Low Dimensional Topology Notes
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1 Khovanov

Good morning, announcements. The morning problem session will be on Khovanov's lecture, the afternoon one on Szabo's. There will be a research program talk at the same time as the first problem session.

Last time we assigned an algebra A to the unknot. Then we assigned $\mathbb{Z}$ to the empty knot. This is an example of a two dimensional TQFT, which is a tensor functor from the category of oriented cobordisms between 1-manifolds to a tensor additive category (e.g., Abelian groups, k-vector spaces, R-modules (where R is a commutative ring)). In this category objects are finite collections of oriented circles and morphisms are oriented surfaces with boundary. So the functor F from this category $Cob \to k$-vector spaces takes A to a vector space A, so we say $F(\bigcirc) = A$. Because the functor is "tensor" it takes a disjoint union of n circles $F(\underbrace{\bigcirc\bigcirc\cdots\bigcirc}_{n}) = A^{\otimes n}$. Then $F(\emptyset) = k$. So a pair of pants should give a map $m : A^{\otimes 2} \to A$ which is associative and commutative because you can decompose the composition of two pairs of pants in either direction, e.g., for associativity.

There is also a map $A \to A^{\otimes 2}$ by a pair of pants in the other direction. There is also the cap, which is a map $k \to A$ which takes 1 to some element of A. By looking at the effect of a cap on a pair of pants, you can see that the image should be a unit in A. The last thing is the trace $A \to k$. So you need A to be an associative, commutative k algebra with a nondegenerate trace (for all nonzero a there exists b with $tr(ab) \neq 0$.) which dualizes the multiplication to a comultiplication. This is called a commutative Frobenius algebra. So if M is an oriented manifold of dimension n then $A = H^*(M,k)$ is a Frobenius algebra with $tr : H^n(M,k) \to k$ integration over the top class. You can avoid supercommutativity problems by looking at $H^{even}(M,k)$ for n even.

$k[G]$ is Frobenius but not commutative.

In our case the ground ring was $\mathbb{Z}$ and the algebra A was $\mathbb{Z}[X]/(X^2)$ which is $H^*(S^2,\mathbb{Z})$. So part of the construction yesterday can be done for any Frobenius algebra. However, this will only be invariant over Reidemeister-I if A has rank 2 over k.

To construct the comultiplication is to use the trace to identify A with $A* = Hom_k(A, k)$. So dualizing multiplication you get $A^{\otimes 2} \to A$ leads to $A^* \to (A^*)^{\otimes 2}$ which leads by dualization back to $A \to A^{\otimes 2}$.

I want to describe an extension of this theory to tangles. Tangles are links with boundary and then you have tangle cobordisms. Links are assigned homology groups and link cobordisms are assigned maps of homology groups.

To tangles are assigned functors. So my tangles will be between $\mathbb{R}^2$ on the top and bottom of an interval. In our construction, for a collection of points on the plain we assign a category, so the tangle will correspond to a functor and tangle cobordisms will lead to natural transformations of functors.

So to n points on a line we assign A_n and to a braid we assigned a particular functor, but now this will work for all tangles.

We're going to restrict to the case where n is even. So now given n we mark $2n$ points on the horizontal line and look at B^n the set of crossingless matchings. This is with n arcs lying in the bottom half-plane which do not intersect. So when n is three there are five crossingless matchings, these ones: [picture]

So $|B^n| = \frac{1}{n+1}\binom{2n}{n}$. So given two crossingless matchings a and b we can reflect b about the horizontal axis, getting "$w(b)$" and then we can attach it to a getting $w(b)a$. Then we can apply F to it and get $A^{\otimes m}$ where m is the number of circles. w is the reflection about the axis.

I forgot to say that our TQFT was graded. The ring A was graded with 1 in degree -1 and X in degree 1.

Exercise 1 *Multiplication is degree one. In general, a cobordism S has $F(S)$ a map of degree $-\chi(S)$*

We'll construct a fancy ring H^n to take the place of A_n. Take $H^n = \bigoplus_{a,b \in B^n} F(w(b)a)\{n\}$. If you're seeing this for the first time, ignore the grading, all of the shifts. So now H^n is a graded Abelian group, but we claim that H^n is a unital associative ring.

So we need to equip H^n with a natural structure of multiplication. We want a multiplication $F(w(d)c) \otimes F(w(b)a) \to H^n$. This map will be zero if $b \neq c$.

In the case $F(w(c)b) \otimes F(w(b)a)$ we will go to $F(W(c)a) \subset H^n$. This will give, summing over the a, c a map $H^n \otimes H^n \to H^n$.

Let me give an example, if here we have $w(b)a$ and here $w(c)b$. We have a one-manifold, and on the left side we have the functor F applied to this one-manifold which is the disjoint union of the two one-manifolds $w(b)a$ and $w(c)b$.

The map must go into $F(w(c)a)$. The idea is to find a cobordism between $w(b)a \sqcup w(c)b$ and $aw(c)$. This is what we do, using the natural cobordism between the two opposite bs and vertical lines, as in this picture.

So then we can shrink the vertical lines, which takes us honestly to $aw(c)$. So this is a cobordism S which induces a functor $F(S)$ which gives us the multiplication map.

One example is $n = 1$. Then there is only one diagram a. Then the ring H^1 is $F(w(a)a)\{1\}$. The composition is the circle. Then the multiplication will be the cobordism from two circles to one circle. So $F(S)$ is the multiplication map $A^{\otimes 2} \to A$. So then $H^1 = A\{1\}$ so that 1 is in degree 0 and X is in degree two and then the multiplication respects the grading.

