Low Dimensional Topology Notes
June 28, 2006

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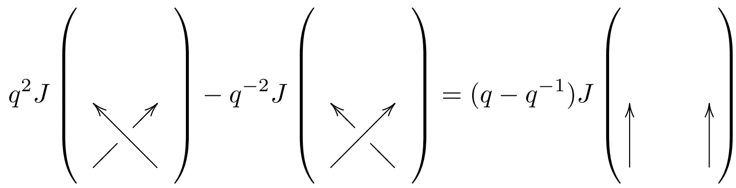
June 29, 2006

1 Khovanov

[Good morning. Before Mikhail begins, an announcement. There will be a problem session this morning on Cameron Gordon's lecture. Joan will be running it. Hopefully by the end of the lecture there will be lecture notes and problems.]

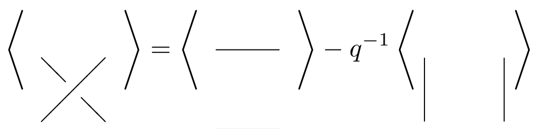
I should give credit to the work of others. The braid group action is due to my work with Seidel. something is due to R. Thomas and myself. Braid cobordism refereneces include Carter-Saito and Kanada.

Today I'm going to talk about categorification of the Jones polynomial. This is a map from oriented links in $\mathbb{R}^3$ to $\mathbb{Z}[q, q^{-1}]$. It is determined by the skein relation



and the condition that J of the unknot is $q + q^{-1}$.

Kauffman found an easy way to show that this is well-defined. For a plane diagram D of a link L without orientation, we are going to define a Laurent polynomial $\langle D\rangle$. This involves the zero and one-resolutions:



and the bracket of a link with a disjoint unlinked unknot is $(q + q^{-1})$ times the brackt of the link without that unknot.

Exercise 1 *If D_1 and D_2 are related by Reidemister moves, then $\langle D_1 \rangle = \pm q^{?} \rangle D_2 \langle$.*

If you count negative (x) and positive (y) crossings (now you orient the link) you can show that $K(D) = (-1)^{x(D)} q^{2x(D)-y(D)} \langle D \rangle$ is a lisk invariant. Show that this is the Jones polynomial.

We want a categorification, so that L will lead to $H(L) = \oplus H^{i,j}(L)$ with $J(L)$ the Euler characteristic, namely $J(L) = \sum_{i,j} (-1)^i q^j \mathrm{rk} H^{i,j}(L)$

So we want D to lead to $C(D)$. We will use [1] and $\{1\}$ to indicate shifting of the homology grading $i \to i-1$ and the other grading $j \to j+1$ respectively in the complex

$$C^{i,j}(D) \xrightarrow{d} C^{i+1,j}(D) \xrightarrow{d}$$

$$C^{i,j-1}(D) \xrightarrow{d} C^{i+1,j-1}(D) \xrightarrow{d}$$

So when D is the standard unknot, well, I should say that Kauffman's relations make things more symmetric. I want to avoid $q^{1/2}$ because q corresponds to a shift in grading and I don't want to shift grading by one half. So $\langle D \rangle$ is $q + q^{-1}$, so why not just take two $\mathbb{Z}$ in i degree zero and j degree ± 1. The graded rank of this group is $q + q^{-1}$, as desired. Denote by A the direct sum, $\mathbb{Z} \cdot 1 \oplus \mathbb{Z}X$. This is the homology of the unknot. It works perfectly, we get the right number.

What are other links? You can take the unlink with K components. That value on the bracket is $(q + q^{-1})^k$. The homology should probably just be $A^{\otimes k}$. Then the graded dimension will be what we want.

Let's start with crossings by looking at the diagram with one crossing. We look at the two resolutions of the once-twisted unknot into two unknots and one. So we get $A^{\otimes 2}$ and A. Since q corresponds to $\{1\}$, the q^{-1} should indicate that A should be $A\{-1\}$. The negative sign means that they are in different parity homological dimension.

So we want $0 \to A^{\otimes 2} \to A\{-1\} \to 0$. How can we find the map, which we will call m? $A \otimes A$ has $X \otimes X$ in degree 2, $X \otimes 1$ and $1 \otimes X$ in degree zero, and $1 \otimes 1$ in degree -2. Then $A\{-1\}$ has X in degree zero and 1 in degree -2. So there is a natural choice, which takes $1 \otimes X$ and $X \otimes 1$ to X and $1 \otimes 1$ to 1. This makes A an associative commutative ring with unit 1 and $X^2 = 0$.

