Low Dimensional Topology Notes
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1 Khovanov

[I have a few announcements to make. Now we have a biorganizer arrangement. Here is the alternative organizer. There will be problem sessions today, after this talk and at 1:00, in the tent. There will be research talks by Nathan Dunfeld and later by Matt Heden.]

[I'm Peter Oszvath, I'm also an organizer. I thought I'd alleviate the problem of space here by showing up a day late. Well, hi.]

In the second lecture today we'll talk about A_n, braids, and braid cobordisms. In three and four we'll do a categorification of the Jones polynomial and an extension to tangles and tangle cobordisms. I'll say that's through 4.5. In the remainder we'll do more advanced topics, like the HOMFLY polynomial, Hochschild homology, and matrix factorizations.

Let's quickly review what we talked about yesterday.

We have rings A_n, projective left and right modules $P_i = A_n(i)$, ${}_iP = (i)A_n$.
$U_i = P_i \otimes_i P$.

R_i is the bimodule complex $0 \to U_i \to A_n \to_0$ where the map is β_i; R_i' is the complex $0 \to A_n \to U_i \to 0$ with map γ_i. Then $C(A_n)$ is the homotopy category of complexes of A_n-modules. The theorem last time was:

Theorem 1 *$R_i \otimes R_i' \cong A_n \cong R_i' \otimes R_i$;*
$R_i \otimes R_j \cong R_j \otimes R_i$ for $|i-j| > 1$;
$R_i \otimes R_{i+1} \otimes R_i \cong R_{i+1} \otimes R_i \otimes R_{i+1}$. So the braid group B_{n+1} acts on $C(A_n)$.

For $M \in \mathrm{ob}C(A_n)$ we have the functors $F_i(M) = R_i \otimes M$ and $F_i'(M) = R_i' \otimes M$. Then $F_iF_i' \cong Id \cong F_i'F_i$. We are distinguishing between the bimodules and the functors of tensoring with them.

Briefly, I'll discuss how to prove one of these. What is $R_i \otimes R_i'$? You tensor pairwise. You get

$$\begin{tikzcd}
U_i \otimes A_n \arrow[r] \arrow[d] & U_i \otimes U_i \arrow[d] \\
A_n \otimes A_n \arrow[r] & A_n \otimes U_i
\end{tikzcd}$$

So we can simplify this right away to:

$$\begin{tikzcd}
U_i \arrow[r] \arrow[d] & U_i \otimes U_i \arrow[d] \\
A_n \arrow[r] & U_i
\end{tikzcd}$$

So summing along the diagonals we get $0 \to U_i \to A_n \oplus U_i \oplus U_i \to U_i \to 0$. Computing the differential we can split this as

$$0 \longrightarrow U_i \xrightarrow{1} U_i \longrightarrow 0$$

$$0 \longrightarrow A_n \longrightarrow 0$$

$$0 \longrightarrow U_i \xrightarrow{1} U_i \longrightarrow 0$$

Since we're in the homotopy category of complexes, this is isomorphic to just the middle factor A_n.

Last time I also claimed that the action was faithful. In $C(A_n)$ we have P_i. So we can apply braids to the P_i. I can have notations, let F_σ be the action of σ as a product of F_i and F_i'.

So what is $F_\sigma(P_i)$? So what is $F_2(P_1)$? You take $0 \to U_2 \to A_n \to 0) \otimes P_1$, which is $0 \to P_2 \to P_1 \to 0$. Here the map is the composition with the path $(2|1)$. So that $F_2^m(P_1) = 0 \to P_2 \to \cdots \to P_2 \to P_1$.

The general answer is the following. First recall that the braid group $B_n + 1$ is the mapping class group of the disk with $n+1$ marked points. This acts on isotopy classes of simple curves on D^2 with marked points. So we start with basic curves c_i and we can apply a braid to these c_i and get some curve in the disk.

Let's do an example: [Picture]. I can drop perpendiculars ℓ_i to the c_i which partition the disk into regions each of which carry one marked point. I take σc_i and express it minimially with respect to intersection with ℓ_j. I want to get a complex from this. I partition the curve along the intersection points, forget the end pieces, and orient each curve to go clockwise around the marked point. Then I mark each point with the index of the corresponding ℓ_j and am left with something like:

$$1 \longrightarrow 2 \longrightarrow 2 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow 5$$

Now I take this diagram and bend it so that the arrows point to the right:

$$\begin{tikzcd}
 & & 1 \arrow[r] & 2 \arrow[dr] & \\
 & & & & 2 \\
5 \arrow[r] & 4 \arrow[r] & 3 \arrow[r] & 2 \arrow[ur] &
\end{tikzcd}$$

Now replace these with projectives:

$$\begin{tikzcd}
 & & P_1 \arrow[r, "(1|2)"] \arrow[d, dash, dotted] & P_2 \arrow[dr, "X_2"] \arrow[d, dash, dotted] & \\
 & & \oplus \arrow[d, dash, dotted] & \oplus \arrow[d, dash, dotted] & P_2 \\
P_5 \arrow[r, "(5|4)"'] & P_4 \arrow[r, "(4|3)"'] & P_3 \arrow[r, "(3|2)"'] & P_2 \arrow[ur, "X_2"'] &
\end{tikzcd}$$

It is a theorem that in general this is what you will get from $F_\sigma(P_i)$.

So to show that this is faithful we want to say $F_\sigma \neq F_1$. Then it is enough to find P_i with $F_\sigma(P_i) \neq P_i$. To do that it's enough to find P_j so that $Hom(F_\sigma(P_i), P_j) \neq Hom(P_i, P_j)$. To do that it's enough to show they have different dimensions. We'll do a combinatorial description. For σc_i you get some complicated curve. Put this in minimal intersecting position with c_j. Let me use $HOM(M, N) = \oplus Hom(M, N[i])$, Then $HOM(F_\sigma(P_i), P_j)$ has dimension twice the number of internal intersections of σc_i and c_j plus the number of common boundary points of $\sigma(c_i)$ and c_j. Any nontrivial braid will have a nontrivial intersection with some c_j. Then you can calculate that the action is sufficiently complicated that this is faithful. I am suppressing all the details.

Any questions? Then let me go on to braid cobordisms.

Why do we care about acting on a category, we already have many things from just acting on vector spaces. There is possibly more information, because functors have natural transformations. So to a braid we assign a functor. What is the meaning of natural transformations? These are assigned to braid cobordisms. We'll have a category of braids with objects braids and morphisms braid cobordisms, which look like, well, what is a braid?