Another special case has $n = 0$ so that a is empty and $H^0 = F(\emptyset) = \mathbb{Z}$.

Then for $n = 2$ there are two crossingless matchings a and b. Then H^2 is the direct sum $F(aw(a)) \oplus F(aw(b)) \oplus F(bw(a)) \oplus F(bw(b))$. So as an Abelian group it is $A^{\otimes 2} \oplus A \oplus A \oplus A^{\otimes 2}$. Then the multiplication map between the second and third elements goes from $A^{\otimes 2} \to A \to A^{\otimes 2}$ via the cobordism. So this is part of the multiplication of H^2. This will always be a composition of multiplication and comultiplication.

I promised this was associative. If you have a third diagram $w(d)c$ You merge b against the reflection of b or d against the reflection of d first. In either case you get the same cobordism, isotopic to the same picture $aw(d)$. That gives you the same map.

We also need the unit element. The construction of the unit element is similar to the unit in A^n, where it was $\sum(i)$. So here it will be a sum over idempotents. Look at $F(w(a)a)$. This is then isomorphic to $A^{\otimes n}$. This contains $\not\Vdash^{\otimes n} = 1_a$. So $x \in F(w(b)a)$ multiplied by 1_a will be multiplication of each part of the diagram with a circle marked by 1. So after merging in the cobordism we get $x1_a = x$ and $1_b x = x$. Of course $1_a 1_a = 1_a$ and $1_a 1_b = 0$ if $a \neq b$. Finally if you define $1 = \sum 1_a$ then this is the unit element.

There are lots of idempotents, summing over them gives 1. This is all built out of $A = H^*(S^2)$. The same A appears in your notes as $Hom(P_i, P_i)$, which were $\{(i), X_i\}$ subject to the relation $X_i^2 = 0$. Of course, H^n is much much bigger.

If you fix a and $P_a = \bigoplus_{b \in B^n} F(w(b)a)$ then $H^n = \bigoplus_a P_a$, these are left projective H^n modules. Then $H^n \otimes P_a \to P_a$ goes by $F(w(c)b) \otimes F(w(b)a) \to F(w(c)a)$.

We can also do this on the right, $H^n = \bigoplus_b {}_bP$ where ${}_bP = \bigoplus_a F(w(b)a)$ which is a right-projective H^n-module. For any flat tangle T you have $2m$ top boundary points and $2n$ bottom end points. You can close off the top and the bottom in all possible ways using $a, b \in B^n, B^m$. Then $F(w(b)Ta)$ is a one-manifold. You define $F(T) = \bigoplus_{\substack{a \in B^n \\ b \in B^m}} F(w(b)Ta)\{n\}$, which is an (H^m, H^n)-module.

In this case the bottom is empty, and you only get an action H^n, and the result is a left H^n-action since $H^0 \cong \mathbb{Z}$. If you take the identity tangle which is just $2n$ vertical lines.

This can be closed off by any a and b, so you get exactly the ring H^n itself which has the right and left actions by multiplication.

If you would look at A_n, we also had bimodules U_i, this is similar but degenerate. If you composed U_i and U_j for $|i-j| > 1$ then the bimodule was zero. So this is a better, nondegenerate version of A_n. We want this all to be as nice as possible. There are at least two ways to see how things relate. If you have two tangles T_1, T_2 then you can compose the two tangles and get $F(T_2T_1)$ and get a H^k, H^n bimodule. But you can also look at $F(T_1) \otimes_{H^m} F(T_2)$ and these two are, by a lemma, isomorphic. When $n = m = 0$ then $F(T) = A^{\otimes r}$.

[Bringing the lines around corresponds to taking Hochschild homology.]

The other way to see an equivalence is, suppose T_1 and T_2 have the same endpoints. Then you can get a cobordism sitting in $\mathbb{R} \times I \times I$. The boundary is T_1, T_2 and two trivial parts. Given such an S we would like a map of bimodules $F(T_1) \to F(T_2)$. We want a bimodule homomoprphism. We always have to close things up. So we can just look at all possible closures, using some diagrams, and then extend by the identity along the cobordism, $w(b) \times [0,1] \circ S \circ a \times [0,1]$, which gives a cobordism from $w(b)T_1a$ to $w(b)T_2a$, but then these are just one-manifolds so we can just apply the functor F to get a homomorphism of Abelian groups, and summing over all a and b we get a map of bimodules which is compatible. If you have two of these, the composition corresponds to taking the composition after the construction has been performed.

So far we only have three dimensions. We still need to go to four dimensions which we'll do tomorrow. We have a 2-functor from the 2-category of flat tangles with objects $0, 1, 2, \ldots$, morphisms $n \to m$ flat tangles with $2n$ bottom points and $2m$ top points, and 2-morphisms flat tangle cobordisms. This goes to the two-category of bimodules. The objects are the same. The one-morphisms are $H^n - H^m$ bimodules and the two-morphisms are bimodule homomorphisms.

This is not the end of the story, you have to add crossings and work with the complex of bimodules.

[What did you say about connecting up top to bottom?]

This I will discuss in the last lecture, we will have a complex of bimodules, so what is the meaning of the closure of such a thing? This will be a linear operator from the bimodule to itself, so we're taking a trace, but what's the trace of a functor, a bimodule, that's the Hochschild homology of the bimodule.

[The next question went by too fast for me to understand.]

[You mentioned that you can use more exotic Frobenius algebras. Can you get around the difficulty of not having Reidemeister 1 invariance?]

I think it's hard to avoid $\mathbb{Q}[x]/(x^m)$.