Then when we take homology, we get $\mathbb{Z}$ in degrees 0 and 2. This is only a small problem, this shift from -1 and 1. We'll deal with it later.

There's one other case when you have a single crossing, the mirror image. Then we need a map $\Delta : A \to A^{\otimes 2}\{-1\}$. A has X and 1 in degrees 1 and -1. Then $A^{\otimes 2}\{-2\}$ has $X \otimes X$, $1 \otimes X$ and $X \otimes 1$, and $1 \otimes 1$ in degrees $1, -1, -3$. So for this to have the right homology we want neither map to be zero. We take X to $X \otimes X$ and then for symmetry choose $1 \mapsto X \otimes 1 + 1 \otimes X$. The homology is two $\mathbb{Z}$ in degrees -1 and -3.

So this gives an algebra and coalgebra structure on A with a trace which takes X to 1 and 1 to zero allowing us to pass from multiplication to comultiplication.

So what if we have a knot diagram with crossings? Here's the trefoil. You are going to get an n-dimensional cube where n is the number of crossings. At each vertex there's a resolution, so at $(0, 1, 0)$ we get the 0-resolution at the first and third crossing and the 1-resolution in the second place.

Can we take homology? We can:

$$\begin{tikzcd}
& A^{\otimes 2}\{-1\} \arrow[rr] \arrow[dd] & & A\{-2\} \arrow[dd] \\
A^{\otimes 3} \arrow[ur] \arrow[rr] \arrow[dd] & & A^{\otimes 2}\{-1\} \arrow[ur] \arrow[dd] & \\
& A\{-2\} \arrow[rr] & & A^{\otimes 2}\{-3\} \\
A^{\otimes 2}\{-1\} \arrow[ur] \arrow[rr] & & A\{-2\} \arrow[ur] &
\end{tikzcd}$$

Then we shift these to look at the diagonals to get things in the right order, and we also need to add an odd number of negative signs on each square to make things anticommute instead of commuting, and we get

$$0 \longrightarrow A^{\otimes 3} \xrightarrow{d} A^{\otimes 2}\{-1\} \oplus A^{\otimes 2}\{-1\} \oplus A^{\otimes 2}\{-1\} \xrightarrow{d} A\{-2\} \oplus A\{-2\} \oplus A\{-2\} \xrightarrow{d} A^{\otimes 2}\{-3\} \longrightarrow 0$$

So this is a complex of graded Abelian groups. Then we can shift it to the left by $x(D)$ and in the vertical degree by $2x(D) - y(D)$. So once we apply the shift $[x(D)]\{2x(D) - y(D)\}$ we call this complex $C(D)$. Then $H(D) = H(C(D))$.

Theorem 1 *$H(D)$ is a bigraded Abelian group $H(D) = \bigoplus_{i,j \in \mathbb{Z}} H^{i,j}(D)$. It's immediate from the construction that $\chi(H(D)) = \chi(C(D)) = K(D) = J(L)$.*

The content of the theorem is that if D_1, D_2 are related by a Reidemeister move then $C(D_1) \sim C(D_2)$ so that $H(D_1) \cong H(D_2)$ with the isomorphism preserving the bigrading. So the groups are invariants of links and can be called $H(L)$.

Let me write again $J(L) = \sum (-1)^i q^j \mathrm{rk} H^{i,j}(L)$.

Next time I will sketch a proof that this is functorial. You want homology to be a functor. The regular homology is a functor **Top** $\to$ **gr Ab** which takes $X \to H_*(X)$ and $f : X \to Y$ to $f_* : H_*(X) \to H_*(Y)$.

We have cobordisms between links, and cobordisms will be taken to homomorphisms of bigraded Abelian groups.

Let's look at some examples. Here's the homology of the knot 10_{121} (in Rolfsen notation). Bar-Natan wrote a program to compute these. For alternating knots you see that the homology is concentrated on two diagonals. You only have homology groups in odd degrees, and this is immediate from the construction. You will always only have odd degrees for a knot. For two component links they are always in even degrees.