It's an embedding of $n+1$ intervals into $\mathbb{R}^2 \times I$ so that the projection onto I has no critical points. Then a cobordism will be a surface in a four dimensional space with boundary two braids and then two sets of flat intervals (fixing the boundary points of the braid). We want the projection onto $[0,1]^2$ to be sufficiently nice. In particular we want it to be a branched covering with simple (double) branch points only. We can compose these by gluing. We add branch points to get changes in topology. As you pass through branch points you change a non-crossing to a single crossing. So we go to or from $\tau_1\tau_2$ to or from $\tau_1\sigma_i^{\pm 1}\tau_2$. At the branch point the arcs kiss.

The reference for this is Carter and Saito. So what is this thing? In one sense it's in four dimensional topology. But in another sense we have some things we can do in smaller dimensions. We can move them around but not get rid of them. So there's also the idea of positive and negative, a branch point is positive if it adds a positive crossing or kills a negative crossing. It's negative if it adds a negative or kills a positive crossing.

Okay, but we want to assign to a cobordism a natural transformation. To go from the functors corresponding to the modules $0 \to A_n \to 0$ to $0 \to U_i \to A_n \to 0$.

We'll use the map 1 :

$$\begin{tikzcd} & 0 \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d, "1"'] & 0 \arrow[d] \\ 0 \arrow[r] & U_i \arrow[r] & A_n \arrow[r] & 0 \end{tikzcd}$$

Similarly

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r] \arrow[d, "1"'] & U_i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_n \arrow[r] & 0 & \end{tikzcd}$$

The negative ones are not as nice, after many tries we get

$$\begin{tikzcd} 0 \arrow[r] & U_i \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d, "X_{i-1}-X_{i+1}"] & 0 \\ & 0 \arrow[r] & A_n \arrow[r] & 0 \end{tikzcd}$$

This is nilpotent, this map $(X_{i-1} - X_{i+1})^2 = 0$. It's an exercise to

Exercise 1 *Do the one remaining case*

So now we have a natural transformation from any cobordism by composing these.

Okay, so some are positive and some are negative. If you have only positive branch points it comes from a holomorphic something in $\mathbb{C}^2$. Take $Y \subset \mathbb{C}^2$ mapping to $\mathbb{C}$. You have a disk which you pull back to get $D \times \mathbb{C}$ and then intersect with Y and get a braid cobordism, positive. It's a theorem of Rudolph that any positive cobordism has this form. The two types have different invarionts, that's an interesting thing. Iy S is positive, then there is $F_S : F_\sigma \to F_\tau$, and this is never zero. For most negative S the natural transformation is zero because the X_i are nilpotent.

If you have two negative branch points next to one another the corresponding map is already homotopic to zero. This is supposedly similar to gauge theory where things act in very different ways on the holomorphic and antiholomorphic things. This is possibly a very naive shadow of those very fancy things.

I'll stop here.

2 Cameron Gordon

[We're almost ready to begin. Two announcements. There are lecture notes and problem sets beginning to appear on the back table. There will be a problem session for Szabo in the big tent at 1:00 PM. Okay? Okay. Cameron Gordon from, where are you from?]

I have an announcement to make too. A distinguished English sailor is going to be giving a special presentation. He's related somehow to Colin Adams.

Thanks for inviting me to participate. But they should have coordinated better with FIFA.

I'm going to talk about Dehn surgery basically, but let me start off by, there's going to be some overlap between what I say today and what John said in his lecture.

Let me say something about 3-manifolds in general. I'll always assume 2 and 3-manifolds are compact, orientable, connected, unless they're obviously not.

Let me talk about incompressible surfaces. If F is a surface in M^3 either in δM or more usually "properly embedded" meaning $F \cap \delta M = \delta F$. Then we have the basic notion due to Haken of a compressible surface. We say F is compressible if there exists a disk D in M with $D \cap F = \delta D$ and δD is essential in F, i.e., doesn't bound a disk.

Writing on these blackboards is a little bit like writing on a boat. Here's a picture of compressibility.

If you have a compressible disk you can cut along it, what's called performing surgery along it, and get a simpler surface.

Theorem 2 *Disk theorem (Papakyriakopoulos, 1957)*
$F \subset M$ is incompressible if and only if $\pi_1(F) \hookrightarrow \pi_1(M)$.

One direction is easy. If you're compressible, then the boundary of the disk is nontrivial and then gets killed in M. The other direction is harder.

I'll be talking about essential surfaces. A properly embedded surface F in M is essential if either

1. it's a two-sphere that doesn't bound a three-ball,
2. F is a disk and δF is essential in δM

3. it's neither a disk nor a sphere, it's incompressible and not boundary parallel (meaning there exists an embedding $F' \times I$ into M with $F' \times \{0\} \subset \delta M$, and $F = F' \times \{1\} \cup \delta F' \times I$.)

If M does not contain an essential S^2 then we call M irreducible. Your first exercise is

Exercise 2 *M is prime if and only if M is either irreducible or $S^1 \times S^2$.*

This irreducibility is a key thing. One thing is,

Theorem 3 *the three-dimensional Schönflies theorem (Alexander 1924)*
S^3 is irreducible.

I should have said, either this is piecewise linear or smooth. Alexander proved, Schönflies proved that every S^1 bounds a disk in S^2. Alexander announced this in three dimensions for S^2 in S^3, and then found a counterexample, the horned sphere. This is true, though, if the S^2 is smooth.

I'm going to deal with surfaces of nonnegative Euler characteristic, that is, S^2, D^2, A^2, and T^2. Every three-manifold can be cut along such surfaces into canonical pieces. Let me repeat what John said. For S^2 we have

Theorem 4 *Prime Decomposition theorom, (Kneser 1929, Milnor 1962) M (oriented) is a direct sum of prime manifolds, with pieces unique up to orientation preserving homeomorphism.*

The spheres aren't unique but the pieces you get are unique. Let me say something about disks. There's a theory of doing that. At every, you might as well assume that you've cut the manifold up into irreducible pieces. Then there's $W^3 \subset M$ unique up to isotopy such that $\delta M \subset W$ and $\overline{M - W}$ is irreducible, but now δ-irreducible, meaning it contains no essential disk. Let me draw a picture.

For example, if M is a handlebody as defined in Zoltan's talk, you can completely compress the boundary, and W is all of M. Well, that's not right. I should say W is useful.

If I defined a compression body, I could have said it was that.

