# Moduli Space of Riemann Surfaces

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## 1 Equivariant Orbifold Structures on [unintelligible]

Thank you to the organizers. We heard already at this conference about Witten's conjecture. It could also be stated about, taking intersection numbers on the Deligne Mumford conjecture and then putting them into a generating function, and then the partial derivatives satisfy some relations.

I am working in more generality and I want to know whether the same is true.

I will start by explaining Hirota Quadratic Equations on the simplest possible example. Take two functions, $a(\lambda) \in \mathbb{C}((\lambda^{-1}))$ and $b(\lambda) \in \mathbb{C}[[\lambda^{-1}]]$. So one has a finite order pole at $\infty$ and the other has a zero there. Then

$$\Gamma^{\pm} = e^{\pm a(\lambda)y} e^{\pm b(y)\frac{\partial}{\partial y}}$$

which acts on $\mathbb{C}[[y]]$, which we see as Fock space. Look now at

$$Res_{\lambda=\infty}(\Gamma^{+} \otimes \Gamma^{-})(\tau \otimes \tau)d\lambda$$

where $\tau(y) \in \mathbb{C}[[y]]$. This is called a vertex operator. This yields

$$Res \left( e^{a(\lambda)y'} e^{b(\lambda)\frac{\partial}{\partial y'}} t(y') e^{-a(\lambda)y''} e^{-b(\lambda)\frac{\partial}{\partial y''}} t(y'') \right) d\lambda$$

and substituting $q = \frac{y'-y''}{2}, x = \frac{y'+y''}{2}$, this is

$$Res\ e^{2a(\lambda)q} e^{b(\lambda)\frac{\partial}{\partial q}} \tau(x+q)\tau(x-q) = \sum_{N} q^N p_N^{a,b}(\tau, \partial_x \tau, \ldots),$$

where $p$ is a quadratic polynomial in the derivatives of $\tau$.

Now Let $\tilde{U}_0$ be $\mathbb{C}$ mapping to $U_0 = \{[z_0, z_1] | z_0 \neq 0\}$ with $z \mapsto [1, z^k]$ and then $U_0 \cong \tilde{U}_0/\mathbb{Z}_k$. Uniformize on the other side with $\tilde{U}_1 = \mathbb{C} \to U_1 = \{[z_0, z_1] | z_1 \neq 0\}$ with $w \mapsto [w^m, 1]$ and $U_1 \cong \tilde{U}_1/\mathbb{Z}_m$.

This gives an orbifold structure. I want to define orbifold cohomology, which is not the ordinary cohomology. In general, if $X$ is an orbifold and $U$ an open set with a uniformization $\tilde{U}/G$ then $IU = \sqcup_{(g)-conj} \tilde{U}^g / C(g)$ where $C(g)$ is the centralizer. This is called a local model of the inertia orbifold and comes with the involution $g \mapsto g^{-1}$.

Now all these local models can be naturally glued, giving something called the inertial orbifold $IX$. This is $X \cup \sqcup_t X_t$, which are called twisted sectors. In our case there is no gluing, the case I want to consider you have $*/\mathbb{Z}_k$ or $*/\mathbb{Z}_m$. I told you that the equations will be for the equivariant theory so I have to tell you what is equivariant.

Take the two-torus $T = S^1 \times S^1$ and then let $\nu_0, \nu_1$ be characters of the representation dual to the standard representation of $T$ on $\mathbb{C}^2$.

Then the cohomology of the classifying space $H^*(BT) \cong \mathbb{C}[\nu_0, \nu_1]$, because the characters determine line bundles which give Chern classes. Now if you have $X$ a manifold with the torus acting with a Hamiltonian action, actually you don't need that, if it just acts then you can define $H^*_T(X) = H^*(ET \times_T X)$, where $ET$ is a contractible space where the torus acts freely.

If the action is Hamiltonian and I have coefficients in a field, $X$ is symplectic, projective, then there is a localization theorem, that this is isomorphic to $H^*(X^T) \otimes H^*(BT)$. In our situation, what is the torus action? $T$ acts on $\mathbb{CP}^1$. This action can be promoted to an orbifold action. Let me denote by $H$ the inertial orbifold cohomology

$$H^*_T(I\mathscr{C}_{k,m}) = \bigoplus_{n=0}^{k-1} H^*(BT) \oplus \bigoplus_{j=0}^{m-1} H^*(BT)$$

where the isomorphism is by restricting to the fixed points. Let $\alpha$ be in $\mathbb{Z}_k \sqcup \mathbb{Z}_m$, then $\phi_\alpha$ restricts to one at the point indexed by $\alpha$ and 0 to all others.