2 Gordon

Can you hear me? So let me begin by saying, the great thing about more than one lecture is you can fix the mistakes of the first lecture. Let me restate the theorem I stated

Theorem 1 *If K is a knot in S^3 then one of the following, exactly, holds:*

1. *K is the unknot (M_k has an essential D_2).*
2. *K is o torus knot (M_k has an essential annulus but no essential torus, it's a small Seifert fibered space)*
3. *K is a satellite knot (M_k contains an essential T_2)*
4. *K is hyperbolic (M_k is simple with hyperbolic cusp)*

The torus knot $T_{p,q}$, its a curve K wrapping around q times in the longitudinal direction, so there are q points of this kind. M_K will be $V \cup_A V'$ where A is the complement of a neigborhood of the knot on the torus T. The core is parallel to A so it's (p, q) curves on δV and (q, p)-curves on $\delta V'$. So you can fiber V by (p, q)-curves and V' by (p, q)-curves, and you have a Seifert fibered space of type $D^2(|p|, |q|)$.

There was one more thing I didn't have time to say about the third part. This had to do with

Theorem 2 *Alexander 1924*
Any torus in S^3 bounds a solid torus.

If you have a satellite knot, the torus boundary of M_{K_0}, suppose you contain a torus, why is that a satellite knot? Well, it contains the knot, otherwise it wouldn't be essential.

How did Alexander prove this without the disk theorem? He was smart. For extra credit prove it without the disk theorem.

Luckily the notes for the lecture were ready before the lecture so I knew what I was supposed to say. Thanks, Joan.

Now I want to talk about Dehn surgery. Look at the exterior of $K \subset S^3$. Here's K and M_K, and we choose a curve we call a meridian which bounds a disk in the neighborhood of K and then λ a meridian in δM_K. You want them to intersect in a single point. μ is determined as an unoriented curve. λ isn't. But $H_1(M_K)$ can be seen to be $\cong \mathbb{Z}$ given by $[\mu]$. You choose λ so that $[\lambda] = 0$ in $H_1(M_K)$. People think of positive things as being right-handed. You orient so that $\mu \cdot \lambda = 1 = -\lambda \cdot \mu$. Then these form a basis for $H_1(\delta M_K)$. But then for any α an esssential simple closed curve in δM_K, then $[\alpha] = \pm(m[\mu] + \ell[\lambda]) \in H_1(\delta M_K)$. Then $\alpha, beta$ are isotopic if and only if $[\alpha] = [\beta]$. The isotopy classes are in one to one correspondence with slopes, and then you take the quotient and get $\mathbb{Q} \cup \{1/0\}$.

So $\alpha = (m/\ell)$-Dehn surgery on KK is $M_K \cup_\delta V$, for V a solid torus, where α goes to the boundary of a disk in V. This is called $K(\alpha) = K(m/\ell)$. The famous example is that if K is the left-handed trefoil, then $K(-1)$ is the Poincaré homology sphere $\Sigma(2,3,5)$ You kill m times the meridian, so that the homology $H_1(K(m/\ell)) \cong \mathbb{Z}_m$. So homology spheres were called Poincaré spaces. You can get infinitely many by doing $1/\ell$ surgery on the trefoil, and Dehn showed they were all different.

A little bit of notation. When we do the Dehn surgery, the knot is K_α, the core of V in $K(\alpha)$. There's one more little bit of basic stuff. If α and β are slopes on δM_K, then we define $\Delta(\alpha, \beta)$, which we call the distance, though it isn't a metric, the minimum $|\alpha \cap \beta|$.

Exercise 2 $\Delta(m/\ell, m'/\ell') = |m\ell' - m'\ell|$.

The trivial Dehn surgery $K(1/0)$ is putting the torus back in how you found it and always gives S^3. So $\Delta(m/\ell, 1/0) = \ell$. In particular if $\ell = 1$ we say α is integral. In fact an integral Dehn surgery is a surgery in the higher dimensional sense. The others aren't surgeries because if you attach a two-handle, the curve that bounds a disk in the new boundary is going to have intersection number one with that meridian.

To attach the two-handle, you choose a homeomorphism of the solid torus to a neighborhood of the knot. The two factors only intersect once. The integer's worth of choices are called framings, they're always integral in this sense.

Dehn surgery on a knot is surgery on some link.

Okay, now so I guess let me take that theorem, and corresponding to that theorem, let me say, satellite knots, here's a particular kind of satellite knot which will come up, that's where you take $K = C_{p,q}$, a (p,q)-curve in the interior of the solid torus. Push it from the boundary into the interior. It wraps q times around longitudinally, p times meridianly. Assume q is at least two, p can be one or whatever. Then $h : S^1 \times D^2 \to N(K_0)$. If you choose this so that h of a longitude of the torus goes to a longitude, something nullhomologous in the complement, then the knot $K = h(J) = h(C_{p,q})$ is called the (p,q)-cable of K_0.

Let me go back to the theorem and say what happens with respect to the four classes of knots under Dehn surgery.

