing $\psi_{i}^{a_{i}}$ and closed under pushforwards by the natural (gluing) maps.

I want to make a brief comment. This is about cohomology classes of the compactification. We can define $R^{*}\left(\tilde{\mathscr{M}}_{g, n}\right)$ where this is an intermediate compactification by restriction from $R^{*}\left(\overline{\mathscr{M}}_{g, n}\right)$. It's a proposition that this gives the first definition if you restrict to $\mathscr{M}_{g, n}$.

Why go to this trouble? The answer is that we can restate Faber's conjecture here in a cleaner way. Let me define an intermediate compactification, the moduli space of genus $g, n$-pointed curves with rational tails, $\mathscr{M}_{g, n}^{r t}$. Here the genus is all in one component. There's a version of Faber's conjecture part of which will imply the intersection number part of the conjecture for $\mathscr{M}_{g}$.

Remember that $\overline{\mathscr{M}}_{g, n} \rightarrow \overline{\mathscr{M}}_{g} \supset \mathscr{M}_{g}$ and the preimage is precisely $\mathscr{M}_{g, n}^{r t}$.

Now let me restate the conjecture. First, I should have said, thanks to the fact that taking monomials in $\psi$ classes tells you the ring. To take intersections, you only need to know pushforwards of monomials in $\psi$ classes.

Conjecture 5 Intersection numbers (take two)

$$
\pi_{*}\left(\psi_{1}^{a_{1}} \cdots \psi_{n}^{a_{n}}=\frac{(2 g-3+n)!(2 g-1)!!}{(2 g-1)!\prod_{j=1}^{n}\left(2 a_{j}-1\right)!!} \kappa_{g-2}\right.
$$

Let me summarize. Inside the cohomology ring of $\mathscr{M}_{g, n}$ or its compactification you get rings with particularly nice structure. If you let $g$ and $n$ go up these stabilize. Even for fixed $g$ you get (conjecturally) nice rings. To understand these rings you need to know about top intersections, which all come from these $\psi$ s which are easy, just first Chern classes of the only natural line bundles in sight.

Tomorrow I will talk about [a bunch of complicated stuff].
Thank you very much for your time.
[Questions?]

## 2 Zogrof

This lecture is elementary. I will explain how to find the determinant of the Laplace operator on a Riemannian manifold and give some examples. So, uh, let me start with a very elementary observation. Let $A$ be an $N \times N$ matrix. The determinant of the matrix is well-known. It is known that it is the product of the eigenvalues det $A=\lambda_{1} \ldots \lambda_{N}$. Now I want to give another presentation that can be used for infinite dimensional matrices. Assume that all eigenvalues are nonzero. Consider $\zeta_{A}(S)=\sum_{i=1}^{N} \lambda_{i}^{-S}$. The derivative of this function with respect to $S$ is

$$
-\sum_{i=1}^{N} \lambda_{i}^{-S} \log \lambda_{i}
$$

So $\exp \left(-\zeta_{A}^{\prime}(0)\right)=\operatorname{det} A$. This formula is actually the starting point in defining the determinant of an operator. This idea was first due to Minakshisundacam and Plejel (1949). They dealt with the Laplace operator on a Riemannian manifold.

So now let $M$ be a Riemannian manifold and denote by $d$ the exterior derivative operator on $M$. The Laplace operator $\Delta$ is $d^{*} d$, where $d^{*}$ is the adjoint of $d$ with respect to the Hodge star. This is

$$
\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{i}} \sqrt{g} g^{i j} \frac{\partial}{\partial x^{j}}
$$

where $g=\operatorname{det} g_{i j}$ and we have $g_{i j} d x^{i} d x^{j}$. This is self-adjoint in $L^{2}(M)$. The spectrum in purely discrete, because these are only compact manifolds. There is always an eigenvalue 0
from the constant function and all others are positive. So $\lambda_{0}=0<\lambda_{1} \leq \lambda_{2} \leq \ldots$ Then define

$$
\zeta_{\Delta}(S)=\sum_{i=1}^{\infty} \lambda_{i}^{-S}
$$

This converges for $R e S>\frac{\operatorname{dim} M}{2}$ and admits an analytic extension to the rest of the complex plane. As in the finite dimensional case, we can define the determinant of $\Delta$ as $\exp \left(-\zeta_{\Delta}^{\prime}(0)\right)$. What I want to mention here is that zero is a smooth point of the $\zeta_{\Delta}$ function.

This is one of the most common definitions of the determinant. There are others, such as Fredholm. For my purposes, this definition is fine. What I want to notice is that this determinant is also called the regularized determinant. That's a well-defined number.