What about when you cut up along tori and annuli? This is important in the context of the geometrization conjecture, but let me say it differently.

Definition 1 *M is a Seifert fibered space if and only if M is a disjoint union of circles (referred to as fibers) such that each fiber has a neighborhood which is a fibered solid torus V, Identify the two ends of $D \times I$ by a rotation of $2\pi p/q$ for p, q relatively prime. The fibers for $x \neq (0,0)$ are images of $x \times I, \rho(x) \times I, \ldots, \rho(x)^{q-1} \times I$. The central fiber is $(0,0) \times I$. So a Seifert fibered space is a union of such solid tori such that the boundaries line up correctly.*

If $q \geq 2$ then the central fiber is a so-called exceptional fiber of multiplicity q. Here's a picture.

On the boundary the fibers are p, q curves. We have to orient things properly, this is one of the pains of this subject.

So that's a Seifert fibered space. It's like a singular circle bundle. If you take each fiber and identify it to a point, you get a projection $M \to F$ to a surface called the base surface. You have a finite number of marked points where the singular fibers map.

At this point let's say that M is simple if it does not contain an essential sphere, disk, annulus, or torus.

Then the theorem about cutting manifolds up along annuli and tori is

Theorem 5 *(Jaco-Shalen, Johannson, 1976)*
Assume M is irreducible and δ-irreducible. Then M contains a disjoint union $\mathscr{F}$, unique up to isotopy, of essential tori and annuli such that each component of M cut along $\mathscr{F}$ is either simple or a Seifert fibered surface, or an I-bundle over a surface.

What does simple mean? It means you don't have annuli and tori. You have to prove that the cutting can't go on forever, which is a classical theorem of Haken and goes back to Knezer.

Haken introduced incompressible surfaces, and then used the normal surface idea to show that there can be only finitely many disjoint incompressible surfaces.

Suppose your manifold is a surface cross S^1. If you cut along simple curves and arcs, you can cut the manifold into being a solid torus. But that's not unique. The clever idea is not to bother to decompose the I-bundles and the Seifert fibered surfaces.

Let me continue with the Seifert fibred spaces.

Let me go back and say something about, this will come up when I finally talk about Dehn surgery.

Let me remind you that there is a base surface F and n exceptional fibers in a Seifert fibered surface.

Definition 2 *A Seifert fibered space is small if and only if*

- $F = S^2, n \leq 3$
- $F = D^2, n \leq 2$
- $F = A^2, n \leq 1$
- $F = P^2, n \leq 1$
- *F is the Mobius band, $n = 0$.*

M is not small implies M contains an essential T^2.

Exercise 3 ***Lemma 1*** *M is an irreducible 3-manifold with δM a collection of tori. If M conains an essential annulus, then either M contains an essential T^2 or M is a small Seifert fibered space.*

Generically Seifert fibered structures are unique, but there are some small counterexamples.

We now are left with the pieces which are simple, don't contain any compressible such surfaces.

Theorem 6 *Geometrization conjecture (Thurston if $\delta \neq 0$, 1980; Perelman 2003) M is simple if and only if either*

1. *$M_0 = M$ minus the torus components of the boundary has a complete hyperbolic structure with the remaining boundary totally geodesic. I will call this M being hyperbolic.*
2. *M is a closed small Seifert fibered space, or*
3. *$M \cong B^3$. But the theory of this manifold is well understood.*

I should have said, well, the round manifolds that John was talking about, the Seifert fibered spaces of the form $S^2, n = 2$. So this is a manifold of Heegaard genus at most one, $S^3, S^1 \times S^2$, and lens spaces. All round 3-manifolds, in fact, are of the form, in fact, let me, it would be useful to say, the n exceptional fibers have multiplicities $q_1, \ldots, q_n$ so I can list multiplicities. All other round three-manifolds are of the form $S^2(q_1, q_2, q_3)$ where these are a plotonic triple, $\sum \frac{1}{q_i} > 1 :\ (2,2,n), (2,3,3), (2,3,4), (2,3,5)$.

[Some background] but these small ones were the motivating example. The $(2,3,5)$ is the first homology sphere.

In the last ten minutes let me start talking about knots and how they fit into this general pattern.

Here we have $K \subset S^3$ and I'll tend to use M_K to be S^3 minus the interior of a tubular neighborhood of K. By the Schönflies theorem, every 2-ball bounds a three-ball. This leads directly to M_K being reducible. Another of your homework problems is

Exercise 4 *M_K is δ-reducible if and only if $M_K \cong S^1 \times D^2$ if and only if K is the unknot, if and only if $\pi_1(M_K) \cong \mathbb{Z}$.*

Definition 3 *The (p,q)-torus knot $T_{p,q}$ is the (p,q)–curve in the boundary of a Heegaard torus in S^3.*

It's not so difficult to prove that

Exercise 5 *the exterior is a Seifert fibered space of type $D^2(|p|, |q|)$.*

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The whole thing will be the two glued together along the complement of the knot. You have two solid tori with p, q and q, p fiberings, and you glue them together and one will have p fibers in the core while the other will have two.

Let me talk about satellite knots. You want $J \subset$ a solid torus V but not sitting in a ball in V and not isotopic to $S^1 \times (0, 0)$. Pick a nontrivial knot K_0 in S^3. Pick a neighborhood and map V to it via a homeomorphism. Then you let K be the image of J. That's a joke to an older generation of topologists, but it's not a joke here, clearly.

You tie the solid torus in a knot, essentially. You call it a satellite of K_0.

Exercise 6 *If you look at the boundary of $N(K_0)$, prove this is essential in M_K, incompressible and boundary parallel.*

Bear with me just another minute.

Theorem 7 *For K a knot in S^3 exactly one of the following holds:*

1. *K is the unknot (contains an essential D^2)*
2. *K is a torus knot (contains an essential annulus but not an essential torus).*
3. *K is a satellite knot (contains an essential torus)*
4. *the generic case, M_K is simple, so has a complete hyperbolic structure, so we call K hyperbolic.*

What I'm going to talk about next is Dehn surgery. We'll mostly focus on hyperbolic knots. We'll start with the others and then start doing Dehn surgery on hyperbolic knots and go on to construct examples from those.

3 Szabo

Last time we were discussing Heegaard diagrams $(\Sigma_g, \underline{\alpha}, \underline{\beta}, z)$. The basepoint is not necessary, it's a technical requirement. We are going to make a chain complex. The generators of $\widehat{CF}$ were $\cup_{\sigma \in S_g} \prod_{i=1}^{g} (\alpha_i \cap \beta_{\sigma(i)})$.