Another amazing thing is that the width is 10. If you look at the construction, the number of terms in the complex is $n + 1$ where n is the number of crossings in the diagram. There is also an overall shift. The width is at most n. For alternating knots this bound is strict.

I should say what these numbers are. These are not supposed to be numbers but Abelian groups. 1 means $\mathbb{Z}$, 4 means $\mathbb{Z}^4.4$, 1_2 means $\mathbb{Z}^4 \oplus \mathbb{Z}_2$. There is only 2-torsion in this example. If you ignore torsion you can match the coefficients diagonally. Most of these patterns were explained by E. S. Lee, on the arxiv.

Let's look at another alternating knot 10_{123}. Again there's only 2-torsion and everything is on two diagonals. This is symmetric because this knot is amphichiral.

Exercise 2 *If $D^!$ is the mirror diagram, then $(C(D))^* \cong C(D^!)$ where the dual is $Hom(C(D), \mathbb{Z})$. Then rationally $H^{i,j}(L^!, \mathbb{Q}) \cong H^{-i,-j}(L, \mathbb{Q})$.*

Here's 11^n_{31}. Here's the $(-3, 4, 5)$-pretzel knot. The width is smaller than the crossing number. So that bound is not sharp. In these two there are smaller ranks than we were seeing with the alternating knots.

A few months ago, it was found that the $(5, 6)$ torus knot has 5-torsion and 3-torsion, which was a big surprise. So somehow the torus knots are perpendicular to alternating knots, the nonvanishing groups lie on many diagonals, and the width is only about half of the crossing number.

These computations are mainly due to Shumakovitch.

Let me give you another exercise.

Exercise 3 *Show that $C(D \sqcup D') = C(D) \otimes C(D')$. This implies the Kunneth formula for homology, so that $H(L \cup L', \mathbb{Q}) \cong H(L, \mathbb{Q}) \otimes H(L', \mathbb{Q})$.*

Exercise 4 *Compute the homologies of the Hopf link and the trefoil.*

I think that's plenty for one day, I'm going to stop now.

2 Szabo

Okay, so last time we were studying homotopic maps $\pi_2(x, y)$. We also want the Maslov index, which is something like the expected dimension of the moduli space holomorphic maps.

You can change your family of complex structures and so the solution space can be smaller, which would make sense of a negative Maslov index.

There is a master solution due to Robert Lipschitz for Heegaard Floer homology. Let me remind you that we assigned $D(\phi) = \sum n_{z_i}(D)D_i$.

Here on the blackboard you have seven different examples. All of them have zero and one coefficients. In general this isn't true. I shaded the regions where it's one. Here's one we've seen before. They are mostly in the second symmetric product, this one is in the third. This is a more complicated example. I am looking at this disk, and then I try to stabilize an additional genus. If you think about the conditions $D(\phi)$ has to satisfy, it is satisfied. At this point when you take the whole region then the boundary is zero, it goes from c to c. This is an even more complicated example. You look at it and take the boundary, that goes from x to y on α, y to x on β, x to y on α and then y to x on β. Is that okay?

Now we want to compute the Maslov index. Here's a combinatorial formula due to Lipschitz for the index.

Definition 1 *You have all these domains and we would like to define some functions on the domains. Let $e(D_i)$ for a 2-gon is $1/2$, for the square is 0, for the 6-gon minus one half, and for the n-gon $1 - n/2$.*

You want this to be additive so if it's a more complicated region, it's okay to cut it into more parts if you like. For something that looks like this, I can cut it into two parts, this is a $2n$-gon and this is too. Take your region, cut it into parts, and use additivity.

This is one definition. It's not quite an Euler characteristic, you also use how many boundary components you have.

So using this, one can write down $e(\phi) = \sum n_{z_i}(\phi)e(D_i)$.

We need some other things. So for instance we nneed a point measure $\mu_p(\phi)$. Suppose you have p an intersection of α_i and β_j. There are four regions around this meeting; they may be different or the same. If you have basepoints in the four regions then $\mu_p(\phi)$ is the average of the n_z : $\frac{1}{4}(\sum n_{z_i}(\phi))$.

So for $\phi \in \pi_2(x, y)$ with $x = x_1, \ldots, x_g; y = y_1, \ldots, y_g$, we have $\mu_{\underline{x}}(\phi) = \sum \mu_{x_i}(\phi)$.