1. So, if K is the unknot, if you do m/ℓ surgery on the unknot, you get a Lens space $L_{m,\ell}$. Some would say it's $\ell, m-\ell$, but we won't worry.
2. In the second case, when you attach a torus to a Seifert fibered space, you almost always get a Seifert fibered space. Generically the core of the solid torus will be an exceptional fiber again. The options are, well, in the first case, how many times does this meet the Seifert fiber? Let's go back to the picture. On the boundary of the exterior of the torus knot, what do you have? You can figure this out, you can say it's how many times this curve, it's the slope of this curve, since the fibers are the annuli. So what's the slope of α which is a boundary component of A? It clearly intersects a meridian. So ℓ must be one, it's an integer, it's the number of times it intersects the longitude, so it's the

linking number with the knot. So if we take the knot, here it is, sitting here, the slope is $m/1$ where m is the linking number of K' with K. When you have an integer slope that corresponds to a framing, which is the same as a froming induced by the knot lying on the surface. So the framing of K induced from this torus T that it lies on δV. So all you have to figure out is what the linking number is. So $K \sim p\mu_{+}q\lambda_0$. These were meridian and longitude for V. The linking number of μ_0 with K', how many times does it link with this guy, it's q. On the other hand, the linking number of the longitude is 0, so the linking number of K with K' is pq since it's p times the meridian. This is the slope of the Seifert fiber on the bonudary. So now this depends on the intersection number $\Delta(m/\ell, pq/1)$. If that number $|m - \ell pq|$ is greater than one it will give a Seifert fibered space with that value, d. If it's one you get a lens space and if it's zero you get a connect sum of lens spaces. Summarizing,

$$T_{p,q} = \begin{cases} S^2(|p|, |q|, d) & d > 1 \\ L(m, \ell q^2) & d = 1 \\ L(p, q) \# L(q, p) & d = 0 \end{cases}$$

3. So what happens when you have a satellite knot? You have this essential torus, let me reinterpret this picture. Now you have a cable, it's the same picture but this is happening inside, here's our curve $C_{p,q}$, and we're tying the solid torus in a knot K_0, but it's a similar picture. Call this annulus, now, A, so if X is $S^1 \times D^2$ the interior of $N(C_{p,q})$ and $A \subset X$, then define $h : X \to X$ to be a Dehn twist along the annulus, as you move across A you twist, and let's twist around ℓ times.

   What does that do to the meridian? This, when you Dehn twist, for each one you add a multiple, so it's $\mu + \ell(pq\mu + \lambda)$. So this has slope $\frac{\ell pq+1}{\ell}$. This filling will then give a solid torus. So going back to the satellite situation, you have the essential torus, and then in this case, T usually remains incompressible in $K(\alpha)$. The exceptions are completely classified. The cables, for any Dehn surgery of this form $(\ell pq + 1)/\ell$ will compress. The cables are the only ones for which it compresses for infinitely many surgeries. There are some others where it compresses, but there's only one surgery for which it compresses, except for one particular magical example where there are two compressing surgeries.

4. If K is hyperbolic then $K(\alpha)$ is hyperbolic for all but finitely many α.

   The goal will be to classify the pairs (K, α) where K is hyperbolic but $K(\alpha)$ is not a hyperbolic manifold.

   All right, so, um, this seems an unreasonable goal until you start looking at examples, let me give a simple explicit example, the figure eight knot. The exterior is a punctured torus bundle over the circle, so when you do zero-Dehn surgery, you're capping this off with a disk so this is a torus bundle over S^1. This is the Solv-manifold that John Morgan talked about, definitely not hyperbolic. So there are at least some examples. Another thing here is if you look ath this surface F which K bounds, it's a once-punctured Klein bottle. You're supposed to compute the slope of this boundary, it has slope four, and I'll give you a hint, this four is not the same as the number of crossings. Nonetheless it's four. So when you do four-surgery, you cap off the boundary and get a manifold which contains a Klein bottle. So either it contains an essential torus or it's a small Seifert fibered space. In fact, it contains an essential torus.

So these two examples are "toroidal," containing essential tori. One thing that makes things more reasonable is that the simplest examples are the worst. The more complicated the knots, the less likely there will be an exceptional surgery.

After that example, let me describe a way of constructing examples of knots with nonhyperbolic surgeries.

1. Imagine $K \subset F \subset S^3$. So $S/X \cup_F X'$. Here's a schematic picture. If you call F_0 the surface F minus a neighborhood of K, then $M_K = X \cup_{F_0} X'$. We saw this before with X, X' solid tori and F_0 an annulus. Call m the framing on K induced by F. Then we can do m-surgery, and get, looking at the topology, you're attaching a solid torus to this guy, you have two curves that bound disk, If you decompose M_K and then V by cutting along the disk, this is the union of two manifolds glued together along their boundary, $X[K]_\delta X'[K]$. So where $X[K] = X$ with a two-handle attached along K and similarly for X'. So as I say, it's an elementary argument, and in particular, here's the beauty of the blackboard, cables become torus knots in a twinkling of an eye, here's a torus knot, and remember the framing was p, q, and we see therefore that p, q-surgery on this guy $T_{p,q}(pq) = X[K] \cup_\delta X'[K]$. This gives a lens space, and so you get $L(q, p) \# L(p, q)$, and this shows the last part of part two of the theorem directly.
2. That's like the first example of this construction if you like. The second example is that there's a similar thing for a cable. If we do pq-surgery on $C_{p,q}$ inside the solid torus you get $L_{q,p}$, connected along, on the outside you get a solid torus. It's an exercise to figure out the slope of the solid torus, in terms of the original slope, but the upshot of this is that if K is a (p, q)-cable of K_0 then $K(pq) = L(q, p)\#K_0(p/q)$. For nontriviality we assume $q \geq 2$, so $K_0(p/q)$ is never S^3 and this is always reducible. We're starting with a nonhyperbolic knot, doing a surgery and getting something reducible.

   That's rare, so here's your next homework. The problem is to show that if $K(\alpha)$ is reducible then K is a (p, q)-cable of some knot. This is not due for quite a while. This is the cabling conjecture. If you want fame and fortune, solve that one. If you put any other structure on the knot, it's probably true. Note that α would be p, q.
3. Let me go back, I talked about torus knots and cables. Suppose $F \subset S^3$ is a genus two Heegaard surface, so X, X' are genus two handlebodies. So take the knot sitting in here, and you have the knot sitting on it here, you have the induced framing m and $K(m)$ will again be $X[K] \cup_\delta X'[K]$, so if we choose K to be nonseparating, you get a torus. So you get two manifolds glued together along a torus. Generically these knots will be hyperbolic. This is a candidate for an interesting torus. Let's try to make these two solid tori. Then their union will be a lens space. Or one solid torus and the other a Seifert fibered space, usually they'll be over the disk with two singular fibers.