There's a much better way to look at the generators. They can be given by this other construction. First take the g-fold symmetric product of Σ_g. This is $\Sigma_g \times \cdots \times \Sigma_g / S_g$. Inside are two tori that you can associate to the two handlebodies $T_\alpha = \alpha_1 \times \cdots \times \alpha_g \subset Sym^g(\Sigma_g)$ and similarly T_β. The generators are $T_\alpha \cap T_\beta$.

An easy observation shows that $Sym^d(\mathbb{C}) \cong \mathbb{C}^d$. I can identify $\mathbb{C}^d$ with polynomials with form $a_0 + a_1 z + \ldots + a_d z^{d-1} + z^d$. Then taking the roots you get d points in $\mathbb{C}$ and that gives the correspondence.

Similarly we get a complex structure on $Sym^g(\Sigma_g)$ from the structure on the product.

There are a few exercises related to this.

Exercise 7
1. *$H_1(Sym^d(\Sigma_g)) = H_1(\Sigma_g)$. It's easy to get a map from the right to the left. It's harder to show it's injective and surjective.*
2. *$Sym^2(\Sigma_2) \cong T^4 \# \bar{\mathbb{CP}}^2$.*
3. *For $g > 2, \pi_2(Sym^g(\Sigma_g)) = \mathbb{Z})$.*

We'll move on to holomorphic geometry soon but for now you can think about easier invariants. Let's look at the algebraic intersection number between T_α, T_β.

Exercise 8 *Show that this intersection number is 0 if $b_1 > 0$. If $b_1 = 0$ then it is $|H_1(Y)|$. This is a torsion group so it has finitely many elements. Really you need the identification from the first exercise.*

This is not really an interesting invariant. So $\widehat{CF}$ needs the boundary map. We have

$$\delta x = \sum_{y \in T_\alpha \cap T_\beta} C(x, y) \cdot y,$$

where $C(x, y)$ is "the number of holomorphic disks from x to y."

So what do I mean by holomorphic disks? Great, so, we're going to look at the unit disk in $\mathbb{C}$. We want maps from here to the symmetric product so that, it's going to be holomorphic. D of it will be a holomorphic map of the tangent spaces. I also want something about the boundary. Inside, T_α and T_β will be sitting, as in this picture, and I want $U(e_1)$ to lie in T_α and $U(e_2)$ to lie in T_β and I want $U(-i) = x, U(i) = y$. For lots of technical reasons we want, so that the moduli space will be smooth, sometimes you want a one-parameter family of complex structures on the symmetric product.

An alternate definition would be to look at holomorphic strips. I would say the same thing, the two sides will map to T_α and T_β.

[Milnor: what happens when $x = y$?]

Excellent question. I have to, well, so, that would be much easier to answer in a different talk.

The constant map is really special, but any other nonconstant holomorphic map, on this infinite strip, there are reparameterizations, and I could precompose and get another map. If it's constant precomposing gives the constant map again, unlike in the other cases. I would like to divide by the reparameterization. What happens is that components of this moduli space will have an expected dimension. I'll talk more about that and write down some formulas, but the bottom line is that between x and y there might be lots of components. I want to count a finite number of solutions. I want a zero dimensional moduli space so I can count the number of solutions.

To make this precise, I can talk about $\pi_2(x, y)$. This would be topological maps satisfying these boundary conditions, and homotopy classes of these maps. After I fix a homotopy class like that, $\phi \in \pi_2(x, y)$, I can look at the space of all holomorphic maps representing ϕ. The Maslov index $\text{Mas}(\phi) \in \mathbb{Z}$ gives the expected dimension of the moduli space. When I say count the holomorphic disks, I'm going to look at homotopy classes $\phi \in \pi_2(x, y)$, and then I will only look when $\text{Mas}(\phi) = 1$. Then after modding out by the reparameterization I can sum with numeric coefficients $\#(\mathscr{M}(\phi)/\mathbb{R})$. There are sign issues and then there are things about when the Maslov index is one and so there's still work to do. When $b_1(Y)$ is zero this is almost exactly what we're going to do. This definition, when it's bigger than 0 we are worried about whether this sum is finite. So we only look at special Heegaard diagrams. For $S^1 \times S^2$ we have these two Heegaard diagrams. This one will be a good Heegard diagram [picture] and this will be a bad one.

I almost have a definition, but if you remember, I have a basepoint. That's important. It's not interesting right now. The first homology is always zero. You have to use the basepoint as well. So how to do that?

This is a point in the complement of all the α and β loops. So if I write down $z \times Sym^{g-1}(\Sigma_g)$, this will be disjoint from T_α and T_β. So $\phi \in \pi_2(x, y)$, for any point not in the α, β circles, I can define $n_z(\phi)$ to be the algebraic intersection number between ϕ and $z \times Sym^{g-1}(\Sigma_g)$. So the actual definition, I want to say $n_z(\phi) = 0$.

[Does this number change with z?]

Yes. If z is here in this picture, there is clearly a holomorphic disk here, if z is here the n_z will be zero, and here it will be one.

Let's look at S^3. Let $g = 1$. Here are α and β and then $\widehat{CF}$ is generating by one element, so it's $\mathbb{Z}$. The boundary map is zero. So if you believe that it doesn't matter which Heegaard diagram we choose, then $\widehat{HF}(S^3) = \mathbb{Z}$, and that's right.

So okay, what about the torus. Here we have three intersection points, a, b, and c. We have a few holomorphic disks. There is this one and then this one which goes between b and a. If I put my basepoint here, then I would have the chain complex

$$\begin{tikzcd} & b \arrow[dl] \arrow[dr] & \\ a & & c \end{tikzcd}$$

So b maps to a and c. If I put the basepoint here I could have

$$\begin{tikzcd} & b \arrow[dl] & \\ a & & c \end{tikzcd}$$

or I could have

$$\begin{tikzcd} & b \arrow[dr] & \\ a & & c \end{tikzcd}$$

I should say lots of things. This is an example of a very general construction. With a symplectic manifold, inside that manifold we have two odd-dimensional submanifolds that are Lagrangian. That setup, and then you intersect them smoothly, and the chain complex is generated by the intersection points, and it goes through as before. Sometimes it's not really a chain complex. You need some knowledge about the symplectic manifold, and the Lagrangians. You can run this argument in Lagrangian Floer homology provided you can find an appropriate (symplectic?) form on the g-fold symmetric product.