The last thing I'll need is the orbifold pairing. The equivariant Poincaré pairing is defined by

$$(\phi_\alpha, \phi_\beta) := (\phi_\alpha, I^* \phi_\beta)$$

where $I^*$ is the involution from before on $IX$ and you pair on the right by representing with forms and integrating. In our case it's

$$(\phi_{0/k}, \phi_{0,k}) = \frac{1}{\nu_0 - \nu_1} \qquad (1)$$

$$(\phi_{i/k}, \phi_{(k-i)/k}) = \frac{1}{k} \qquad (2)$$

with switching $k$ and $m$ corresponding to switching $\nu_0$ and $\nu_1$.

Now let me give you equations. So start with $\mathbb{C}_\epsilon[[q_n^\alpha]]_{n \geq 0}$ with $\alpha \in \mathbb{Z}_k \cup \mathbb{Z}_m$. Here $\mathbb{C}_\epsilon = \mathbb{C}((\epsilon^{-1}))$ is the genus expansion parameter.

These will be formal series in $\alpha$ except for $q_1^{0/k} + 1, q_1^{0/m} + 1$. This is the shift because $\phi_{0/k} + \phi_{0/m} = 1 \in H^*\mathscr{C}_{k,m}$

The vertex operators are the following:

$$\Gamma^{\pm} = exp(\pm \sum_i \frac{\prod_{\ell=-\infty}^{n}(\nu + (\ell - 1 + \frac{i}{k})z)}{\prod_{\ell=-\infty}^{0}(\nu + (\ell - 1 + \frac{i}{k})z)} \lambda^{-nk+i}\phi_{i/k})^{\wedge}$$

with another such equation switching $k$ and $m$ and also $\nu_0$ with $\nu_1$ (here $\nu = \frac{\nu_0 - \nu_1}{k}$).

Let $(\phi_\alpha z^n)^\wedge = \epsilon \frac{\partial}{\partial q_n^\alpha}$ and $(\phi^\alpha(-z)^{-n-1})^\wedge = \frac{q_n^\alpha}{\epsilon}$.

Here is the conjecture:

The generating series of the Gromov Witten invariants (equivariant) of $\mathscr{C}_{k,m}$ satisfies

$$Res_{\lambda=\infty}\frac{d\lambda}{\lambda}(\lambda^{n-\ell}\Gamma^{-}\otimes\Gamma^{+} - (\frac{Q}{\lambda})^{n-\ell}\bar{\Gamma}^{+}\otimes\bar{\Gamma}^{-})\mathscr{D}(q_0^{0/k}+(n+1)\epsilon, q_0^{0/m}+n\epsilon,\ldots)\otimes\mathscr{D}(q_0^{0/k}+\ell\epsilon, q_0^{0/m}+(\ell+1)\epsilon,\ldots) = 0$$

for all $n, \ell$, where $Q$ is a constant and I'll tell you what $\mathscr{D}$ is tomorrow, To prove this you need to generalize the localization argument to orbifolds. If $k$ and $m$ are not relatively prime we don't know how to do this, although the formula we have makes sense for all $k, m$.

Okay, now there is an analogue of Getzler's change. Each sequence $q_0^{i/k}, q_1^{i/k}, \ldots \mapsto y_i, y_{i+k}, \ldots$ and $q_0^{j/m}, q_1^{j/m}, \ldots \mapsto \bar{y}_j, \bar{y}_{j+m}, \ldots$

These linear changes are such that

$$\sum(-w)^{-n-1}\frac{\partial}{\partial q_n^{i/k}} = \sum \frac{1}{\prod_\ell(\nu - (\ell + \frac{i}{k})w)} g_{i/k}\frac{\partial}{\partial y_{nk+i}}$$

where $g_{i/k} = \begin{cases} 1/k & , \quad 1 \le i \le k-1 \\ \frac{1}{\nu_0 - \nu_1} & , \quad k = 0 \end{cases}$ If we write $\tau_n = Q^{n^2/2}\mathscr{D}(q_0^{0/k} + n\epsilon, q_0^{0/m} + n\epsilon, \ldots)$ then we get the HQE of the 2 Toda hierarchy.

## 2 Dubrovin