   So if you do that you'll be getting a Seifert fibered space with a solid torus glued in, so it's a lens space or a Seifert fibered space with three exceptional fibers, or if you're lucky it's a connect sum and you have a counterexample to the cabling conjecture, but you won't.

   You could try to make both of them Seifert fibered spaces, so you get a Seifert fibered space or maybe a graph manifold. It turns out that conjecturally, roughly speaking, most of the nonhyperbolic surgeries come from this structure. That gives hope that one might be able to prove that these are the only nonhyperbolic surgeries. I'll give a way to construct many, many examples using tangle surgery next time.

3 Szabo

[Hello, I'm Bob Edwards from UCLA, I brought this camera here to try it out and I really think I've started something. You can read the board, hear the audio, they're really good, but how can we distribute them? The question is, can they be distributed in a good way? Presumably I can convert this to a format, but I'm a novice and can't really proceed without more and better experience and computer equipment. This is a good chance for me to try my other new toy, too, and take some pictures of you.]

Last time we were sketching the construction of the various $\widehat{HF}(Y)$. All of them were counting holomorphic disks. In some of these other constructions we had

$$\begin{tikzcd} HF^-(Y) \arrow[r] & HF^\infty(Y) \arrow[r] & HF^+(Y) \arrow[ll, bend left] \end{tikzcd}$$

$$\begin{tikzcd} widehat HF(Y) \arrow[r] & HF^+(Y) \arrow[r] & HF^+(Y) \arrow[ll, bend left] \end{tikzcd}$$

And we had related these, although there are four different versions they are closely related. When the first homology of Y is torsion the same construction goes through, but you have to choose only special Heegaard diagrams for $H_1(Y)$ not torsion. But the chain complex decomposes into different components, but we also have a decomposition $\widehat{HF}(Y) = \bigoplus widehat HF(Y, \mathfrak{s})$ where $\mathfrak{s} \in Spin^c(Y)$. Here $Spin^c(Y)$ is roughly $H^2(Y, \mathbb{Z})$.

The same constructions go through in all the terms of the long exact sequence. And why is this not necessary when H_1 is zero? Then I have a unique $Spin^c$ structure, so I don't need to bother.

So what is the role of $H_1(Y)$?

Suppose we have two intersection points x and y. There is an obstruction for there being a holomorphic disk connecting them, and that is first there should be a homotopic disk connecting them. So is $\pi_2(x, y)$ empty or nonempty? Well, what is that obstruction? Here are t_α and t_β. I can connect the same points to get another path connecting them and in that way I will get a closed loop $\gamma_1 - \gamma_2$ in $Sym^g(\Sigma_g)$. I should be able to choose this to be a nulhomotopic loop. So $[\gamma_1 - \gamma_2] \in H_1(Sym^g(\Sigma_g))$, but this is well defined if we mod out by the indeterminacy $H_1(T_\alpha)$ and $H_1(T_\beta)$. This uses the exercise. It is really in $H_1(\Sigma_\alpha)/([\alpha_i], [\beta_i])$ which by Mayer-Vietoris is just $H_1(Y)$. If that, ϵ, is nonzero there will be no homotopy class connecting these things. So in $T_\alpha \cap T_\beta$ we have $x \sim y$ if $\epsilon(x, y) = 0$.

So our chain complex decomposes into lots of different pieces. You may remember the family of examples Y_n from the first day. If you want a nice homework, you have to get your hands dirty, but choose the Heegaard diagram for Y_{12} and find equivalence classes.

Some of these equivalence classes contain many fewer elements. So sometimes it's easier to compute certain parts of the Floer homology.

Well, we haven't yet decomposed according to $Spin^c$ structures. So in order to do that I have to define them.

Definition 1 *There is a similar definition in every dimension. $Spin(n)$ is the connected (double) cover of $SO(n)$, and $Spin^c(n)$ will be almost $Spin(n) \times S^1$, but we will divide by the $\mathbb{Z}_2$ action on both sides. This is a Lie group, and if I forget about the last factor, it is a map to $SO(n)$, and the $Spin^c$-structure is a lift, I can think of the tangent space giving a $SO(n)$-bundle, and there may or may not be some number of lifts to a $Spin^c$-bundle.*

This is one definition. For oriented closed manifold with trivial tangent bundle, there are lifts, but maybe a good exercise if you haven't done it before is, in dimension three, show that these are in correspondence with H^2.

Definition 2 *This is an alternative definition in dimension three due to Turaev. Take nowhere vanishing vector fields on Y modulo the relation $V_1 \sim V_2$ if they are isotopic in a complement of a ball.*

A different definition would be for them to be isotopic; this is not what we want. The Euler characteristic is zero so there are always nowhere vanishing vector fields. There are some really nice things you can do with this definition. If I fix a trivialization, I get a map $V_1 : Y \to S^2$. I can look at the direction of this vector field and it gives us a map. So I could take the cohomology class $e \in H^2(S^2)$, and using the trivialization I can pull back and get $V_1(e) \in H^2(Y, \mathbb{Z})$. So it looks like we can identify the $Spin^c$ structures with second cohomology classes.