We have to resort to tricks to show that the boundary maps are well-defined, that the moduli spaces are compact. You use the symplectic form upstairs to get energy bounds on the moduli space.

Before I go on to other results, I'd like to go over some simple definitions. There's a notion of looking at the holomorphic disk in the g-fold symmetric product, and there's also the notion of working in the surface itself. It's complicated to do the example from last time by hand. We need some methods, some tools for topological and holomorphic disks.

Let's start with topological disks first. We can take $\Sigma_g - \alpha_1 - \ldots - \beta_g$. Let $D_1, \ldots, D_n$ denote the connected components (not necessarily disks). In each component I will choose a reference point z_i. I can associate to a homotopy class $\phi \in \pi_2(x, y)$ the two chain $D(\phi) = \sum_{i=1}^{n} n_{z_i}(\phi) D_i$. We call this the domain of ϕ, maybe this is a bad name.

I want to write down the two-chains that correspond to homology classes and try to work out the Maslov index from this kind of formula.

So here we have T_α and T_β and ϕ between them. So this is x and this is y, and x has components $x_i \in \alpha_i$ and likewise for y. We have a map which is an arc here on T_α. It's an arc in $\alpha_1 \times \ldots \times \alpha_g$. So the components in this arc will connect x_i to y_i on α_i. You will get g arcs on the β circles connecting them as well.

So what we get, eventually, is the following:

Exercise 9 *$\delta D(\phi)|_{\alpha_i}$ is a one-chain on α_i. The boundary of this one-chain is $y_i - x_i$. On β_i it will be $x_j - y_k$, where the indices depend on which x and y meet in β_i.*

So in this picture, here $x = (x_1, x_2)$ is one of our generators. $y = (y_1, y_2)$ is our other generator. Then D_7 is a two-chain that satisfies the properties I wrote down. I get, if I restrict to α_i or β_i, an arc connecting x and y.

When I think about homotopy classes, I can go back and write down the domain. Most of the time I can recover the homotopy class from the domain. Note that the square has the involution of 180° rotation. The quotient gives me a map to the unit disk. This is a two-fold branched cover. Here I have a map to Σ which is basically like the identity. Then I want to use these to define a map to the second symmetric product.

A.

$$\begin{tikzcd} \square \arrow[r, "f"] \arrow[d, "\Pi"'] & \Sigma \\ \bigcirc \arrow[r, dotted, "U"] & Sym^2(\Sigma) \end{tikzcd}$$

B. There are lots of interesting examples, such as this one, just in the plane. α and β intersect each other in four points. There's a homotopy class connecting x to y so that $D(\phi)$ is $D_1 + D_2 + D_3$ and you just take the homotopy class of this disk.

C. Let's see some other examples still in the second symmetric product. Here are α_1 and β_1 and here are α_2 and β_2. I have a, b, and c in the picture. I pick $x = a \times c$ and $y = b \times c$.

   Choosing $a \times b$ will not give us one because we don't use α_2 or β_2. Then $D(\phi) = D_1$.

D. Here we have α_1, β_1 and here are. x_1, x_2 and y_1, y_2.

Exercise 10 *The exercise is to study the moduli space of these homotopy classes $M(\phi)$ for these four examples and see if you can understand it.*

Maybe one more thing. The Maslov index of ϕ could be negative. It's a very useful notion and works nicely to connect homotopy classes. What about homotopy classes ϕ_1, ϕ_2, like this, connecting x te y and y to w. I can connect these to get one disk from x to w, and we have $M(\phi_1 \# \phi_2) = M(\phi_1) + M(\phi_2)$.

You can look at the spaces of Lagrangians going through the origin and [unintelligible]is $\mathbb{Z}$, and so this is a Lagrangian subspace and another intersecting transversally. Use the disk to fix a trivialization of the tangent bundle. Think about working in $\mathbb{R}^{2n}$. Then you have two paths of Lagrangians connecting them to one another. I can make this path to be constant, so then I have no control of the other path, and so it comes back and I get a loop in the space of Lagrangians. So I just recall how many times I get a postive generator and that's the Maslov index. So in this picture I get something like this, and then the tangent space to this line is always the same and the tangent space changes. In $\mathbb{C}$ everything is Lagrangian so I get an S^1 worth of Lagrangians and I go around and see that the Maslov index is one. This is almost the same picture. So here I use this and get an antiholomorphic disk. So then I get the constant homotopy class so that one should have index -1. So study these spaces and compute the Maslov index. Next time we'll learn a nice formula from Robert Lipschitz which will allow us to check this combinatorially.

Thank you very much.

4 Milnor

Okay, well, my ambition was to talk about topology of manifolds in the fifties and sixties, but I realized if I can get through the fifties I'll be lucky, so let me get started.

Well, nowadays we know that there are clear separations between low dimensions (< 4,), high dimensions (> 4) and 4 (the jungle). Back then we assumed that one-manifolds were easy, two manifolds were pretty easy, three manifolds were hard, we just assumed it would get harder as we went up, so it was a shock when we discovered that higher dimensions were often easier to understand. My lecture will be divided into two parts, low dimension and high dimension. Dimension four is too hard, but as far as I know nothing was discovered about four dimensions in the fifties so I'm off the hook.

Okay, so first I want to talk about three-manifolds, here a lot was done but I think there was one really important contribution, by Papakyriakopoulos. As I said, he was working completely by himself, he didn't have a regular academic position, he worked on old hard problems, not talking to anyone, I was in Princeton at the time and I didn't really talk to him, we were probably both too shy. I had no idea he was doing anything so important. Finally he came out. Let me try to explain what he accomplished.

Max Dehn in 1910 proved that if you have a piecewise linear map from a 2-simplex Δ onto $\mathbb{R}^3$ which is one to one near $\delta\Delta$ then there exists a piecewise linear embedding of Δ which agrees with the original map near $\delta\Delta$.

As an easy corollary he proved that if the fundamental group of a knot complement is free cyclic, you can get a spanning disk, so that the curve was unknotted. This was a happy state of affairs for twenty years or so until in 1929 Kneser was publishing a paper and wanted to apply this proof and found that the proof was just wrong. So the situation remained open for another thirty years or so when Papakyriakopoulos, working by himself using classical methods, finite simplicial complexes, and gave a complete proof. I'd like to show you a picture of him, but the only picture I could find was this snapshot of him reclining on a couch by Ralph Fox, who was my advisor and brought him to Princeton.