Now, ah, let me consider soe examples. The first example is rather trivial. Take $M=S^{1}$, and consider a flat metric such that the length of the circle is $2 \pi$. Then the Laplace operator is $-\frac{d^{2}}{d x^{2}}$ and the spectrum of $\Delta$ is $\left\{0, n^{2}, n^{2}\right\}$, where $n$ varies from 1 to $\infty$. Then

$$
\zeta_{\Delta}(S)=2 \sum_{i=1}^{\infty} n^{-2 s}
$$

where the coefficient is because each eigenvalue is of multiplicity two. This is the Riemann zeta function $2 \zeta(2 s)$.

What is the determinant? In this case

$$
\operatorname{det} \Delta=\exp \left(-\zeta^{\prime}(0)\right)
$$

and using this formula, we can write it as $\exp \left(-4 \zeta^{\prime}(0)\right)$, and $\zeta^{\prime}(0)$ for the Riemann $\zeta$ is $-\frac{1}{2} \log 2 \pi$, so the determinant is $4 \pi^{2}$. This is the regularized product of the squares of the integers, $\prod_{n=1}^{\infty} n^{4}$.

Now another example, in this example there was no moduli. Now let $M=\mathbb{C} / \Lambda$, a complex torus, where $\Lambda=\mathbb{Z} \oplus \tau \mathbb{Z}$ where $i m \tau>0$. There is a natural flat metric on each of these, and it is convenient to normalize it to make the area equal to one, by dividing the metric by $i m \tau$. The Laplace operator $\Delta=d^{*} d$ can be written as $2 \bar{\partial}^{*} \bar{\partial}$. In terms of the standard coordinate on $\mathbb{C}$, half of this is

$$
-i m \tau \frac{\partial^{2}}{\partial z \partial \bar{z}}
$$

The spectrum can be computed. It's not much harder than in the case of the circle. The spectrum is the collection of $\left\{\lambda_{\ell}=\frac{\pi^{2}|\ell|^{2}}{i m \tau}\right\}_{\{\ell \in \Lambda\}}$. Then

$$
\zeta_{\Delta}(S)=\sum_{\ell \neq 0} \lambda \ell^{-S}
$$

This is a rather simple expression. We can write this in terms of, well, This is Kroenecker's first limit: $\Delta_{\tau}=4 i m \tau|\eta(\tau)|^{4}$. Here $\eta(\tau)=q^{\frac{1}{24}} \operatorname{prod}_{n=1}^{\infty}\left(1-q^{n}\right)$. Here $q=e^{2 \pi \sqrt{-1} \tau}$. This example was considered by Ray and Singer.

This example shows that determinants are related to moduli. The determinant is a nontrivial function on the moduli space.

I want to give another example which is less trivial. This is the example of the Laplace operator on high genus Riemann surfaces. Now $M$ is a compact Riemann surface of genus $g \geq 2$. There is a natural metric on each such Riemann surface, namely the hyperbolic metric. Consider the corresponding Laplacian on $M$. Its determinant is well defined as is a function $\mathscr{M}_{g} \rightarrow \mathbb{R}$. This is so because the eigenvalues of the Laplacian are real. Smooth is in the sense of orbifolds. Denote by $\Lambda \rightarrow \mathscr{M}_{g}$ the so-called Hodge bundle on the moduli space. The fibers of this bundle are the spaces of holomorphic one-forms on the corresponding Riemann surface.

Let's consider the determinant of this bundle, $\operatorname{det} \wedge \rightarrow \mathscr{M}_{g}$. This is not simply a holomorphic line bundle, it is a holomorphic hermitian line bundle. It has a natural metric. The Hermitian metric is given by $\operatorname{det} i m \tau$, where $\tau$ is the period matrix of $M$. Now denote by $\partial$ and $\bar{\partial}$ the components of the exterior derivative operator on $\mathscr{M}_{g}$. Then $\frac{\sqrt{-1}}{2 \pi} \bar{\partial} \partial \log \frac{i m \tau}{\operatorname{det} \Lambda}=\frac{1}{12 \pi^{2}} w_{W P}$. The ratio det $\Im \tau$

My next objective is to describe, present, an analogue of this formula for the space of meromorphic curves. I will start this today.

I will need a metric on the space of meromorphic functions, and then I will need Laplacians.
Spaces. The next example is spaces of meromorphic functions. I will do it first with a fixed surface, and consider functions of a certain degree. Then I will generalize to when the complex structure always varies.