Exercise 3 *If I have two $Spin^c$-structures I can define these cohomology classes. The difference of them is independent of the trivialization, so $S_1 - S_2 \in H^2(Y)$.*

So the second cohomology of Y acts on $Spin^c$ structures, and it's not hard to show that this gives you an identification.

What is less clear with this definition is how it fits into the Heegaard Floer homology. We have these two subspaces T_α and T_β, and we have the point $x \in T_\alpha \cap T_\beta$ and the basepoint $z \in \Sigma_g - \alpha_i - \beta_j$. We need $S_z(x) \in Spin^c(Y)$. This is the additional knowledge that we need. How will we do this? Being isotopic on the complement of one ball and on the complement of several balls is the same thing. Here int may not hurt to come back to the picture of Morse functions. Recall that the Heegaard diagram can be reconstructed with a self-indexing Morse function. So x has several components, and it's in the intersection of some α and β, so it's up here, and it's going to flow down with $-\vec{\nabla}_f$ and up to this point with $\vec{\nabla}_f$. This is the gradient vector field, if it's nowhere vanishing you'd be done but it's not, it vanishes precisely at the critical points. You also have the basepoint, z, which will not be taken by the gradient to the minimum and maximum. You simply delete a neighborhood of all these things, these trajectories, and you have a nowhere vanishing vector field. Around every critical point there's an index depending only on the parity. Because we're connecting points of different parity, it follows that this vector field can be extended to these balls lots of different ways, which doesn't matter because of the definition of $Spin^c$ structures. You cauld look at the obstruction between them in $H_1(Y) \cong H^2(Y, \mathbb{Z})$.

Exercise 4 $S_z(x) - S_z(y) = PD(\epsilon(x, y))$.

So $\widehat{HF}(Y, \mathfrak{s})$ is the homology of the chain complex $\widehat{CF}(Y, \mathfrak{s})$ where you only use the indicated $Spin^c$ structure.

So all of these are α_i I can also define, I can choose another curve in the handlebody which meets this exactly once and misses all the other disks. That's in the first homology of Y, it's $\alpha_i^* \in H_1(Y)$. So if I take $S_{z_1}(x) - S_{z_2}(x) = PD(\alpha_i^*)$.

Why do we care about this at all? Let's go back to the family Y_n. Now Y_3 is a very hard computation. Take n to be rather large, say $n > 12$, although we don't need that big. This creates a more complicated Heegaard diagram, and after each twist you introduce three more generators, so eventually you get only one equivalence class which is just three generators.

Exercise 5 *Compute $\widehat{HF}(Y_n)$ and $HF^+(Y_n)$ for each $Spin^c$ structure. You can always use the same generator, and vary the basepoint. Look at the three generators, you always get the same holomorphic disks, and it's just a question of which ones you're counting at any particular case.*

This was the first nontrivial calculation we made and it sort of applies in this setting.

When $g > 2$ everything is easy to do, $\pi_2(x, y)$ is empty if $\epsilon(x, y)$ is nonzero, and $\pi_2(x, y)$ is $H_2(Y, \mathbb{Z}) \oplus \mathbb{Z})$. In the $g = 2$ cases you could have different homotopy classes corresponding to the same domain. You could stabilize to increase genus. You could also look at slightly different equivalence classes. The reason for this last isomorphism is, well, look at $D(\phi)$. These are periodic domains, that is, two-chains. The boundaries will be loops which lie in α or β. You want $n_z = 0$, which uniquely specifies it.

Typically this will be like $S^1 \times S^2$. A typical domain in this picture would be $D_2 - D_3$, and of course you can take any multiple of this. This is a definition, but if you have such a domain, and when you study $\pi_2(x, x)$ then any two-chain like that will give you homotopy classes, so that there is a reason why we have lots of homotopy classes. A really easy homework is

Exercise 6 *P is the space of periodic domains (the part of the boundary of a periodic domain on α is a multiple of the circle). Show it is naturally isomorphic to $H_2(Y, \mathbb{Z})$*

How about this $\mathbb{Z}$? The domain of π is $D(\phi) = \sum b_i D_i$, then I can write $H(\phi) = \sum (b_i + 1) D_i$. If I add one to each coefficient, the boundary will still be the same, so that picks out this other part. If you write the definition of the boundary map, you are making a summation. You want only finitely many terms, so you have to look at special Heegaard diagrams.

Definition 3 *A Heegaard diagram is admissible if for each $p \in P$ which is nonzero we have that p has both positive and negative coefficients. This is one definition. Or you can put areas to each of the domains D_i. Then you could equivalently fix them so that the area of any period domain is zero.*

You will see that the nontrivial elements have both negative and positive elements in this diagram, which means that they are all admissible. In this diagram, then you have kD_1 which just has negative ones and so is not okay. Similarly, if I put the basepoint here it won't work, nothing will be admissible.

Unfortunately in general we have to deal with this issue.

[The places where the basepoint can no longer go, that can't cut the surface into two pieces, can it?]

You have to say, first, there is an admissible Heegaard diagram. Second you have to show that two admissible Heegaard diagrams have the same invariants, so you can move them and the Floer homology don't change.

So far this is more like symplectic geometry. It's not clear at all that this is a three-manifold invariant. It's sort of obvious what we have to check.

First you have to check Heegaard moves, which change the diagram but not the manifold.

1. isotopy. So you might want to create an intersection between α and β where one did not exist before.
2. handle-slide, which could be between α and β. I did not allow the two α to intersect, and then you can turn α_1 and α_2 into one picture which is like the sum of them, α_1'. For a symmetric picture, look at a pair of pants, and it is turning (α_1, α_2) to (α_1', α_2). So these are some operations which do not change a three manifold. They are maybe more obvious in this picture. You could also try to make a change to that circle. So the handlebody itself does not give these uniquely, so you have to deal with it.
3. stabilization This is when you have a Heegaard diagram and change it by adding a genus, and then $\alpha_{i+1}, \beta_{i+1}$.