The same methods proved, say you have an essential map from $S^2 \to M^3$. Then he proved there exists an essential embedding which can't be shrunk to a point, which means M^3 is reducible, it has an embedded sphere which can't be shrunk to a point, bounds a ball. If M is irreducible, it follows that $\pi_2(M^3) = 0$. You may have heard of the disk theorem, the loop theorem, essentially restatements of the same thing, making a map of a simplex with singularities, playing around to make it without singularities.

I want to contrast this with progress in higher dimension. There were many different fields and ideas that converged, so most of these tools had been established earlier or were just coming into being. Cohomology theory had been established by Whitney among others. Cohomology operations had been studied by Steenrod among others. Fiber bundles were developed by Whitney, Steenrod, so on, characteristic classes by Whitney, Stiefel, Pontrjagin, Chern, homotopy groups had been studied but were very poorly understood. Morse theory had been developed but its many applications had not been realized.

Serre's thesis in 1953, he wasn't interested in geometry. He developed spectral sequences and applied it to fibrations in homotopy theory. He proved that the homotopy classes of maps of spheres into each other is finite, it's known to be Abelian except for the fundamental group. It's finite except for special cases. It played a very big part in what follows.

In 1954 Rene Thom came out with cobordism theory. If you can join two compact manifolds with a compact smooth manifold with boundary, they're cobordant. You can also do it with orientation. He showed that cobordism classes formed a group, which isn't so special, but then he proved a lot of things about it using the algebraic techniques pioneered by Serre, Steenrod, among others.

Theorem 8 *Cobordism classes form a finitely generated Abelian group, and you can use the topological product to make this an algebra over $\mathbb{Z}$. He couldn't deal with torsion so he tensored with $\mathbb{Q}$ and got a polynomial ring on the complex projective spaces of even complex dimension.*

He also gave an effective test to show if a manifold is zero in this group. If you take a product of Pontrjagin classes of the correct total dimension you can apply it to the fundamental class and get an integer. Thom showed that the manifold is cobordant to zero, is a boundary modulo torsion if and only if all of these integers are zero.

Let me show you a picture of Thom. This is later, in the seventies. Here's another picture, not as good, but I have nostalgic interest in the picture. Here is a group of mathematicians working hard, here's Thom, Kurt Weyl, Grothiendieck, and then [unintelligible], myself, and [unintelligible].

Let me say a little more about what Thom did. He defined a signature of a $4n$-dimensional manifold. Let me say it in terms of homology. Two middle dimensional homology classes represented by manifolds, they intersect transversally in a finite number of points, an integer intersection number. For what I want to do next it's better to take rational coefficients. Pick a basis for the vector space so that the quadratic form is diagonalized, then the sum of the signs of the diagonal elements, that's called the signature. He proved by a geometric argument that if the manifold is a boundary then the signature is zero, so it could be expressed as a linear combination of Pontrjagin numbers with rational coefficients.

Hirzebruch had worked out what this formula would look like, we had a formula in every dimension divisible by four. The case we're interested in is the eight-dimensional case. There are two Pontrjagin numbers, p_2 and p_1^2. The coefficients are $7/45$ and $-1/45$. You can work these out easily. You know that all of these, rationally, are sums of complex projective four-space and products of complex projective two-spaces. You can work out what the unique possible formula. You can restate this as p_2 being given in terms of p_1 and the signature with rational coefficients (denominator 7). I want to apply this last formula, if we consider a manifold with boundary of dimension 8. Assume the boundary is a homology seven-sphere. The signature still makes sense in a manifold with boundary. It's not hard to show that the first Pontrjagin number is a well-defined class in H^4. You can get a second Pontrjagin class working modulo boundary.

Now suppose this is actually a sphere, this boundary. Then we can paste on an 8-ball and get a closed manifold for which this formula will have to be true. If we have a standard sphere then this expression must be an integer, so $45 \text{ sgn} + p_1^2$ must be $0 \mod 7$. So if we can find any example where this fails, then we know we've found a homology sphere which cannot be diffeomorphic to the standard sphere. I came upon such an example in the mid-fifties and was very puzzled. I didn't know what to make of it. I thought I'd found a counterexample to the generalized Poincaré conjecture in dimension seven. But looking carefully I saw that the manifold really was homeomorphic to the seven-dimensional sphere, so there was a differentiable structure on S^7 not diffeomorphic to the standard one. The same argument shows that there are at least seven differentiable structures on that sphere; in fact there are twenty-eight.

I seem to have gone through the fifties rather rapidly. I do want to mention one other contribution, that of Raoul Bott, who exploited Morse theory in a way that no one had thought possible to study homotopy groups of classical groups. Again, this seems to have nothing to do with manifold topology but turned out to be very important for developing the topic. The easiest case is with $U(1) \subset U(2) \subset \cdots \subset U$. He proved that the homotopy groups of U are $\mathbb{Z}$ in odd dimensions and 0 in even dimensions. This is a fantastic achievement; at this point there were practically no homotopy groups that were completely known, and having such a large family with such a simple answer was simply mindboggling.

I do have some slides here that I managed to skip over. One big surprise that developed in the fifties is that higher dimensions are often easier than lower. This is a bit hard to understand and explain, but one simple reason that can be stated is that in all dimensions it's important to study maps of S^1 into the manifold. Even if you know a map can be shrunk to a point, you can't make geometric use of that. But in dimension four or more, it's quite easy to see if you have a map into the manifold you can first approximate it by an embedded circle, and then if it can be shrunk to a point it bounds a singular disk. In high dimensions it's easy, you put it in general position and then by an argument that goes back to Whitney it bounds an embedded disk.

In dimension three this obviously doesn't work. Suppose our three-manifold is the three dimensional Euclidean space with this black circle removed. It's true that you can't embed a disk so that its boundary is the red circle and it doesn't intersect the black circle. You can immerse a disk, and you can find a disk that it bounds if you forget the black circle.

M^4 doesn't look so bad at first, but you may find your immersed disk has self-intersection, transversal self-intersection at a point. So that's a brief idea of why the high dimensional methods don't work in low dimensions, and the low dimensional ideas don't work, because, well, they don't work.

So that's the end of the slides, I can move to the blackboard. That more or less carried us to the end of the fifties. Things really came alive in the sixties. It started with Steve Smale, who proved the generalized Poincaré conjecture in dimension greater than four, if $M^n \sim S^n$ then M^n is diffeomorphic to S^n.

[Don't you mean homeomorphic? I think there's a counterexample due to Milnor.]

Right. It's late in the day or something.