Now the formula for the Maslov index is $\mathrm{Mas}(\phi) = e(\phi) + \mu_x(\phi) + \mu_y(\phi)$. So that's a great help for us. Here I know the answer should be one. I have coefficients $1, 0, 0, 0$ and $1, 0, 0, 0$ with $e = 1/2$ so here I get Maslov index one. Here I also get one, because e is zero. This is more complicated in the third symmetric product. Here e is $-1/2$ and pick up $1/4$ in each of the six points so again get 1. In this one you get some problems studying holomorphic disks. You have to fix the structure, and you have three too many points for the holomorphic structure. You might think it depends on the complex structure but it's not true.

The Maslov index is one here. And here, again. Here it should be two. This one can be decomposed into the sum of these two homotopy classes. Do we get that from the formula?

Here $e = 0$. Here it's 1. Here we get some $1/2$ and get index two. Here it's also one.

Sometimes it's easy to compute and sometimes it's a challenge.

Okay, so this is a beautiful formula but it's really a theorem, not a definition. Nevertheless, if you look at this formula, some characteristics are not obvious. It's always an integer.

Exercise 5 *The right hand side is always an integer.*

Another useful property is that the Maslov index is additive. If I have one from x to y and another from y to w then $M(\phi_1 \# \phi_2) = M(\phi_1) + M(\phi_2)$.

I finished the part of the lecture that was supposed to last ten minutes so I'll speed up.

What do we require from holomorphic disks? For some homotopy classes the Maslov index is negative. In this case what do we want? We want the moduli space to be empty. It's negative dimensional. This is not necessarily true. Now what happens when the Maslov index is zero? It could be that $x = y$ and ϕ is the constant homotopy class. Then there's no way to reparameterize the constant map. Then $M(\phi)$ is a point. The other case is that it's supposed to be zero dimensional with a free $\mathbb{R}$ action on it so we expect it to be empty.

These are the easy cases where we don't have to deal with homotopy classes. There's the case where $\mathrm{Mas}(\phi) = 1$, so $M(\phi)/\mathbb{R}$ is zero dimensional. Then you also want it to be compact. That is, if you manage to get the two conditions right, then the last part about compactification, and it's really important that we are using Lagrangian submanifolds. There's a nice theorem of Gromov about studying degenerations of surfaces and then disks (very soon) and that shows that there's nothing to converge to. You want to cut the number of points so you want a finite number of points.

When the Maslov index is two, when we divide by the $\mathbb{R}$ action it should be one-dimensional, but we can no longer ensure compactness. I am going to erase everything except the index two example.

Okay, so we have this example. Maybe I can write the whole thing slightly larger. The class we're looking at goes from x to y. So we would like to understand the moduli space. It's very easy to see one solution in that space. Look at this heart-shaped region. It's simply connected, and then I have $\mathbb{R}$ worth of maps once I stabilize two points, and once I mod out by $\mathbb{R}$ I get one map. But are there other solutions? There should be one dimension worth of solutions. I could overshoot and come back. I can make this arbitrarily long, so if I fix the length I get a unique solution. I can do it in the other direction. If I cut in both directions the cuts overlap inappropriately and I can't use the Riemann mapping theorem any more. So you can get $(-1, 1)$ with varying length cuts on the two edges. It has a compactification as an interval and also in terms of holomorphic disks. What happens when I try to cut almost completely to here? I get two holomorphic disks. This corresponds to a decomposition of the original homotopy class into two homotopy classes.

These correspond to going from x to p by D_2 and D_3 and then p to y by D_1. You can also go the other way by going from x to q with D_1 and D_3 and q to y by D_2.

The trick will be that when you take limits, you can no longer use the $\mathbb{R}$ actions. Instead you fix the structure before taking the limit. This is called a broken flow line. This is an example of a Maslov index two homotopy class having an interesting compactification. Unfortunately you need extra conditions or not everything will be defined.

Here is a more interesting, ugly example here. Luckily it won't happen in our case. Here two of the α will be homologous, where for us they will be linearly independent. This is in the second symmetric product. I want to look of $x = (a, b)$ to $x = (a, b)$. Then $D(\phi$ will be $D_1 + D_2$ where the region is an annulus with two cuts. You can use the formula to see the Euler measure is zero here and here, so e is zero, and μ_x is $1/2 + 1/2$, and the same for y, so it's two.