Let $f: X \rightarrow \mathbb{C}$ with $\operatorname{deg} f=n$ and $g(X)=g, n \geq 2 g-1$.
Now denote by $\mathscr{F}_{n}(X)$ the space of meromorphic functions on $X$ of degree $n$. I want to consider generic meromorphic functions, those with only simple critical values. The space of moromorphic functions with only simple critical values I denote by $\mathscr{H}_{n}(X)$. That's actually a manifold of dimension $2 n-g+1$. So now to finish this lecture I will write the natural isomorphism

$$
T_{f} \mathscr{H}_{n}(X) \cong H^{0}\left(X, f^{*} T \overline{\mathbb{C}}\right)
$$

In other words, this is the space of holomorphic sections. Why this is so I will explain next time.
[We have time for one quick question.]
[The distribution of the eigenvalues?]
It is possible to say, but for varying the complex structure on the Riemann surface, it is more complicated. Hopefully the general theory [unintelligible]and the eigenvalue distribution is as it should be. But I don't know.

## 3 Chekhov

I do not take notes on projector talks.

## 4 Brad

I want to start with a very general principle in algebraic geometry. Suppose you have a compact Lie group acting on a manifold and you're interested in the cohomology ring of this manifold. A general principle says that you can recover $H^{*}(M)$ from the fixed point set $M^{G}$ of your group action, along with the Euler class of the normal bundle of this embedding. Let's look at a simple picture.

Take $S^{2}$ with a very nice circle action by rotation around the vertical axis. The projection tells you the $z$ coordinate, and this group action has two fixed points. The area of this sphere, which is of course just $4 \pi$, is $2 \pi(h(n p)-h(s p))$. To understand these signs, take a disk around the fixed points. You have the orientations from the sphere and from the rotation. At the south pole these differ, at the north pole they agree. So that's the signs. This isn't a coincidence, it's an example of localization.

The moduli space of curves appears to obey this kind of principle even though there's no group action on it.

So $\int_{\mathscr{M}_{g, n}} \alpha$ is what you want to evaluate. You'll have some subsets $X_{i} \subset \overline{\mathscr{M}}_{g, n} \backslash \mathscr{M}_{g, n}$ and you'll be able to write this integral as $\sum_{i} \int_{X_{i}} \alpha_{i}$. This almost looks like $\left.\alpha\right|_{X_{i} / e\left(N_{x_{i}}\right)}$.
Some examples of this phenomenon:

- Witten Kontsevich, where $\alpha$ is a product of $\psi$ classes and so is $\alpha_{i}$.
- $\lambda_{g}$-theorem
- The Faben conjecture
- The generalized Witten conjecture for $r$-spin moduli
- the Virasoro conjecture.

I want to give an explanation of this behavior. Let me introduce notation.
We still have $S$ a surface and $M_{S}$ is the moduli space of hyperbolic metrics on the surface. I mean a complete finite area constant -1 curvature. The equivalence classes are isometries. If $S$ is a surface of genus $g$ with $n$ punctures I'll revert to $\mathscr{M}_{g, n}$. If you talk about integration, this is problematic because it isn't a compact space, so one introduces the Deligne Mumford compactification $\overline{\mathscr{M}}_{g, n}$, where you let metrics degenerate along geodesics. If you have a surface with a hyperbolic metric and a simple closed curve on your surface, then you imagine
a space where the geodesic representative of this curve gets smaller and smaller, and then you add the limit point.

It's easier to work with $\mathscr{T}_{g, n}$, where you add in a marking, a fixed diffeomorphism $f$ from $S_{g, n} \rightarrow X$. So now the equivalence is isometries that preserve the markings, that is, that are isotopic to the identity. We can embed this in a closure $\overline{\mathscr{T}}_{g, n}$ which is no longer a compact space, marked nodal surfaces.

The final piece of notation I'll need is the mapping class group $\operatorname{Mod}_{g, n}, \pi_{0}\left(\operatorname{Diff}^{+}\left(S_{g, n}\right)\right)$. There is a natural action on $\mathscr{T}_{g, n}$ by precomposition with the markings, and the quotient gives you your moduli space. This applies with $\mathrm{a}^{-}$as well. Next let me construt a space with a torus action on it. We'll need the Fenchel-Nielsen twist. Start with a hyperbolic surface and a simple closed geodesic on it. Chop the surface open along the geodesic, twist one, and glue them back together. In the complement of the geodesic you have the metric you started with. Your metric extends smoothly over the surface. If you had a geodesic that intersected the curve you cut along, then it's now a split geodesic.

So you really get a vector field on hyperbolic space. An infintisemmial deformation gives a vector field on $\mathscr{T}_{g, n}$. In moduli space, well, this doesn't work, the vector space is not mapping class invariant.

But there's a useful compromise (Mirzakhani). Suppose there's a small class of curves you want to twist along. Say they're disjoint. We can talk about the stabilizer, namely $[h] \in$ $\operatorname{Mod}_{g, n} \mid h X_{1} \sim X_{2}$.

This is an intermediate group. In particular, you have these geodesics on the space. [Too fatigued]