You can take a level set in a Morse function. and look at the flow lines running to it and that's homeomorphic to a disk. This shows that you can obtain the manifold by taking two disks, and then taking a diffeomorphism from the boundary sphere to itself. It's a manifold M^n which depends on this diffeomorphism. I call this a twisted sphere. Smale showed that having the homotopy type of a sphere implies that you have a twisted sphere.

We can form a group called Γ_n equal to the set of twisted n-spheres modulo orientation preserving diffeomorphisms. Then we get an easy exact sequence like

$$\pi_0 \mathrm{Diff}^+(D^n) \to \pi_0 \mathrm{Diff}^+(S^{n-1}) \to \Gamma_n$$

A few years later he proved the h-cobordism theorem, which said that two twisted homotopy spheres of dimension greater than four are diffeomorphic if and only if they are h-cobordant. So the cobordance was shown to be diffeomorphic to a product, so that the two manifolds at the end were actually diffeomorphic to one another.

What else, Munkres and Hirsch had constructed an obstruction theory for passing from combinatorial to smooth manifolds. These lay in $H^{k+1}(M^n, \Gamma_k)$ for existence and $H^k(M^n, \Gamma_k)$ for uniqueness. The first few are easy. $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$. Cerf showed that $\Gamma_4 = 0$. Similar methods showed they were finite. So some of us studied Θ_n, homotopy spheres modulo h-cobordism. Putting all of the work together, we had finite obstruction groups in all dimensions. $\Gamma_k = 0$ for $n < 7$ (assuming Perelman) and Γ_7, which we saw had to be at least 7, was actually 28.

For dimension four we don't know that every manifold with homotopy type of the sphere is actually a differentiable sphere. If an exotic four-sphere exists, you could puncture it and get a standard four space or an exotic one. In spiteof the tantalizing fact that $\Gamma_n \cong \Theta_n$, dimension four is worse. There's no h-cobordism and you also don't know that every homotopy sphere is a twisted sphere.

I've talked about obstruction theory for combinatorial to smooth manifolds. Near the end of the sixties there was an obstruction theory by Kirby and Siebenmann to go from topological manifolds to PL-manifolds. There's only one obstruction, assuming dimension at least five, with existence obstruction in $H^4(M^n, \mathbb{Z}/2)$ and a uniqueness one in $H^3(M^n, mathbbZ/2)$.

[Are there any themes from the fifties and sixties that have been forgotten and shouldn't have been?]

It's hard, I think, to find any branch of mathematics that has been completely neglected.

[One thing you're famous for is your list of problems from the fifties. Do you have any problems that you want to see solved now?]

I mentioned exotic structures on the 4-sphere. I don't have any idea what one could do with that.

[I heard you were once challenged to write a limerick involving Papakyriakopoulos?]

I've heard many versions of it, I can't remember it exactly.

The perfidious lemma of Dehn drove many a good man insane but Christos Pap akyriakop oulos proved it without any strain.

Very satisfying. No more questions? No more mistakes to point out?

[Is Perelman related to $\Gamma_4 = 0$?]

Cerf proved that the group of diffeomorphism of the 3-sphere, orientation preserving, is connected. Hatcher showed that the group has the same homotopy as the rotation group $SO(4)$, much sharper. I should mention also that Eliashberg used symplectic methods to prove this result. Yasha, are you here? Can you say anything about that?

[[unintelligible]]

[Can you give us an update on the list of questions you asked in the fifties?]

I was listing the hardest and most difficult and important problems in topology. Someone in Moscow got a hold of this list and immediately gave answers to many of the problems.

5 Matt Hedden

Okay, so, is this good? So, I don't really know how to follow the last talk.

[Just talk about your results from the fifties.]

I'll just talk about my results from the last two years. So "Knot Floer homology and something about Satellite knots" is my title.

I'll give some background and motivation, but not definitions, which Zoltan will do. I'll give some flavor of how the knot invariants are organized within the Floer homology package.

I feel like there's a twilight zone episode that made an impression on me. They offered this guy $100,000 but someone he didn't know would die. So he spends it, but then the guy comes, takes the little ball, and now we'll take this ball to someone who you don't know. I sort of feel like I'm that guy. I'm the guy who sort of had to give the talk, so I should get to suggest to give the next talk. I have a list of people, I'm going to give to Jake. You know who you are.

Let's get going. You have 3-manifolds and Abelian groups. So you start with Y and get $\widehat{HF}(Y)$. Let's say, now, knot Floer homology, what is that, for integer homology three-spheres these are $\mathbb{Q}$-graded Abelian groups. Is a very general thing that covers all knots. It takes $K \hookrightarrow Y = \mathbb{Z}HS^3$, an integer homology three-sphere, to a bigraded collection of Abelian groups. You have $K \hookrightarrow Y$ leads to $\oplus_{i,j}\widehat{HFK}_i(Y, K, j)$. These groups have a lot of important applications. One thing they excel at is problems of genus. For the case of closed three-manifolds you can define the genus of a homology class. Let $a \in H_2(Y)$ then $g(a)$ is the minimum over all smoothly embedded surfaces $F \hookrightarrow Y$ of the genus of F where the homology class inside the manifold is a. What's the minimal genus? It's an exercise that you can find a smooth representantive of every second homology class.

In the case of knots you have $g(K)$ is the minimum over all surfaces with boundary (this is for knots in S^3) on the knot, that is $(F, \delta F) \hookrightarrow (S^3, K)$. This is the Seifert genus.

We also have the four-ball genus (smooth), which is $g_4(K)$ the minimum over all surfaces $(F, \delta F)$ embedded in (D^4, S^3), with boundary the knot.

So you can always push a surface down into the four-ball, so $g_4(K) \leq g(K)$. One thing that's not obvious is if you relax the smooth embedding condition, say instead it's a topological embedding, then the answer to the four-dimensional genus is very different. A lot of people have worked on that topological definition. Michael Friedman was one of the people who founded this area, but until recently the smooth genus was one of the best things we had.

More precisely it's the filtered chain homotopy type of a filtered chain complex.

Let's motivate a little more. There's talk about cutting three-manifolds along two-manifolds and gluing along boundaries like this. You can construct all three manifolds by Dehn surgery on links. You can chop up along tori as well, as John Morgan said, and get geometric structure.

I'd like to understand how knot Floer invariants behave when you glue along tori boundary components. It's not necessarily obvious that such a relation should exist, but that there should be such a relationship is roughly a topological quantum field theory. That seems natural to us thinking we're in a path space, three dimensional,... configuration... topological Chern-Simons... geometric Langlands.