So these are moves that don't change a three-manifold.

How do we get these Heegard diagrams? They are from self-indexing Morse functions. You also fix a metric, which you can vary if you like. You also have to fix the basepoint. Great, so

Exercise 7 *Suppose I have two self-indexing Morse functions f_1 and f_2. I want you to show that you can move through finitely many Heegaard moves to the other diagram.*

It's obvious how to start. You want to connect the two functions in a one-parameter family of functions. They are not necessarily Morse functions. So isotopy does nothing, really, handle slides correspond to Morse functions that aren't really self-indexing, and stabilization corresponds to acquiring singularities.

There is a stronger result that says that you can connect them with pointed (away from the basepoint) moves.

In the remaining five minutes, we are only using Legrangian Floer homology. When I change complex structures, there is that notion, and there's a very similar version where [unintelligible]corresponds to isotopy. Let me talk about handleslides. I can do this where the genus is two, or in higher genus as well. This is the genus two surface right here, and here is a Heegaard diagram of something, and it's easy to see it's $S^1 \times S^2 \# S^1 \times S^2$. Here you can compute the Floer homology and write down a new diagram, here I've changed, made a handleslide between these two circles. Both of those pictures are admissible Heegaard diagrams, and they are connected by handleslide and isotopy.

Exercise 8 *This should give the same Floer homology as the previous picture. This is not a hard exercise.*

When you want to prove in general that the handleslide doesn't change the Floer homology, try the follawing trick. You have $\underline{\alpha}, \underline{\beta}$, and $\underline{\gamma}$ which is what I get after handlesliding some of the β. So I want to show $\widehat{HF}(\alpha, \beta) \cong \widehat{HF}(\alpha, \gamma)$. So the way out is to think of holomorphic triangles mapping to T_α, T_β, and T_γ. This really gives a map $\widehat{HF}(\alpha, \beta) \otimes \widehat{HF}(\beta, \gamma) \to \widehat{HF}(\alpha, \gamma)$. So to do this you really have to work out the one example. It's hard to prove that these give the same thing, but this is the basic idea.

Stabilization is very easy if the basepoint is here and we are working in $\widehat{HF}$. The generators have to have intersection points one here. The generators are the same. If the basepoint is here, we don't allow anything to go through the basepoint, so then the disks that don't intersect it are the same.

it's harder if we're in HF^+, HF^-, or HF^∞. So remember this picture. This point might be in the middle of a disk like that, and I have to be able to deal with this other kind. I have to fix the complex structure, and thath means that I could really try to better relate the complex structure on the circle, and what you would like to see is what happens in the limit. The statement is that when you make this neck really long, you get an isomorphism.

In a really long and ugly paper we check all these details. Next time I'd like to show you how to use this to onswer some problems, and there's an easier version involving knots.

4 Wu

So today I will talk about the relation between Legendrian links and spanning tree models. The first part will be about the spanning tree models for Khovanov homology due to Champanerkov-Kofman-Viro and Wehrli. The second part is an upper bound, due to Lenny Ng.

The spanning tree model was discovered by Thistlewaite. Humans compute the Jones polynomial by means of the Kauffmann bracket. I am using Khovanov's convention. So here is the right-handed trefoil knot. In the process we keep track of which splice we use, which resolution. So we use the defining relation to compute the Jones polynomial for the bottom things. But only stupid computers do this. In general people just look to see that they have reached the unknot and then plug in for that.

This depends on the order in which I splice these three crossings. Then the ending unknots I come up with look different. Then the question is whether there's a good and rigorous way to give this smarter algorithm. This was figured out by Thistlewaite. We need to, that's the spanning tree expansion of the Jones polynomial. In order to introduce this I first need to introduce the Tait graph. The regions outside the diagram in the plane can be bicolored. Any two regions of the same color do not share a common boundary curve. We put a vertex in each of the black regions, and at each crossing we put an edge that connects the vertices in the two regions connected by the crossing. We need to know which strand is on top and which one is on the bottom. So the sign convention looks like this. This edge is positive and this edge is negative.

From this graph we can uniquely reconstruct the knot diagram. But we have two choices of colorings. So we can get different diagrams. This is a very different Tait graph. They are dual to each other. Two embedded planar graphs, we can take duals, it doesn't really matter which graph we use.

In order for the graph to have spanning trees we need the graph to be connected. We can always make a link diagram connected with Reidemeister II moves. We denote the Tait graph by G. Let T be a spanning tree of G.

Let $e \in T$. Then $cut(T, e)$ are the edges of G that connect the two components of $T \backslash e$.

Say $f \notin T$. Then $cyc(T, f)$ are the edges of G in the unique simple closed curve containing $T \cup f$. It's easy to see that if $f \in cut(T, e)$ then $e \in cyc(f)$.

Fix an ordering of the crossings. e is of the type L if e is a positive edge and $e \in T$ and e has the lowest ordering in the set $cut(T, e)$. This is for positive edges. We say that it is the type $\bar{L}$ if it's negative and has the other properties.

We call it type $D, \bar{D}$ if it is positive, negative, and not type $L, \bar{L}$. We have the same definitions for things not in the spanning tree.

We say f is type ℓ if f is positive, not in T, and has the lowest ordering in $cyc(T, f)$, and $\bar{\ell}$

if negative with the other hypotheses. We also have the corresponding definitions for d. f is type d if positive and not ℓ, and f is $\bar{d}$ if negative and not $\bar{\ell}$.