So we want something like

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \Sigma \\ D^2 \arrow[r] & Sym^2(\Sigma) \end{tikzcd}$$

There is a dictionary between maps from D^2 to the symmetric product and F into Σ and D^2.

Some of these annuli will not come from branched covers. It's a one-dimensional manifold after the $\mathbb{R}$-action.

Exercise 6 *In the compactification, I use longer and longer cuts on one end, this broken flow line type. But were I make smaller and smaller cuts, that's very different. That's like cusping a new disk off of the disk from x to x. If you look at this region, with no cut at all, there is a map in the second symmetric product with boundary completely in p_α.*

In this case it's a one-dimensional manifold. These are the things where we want to say, like Maslov index negative means empty, Maslov index zero means compact, in two you have a compactification.

[unintelligible][unintelligible][unintelligible]So fixing a complex structure, varying your complex structure, so everything shifts up or down. If you do it this way the moduli space will vary by the $\mathbb{R}$-action still.

[A question about branched covers]

The correspondence is that when you take the map, that also gives you a two-chain, and the multiplicity of that map over each region is n_z. In this example here $d(\phi)$ is one. In this case it has to be an annulus. In general you have to consider the topological type of the surface.

Why do we care about this? We wanted to define the boundary map. Definitely having a compact manifold lets us count the number of points. For $\delta^2 = 0$ we need to look at Maslov index two. It's enough to show this on a generator. I count with multiplicity (in δx), it maybe uses p, and maybe δ of that gives me y. So there was a homotopy class here with

Maslov index one from x to p and again from p to y. So then the Maslov index of the class from x to y is 2. Suppose I would like to show $\delta^2 x = 0$. Then I just have to show that the coefficient of $\delta^2 x$ at any given point y. Take $\pi \in \pi_2(x, y)$. Here the Maslov index is two. Take this moduli space with the $\mathbb{R}$-action. This is a one-manifold with a compactification. There is the broken flow line kind and then the other uglier kind. If I can show that these flow line degenerations are the only ones, then the number of ends would be even, and they would be the number of ways to go from x to p times the number of ways to go $p \to y$. This is exactly what happens here,

$$\begin{tikzcd}
& x \arrow[dl] \arrow[dr] & \\
p \arrow[dr] & & q \arrow[dl] \\
& y &
\end{tikzcd}$$

Anyway, the point is that the proof of $\delta^2 = 0$ is easy if we have only the good kind of degeneration, because in the one kind of degeneration all of the boundary will lie on the α.

I wanted to say why we have a chain complex. We don't have time, so say $H_1(Y) = 0$ and $g > 2$. In this case $\pi_2(x, y)$ is $\mathbb{Z}$ for every two points. Then you can look at $n_z : \pi_2(x, y) \to \mathbb{Z}$. In $\widehat{CF}$ we only looked where $n_z = 0$. So there's a unique choice for any x, y. So we have a grading $gr(x, y) = \mathrm{Mas}(\phi)$. What you still have to think about is that the degenerations are counted in d^2. We only counted those that did not intersect z. Maybe in both cases here n_z was zero, and so the homotopy class of the sum is zero. But you could also have $\pm k$ as n_z. But n_z is the intersection number of something holomorphic with something holomorphic and it won't be negative. If the moduli space is nonempty, then $n_z(\phi) \geq 0$. This allows us to show that $\widehat{CF}$ is a chain complex and the corresponding thing is $\widehat{HF}$. There is a more general thing called CF^∞. The generators are $[x, i]$ where i is an integer. The boundary map is the sum over the homotopy classes with Maslov index one of the number of solutions once I've gotten rid of the $\mathbb{R}$-action times $yi - n_z(\phi)$:

$$\sum_y \sum_{\substack{\phi \in \pi_2(x,y) \\ \mathrm{Mas}(\phi) = 1}} \#\left(\frac{M(\phi)}{\mathbb{R}}\right)[y_j i - n_z(\phi)]$$

So we can look at the subcomplex CF^- where $i < 0$. Then $CF^+ = CF^\infty / CF^-$. There is a long exact sequence

$$\begin{tikzcd}
HF^-(Y) \arrow[r] & HF^\infty(Y) \arrow[r] & HF^-(Y) \arrow[ll, bend left]
\end{tikzcd}$$