L means live and D means dead. The capitals mean they're in the spanning tree. A bar means the edge is negative. Now we use the following procedure to get our unknot. We want to kill all the dead crossings. There are four types of dead crossings. We use particularly signed crossing resolutions for the four kinds, the same for d as for $\bar{D}$ and the same for D as for $\bar{d}$.

After you're done with that you get the unknot corresponding to the spanning tree T. Termnal unknots are in one to one correspondence with spanning trees T.

Let's do an example. Start with the highest ordered crossing to splice. Note that this graph has only two vertices and three edges. I chose this to be my spanning tree. So e_1 is $e\bar{l}l$ and e_2 and e_3 are $\bar{d}$. You can try to use the other Tait graph and you will get exactly the same three unknots. If you change which crossing you start with, you can draw the corresponding ordered Tait graph, and you will see that the edges correspond to these three unknots.

I hope this example convinced you that there is a one to one correspondence between the spanning trees and the ending unknots.

All of these unknots are twisted unknots, so they don't need type two and three moves, so it's easy to write down the spanning tree expansion for the Jones polynomial, it's just, so the Kauffman bracket is the sum over the spanning trees

$$\langle K\rangle = \sum_T \langle K|U(T)\rangle\langle U(T)\rangle$$

where

$$\langle K|U(T)\rangle = (-q^{-1})^{\#d+\#\bar{D}}$$

and

$$\langle U(T)\rangle = (q+q^{-1})(-1)^{\#L+\#\bar{\ell}}q^{\#\ell+\#\bar{L}-2\#L-2\#\bar{\ell}}$$

For T let $u(T) = \#L - \#\ell - \#\bar{L} + \#\bar{\ell}$ and $v(T) = \#L + \#D + \#e\bar{l}l + \#\bar{d}$

Let $\mathscr{C}_T = \langle \xi'_T(u(T), v(T)), \xi^L_T(u(T)+2, v(T)+2)$ and let $\mathscr{C}_X = \bigoplus_T \mathscr{C}_T$.

Theorem 3 *Champanerkov-Kofman-Viro, Wehrli*
There is a boundary of degree $(-1,-2)$ on $\mathscr{C}_K$ that makes the resulting complex a deformation retract of the Khovanov chain complex. Here $u = j - i = wr(K) + 1$ and $v = j - 2i = \frac{w(K)+\#\text{crossings}}{2} + 1$.

The proof is kind of straightforward. These are all unknots so their homologies are all A. The chain complexes are just $A \oplus B$ where B is a contractible chain complex. We know that the Khovanov chain complex for this diagram is basically a mapping cone of these two Khovanov chain complexes. This decomposition is preserved through the mapping cone construction. So this is the mapping cone of A and A summed with B_1 and B_2, contractible complexes.

These two indices are very interesting, for example v is constant when K is alternating, doesn't depend on T. So the $j-2i$ in the Khovanov homology can only come from these two generators. This reproves the result of Lee.

Theorem 4 *(Ng)*
Let K be a Legendrian link in $\mathbb{R}^3, \xi_{st}$. Then $tb(K) \leq min\{k| \bigoplus_{j-i=k} \mathscr{H}^{i,j}(K) \neq 0\}$.

It's clear that if we can find a good lower bound on u we can reprove this theorem.

Theorem 5 *For any spanning tree T of G we have $tb(F_T) \leq -1-(\#d+\#\bar{D})$. This is equivalent to $u(T) \geq 1-c(F)$.*

Here F is a front projection of the Legendrian link K. At each crossing we can resolve the crossing. In a front projection the strand with lower slope is always on top. We change the B-splicing because we allow cusps but not vertical tangents. Then $F(T)$ is the front projection of T. Then $c(F)$ is half the number of cusps.

First let me explain how these two inequalities are equivalent. Note $\#d+\#\bar{D}$ is equal to the number of B-splicings used, which is $c(F_T)-c(F)$.

We also know $u(T) = -w(F_T) = -tb(F_T) - c(F_T)$. Looking at these two equations it's easy to see that these two inequalities are true. Let's explain why the second inequality implies this upper bound. We know $j-i = u+w(F)-1 \geq w(F)+1-c(F)-1 = w(F)-c(F) = tb(F)$. So this implies the Khovanov upper bound given by Ng.

I don't have time to go into the proof. So you need to find a good ordering of the crossings in the front projection. First you perturb this so that all of the crossings are at different x-coordinates and use that ordering. You need to deal with a bunch of things. This divides the proof into several situations. There are several easy situations and dual harder situations. It's long but easy to read.

So first we see that if the Khovanov bound should be sharp, you need a spanning tree where this inequality in the theorem is sharp, called a good spanning tree.

Say K is an unsplit alternating link. Then the claim is that the minimal spanning tree is a good sponning tree. The correct front projection is constructed by Lenny. The minimal spanning tree means the sum of all the x coordinates is the minimum among all spanning trees. There is nothing mapping into this by the boundary map so it won't be killed when computing the homology.

So this shows that this is sharp for alternating links. I hope that we would be able to find a sufficient condition. There is a nice analogous result for the Kauffman polynomial by Dan Rutherford. This will be harder. In Lenny's original paper, his proof uses exact triangles and I want to redo this using spanning trees.

[The Alexander polynomial has a similar spanning tree expansion, that lifts for Oszvath

Szabo homology. Is there something similar you can do in that case?]

There should be but it's not in this setting.

[Do you have a reformulation of your theorem in the setting of twisted unknots. [unintelligible]uses the same spanning trees, but he uses different invariants. He uses a smaller number of notations.]

I can't answer that. Other questions? I probably won't be able to answer.

[Let's thank the speaker again.]