I'm going to say some examples where I actually know what happens.

On to the algebraic structure of $\widehat{HFK}$. Really the invariant is a filtered chain complex, precisely its chain homotopy type. We have an increasing sequence of subcomplexes, $0 \subset \mathscr{F}(K, j) \subset \mathscr{F}(K, j+1) \subset \cdots \subset \mathscr{F}(K, j+n)$, where things stabilize and the rest are identity. The last one is $\hat{C}F(S^3)$. So we have this increasing sequence of subcomplexes, and eventually we just get the homology of the sphere. The last one has as homology $H_*(\widehat{CF}(S^3)) \cong \mathbb{Z}$ in degree zero.

But we have lots of complexes. We can look at the intermediate homologies and the homologies of the quotient complexes.

So $\widehat{HFK}_i(K,j) \cong H_i(\frac{\mathscr{F}(K,j)}{\mathscr{F}(K,j-1)})$. These were introduced by Oszvath-Szabo and independently Jake Rasmussen. I'm sorry for using their initials, but I have a history of misspelling them and I prefer not to offend them again. I can spell their names, just not always on the spot.

So the Euler characteristic, in the right interpretation, is familiar, so $\sum_i (-1)^i \mathrm{rk}\, \widehat{HFK}_i(K,j)T^j = \Delta_K(T)$, the Alexander polynomial.

Two important parts of knot Floer homology involve these geni notions.

Theorem 9 *(Oszvath-Szabo)*
Let $K \hookrightarrow S^3$ *then* $g(k) = \max_{j\in\mathbb{Z}}\{\widehat{HFK}(K,j) \neq 0\}$

There's another invariant that gives nice new information for g_4.

Definition 4

$$\tau(k) = \min_{j\in\mathbb{Z}}\{j|(\mathrm{inc})_* : H_*(\mathscr{F}(K,j)) \to H_*(\widehat{CF}(S^3)) \text{ is surjective}\}$$

Theorem 10 *(Oszvath-Szabo, Rasmussen)*
$|\tau(K)| \leq g_4(K)$.

So now I'd like to present a formula for the knot Floer homology for certain satellite constructions. This will use the subcomplexes and not just the quotient complexes.

Let's set up some notation. Say I have a knot P in a solid torus V^3. You can identify a neighborhood of another knot, which I'll call K with the solid torus by a diffeomorphism that sends the longitude to the longitude. On the right this depends on the framing, so I have an integer's worth way of doing this. I'm really looking at framed constructions. If I just take an oriented projection, I determine a framing by taking a parallel copy and then computing the linking number of one of these knots with the other. In this case I probably drew the left-handed trefoil, so the framing, well, in general it's just the writhe of the projection. You can increase or decrease this by adding twists. Now a way to come up with a projection af the satellite knot, it wall have three parameters: P the satellite knot, n the framing, and K the knot that it is a satellite of. This is written $P_n(K)$. So I come up with a knot that looks like this.

For this very specific satellite knot P, $P_n(K)$ is what people have called the n-twisted Whitehead double of K. Usually they say the n-twisted positive class. The negative class reverses the crossings of P.

I should mention that [unintelligible]studied this problem before me, for the untwisted double. There's a good motivation to study this knot. It has trivial Alexander polynomial, always.

Let me say, $\Delta_{P_n(K)}(T) = \Delta_{P_n(V)}(T)\Delta_K(T^{\mathrm{wind}\ P\hookrightarrow V})$. If it represents c times a generator we replace T with T^c.

But P is nullhomologous in the solid torus So now we raise T to the power zero, which is one. So $\Delta(1) = 1$ always.

I should have said $P_n(V)$ is the satellite of the n-framed unknot. The Alexander polynomial is not telling us anything, really. But we should have some information. The precise way we get information is as follows.

Theorem 11 *Let $WD_n(K)$ be the n-twisted Whitehead double of K. I mean the positive class. Then for $n \geq 2\tau(K)$, (we already, then, need more than knot Floer homology groups), we have*

$$\widehat{HFK}_*(WD_n(K), i) \cong \begin{cases} \mathbb{Z}_{(1)}^{n-2g(K)-2} \bigoplus_{j=-g(K)}^{g(K)} [H_*(\mathscr{F}(K,j))\{1\}]^2 \ (i=1) \\ \mathbb{Z}_{(0)}^{2n-4g(K)-3} \bigoplus_{j=-g(K)}^{g(K)} [H_*(\mathscr{F}(K,j))]^4 \ (i=0) \\ \mathbb{Z}_{(-1)}^{n-2g(K)-2} \bigoplus_{j=-g(K)}^{g(K)} [H_*(\mathscr{F}(K,j))\{-1\}]^2 \ (i=-1) \end{cases}$$

So for $n < 2\tau(K)$ we have

$$\widehat{HFK}_*(WD_n(K), i) \cong \begin{cases} \mathbb{Z}_{(1)}^{2\tau(K)-2g(K)-2} \oplus \mathbb{Z}_{(0)}^{\tau(K)-n} \bigoplus_{j=-g(K)}^{g(K)} [H_*(\mathscr{F}(K,j))\{1\}]^2 \ (i=1) \\ \mathbb{Z}_{(0)}^{4\tau(K)-4g(K)-3} \oplus \mathbb{Z}_{(-1)}^{2\tau(K)-2n+1} \bigoplus_{j=-g(K)}^{g(K)} [H_*(\mathscr{F}(K,j))]^4 \ (i=0) \\ \mathbb{Z}_{(-1)}^{2\tau(K)-2g(K)-2} \oplus \mathbb{Z}_{(-2)}^{\tau(K)-n} \bigoplus_{j=-g(K)}^{g(K)} [H_*(\mathscr{F}(K,j))\{-1\}]^2 \ (i=-1) \end{cases}$$

The theorem is stronger than what I've stated. It sounds pretty bad, but it actually, I've been stressing that the knot Floer homology groups are much weaker than the whole filtered chain complexes. I know the filtered chain homotopy types of the knots themselves. The knot Floer homology groups have a differential that strictly lowers the filtration and lowers the homological dimension i by one. If you let that differential act on the knot Floer homology groups you again get the homology of S^3. That differential actually equips the knot Floer homology groups themselves with the structure of a filtered chain complex. The full content of the theorem is that I know the filtered chain homotopy type. It's getting technical, I want to encourage everyone to leave, it's about to get technical.

When I add a group of negative rank, it means quotient out by such a group of the corresponding positive rank in the indicated grading. We don't have any negative rank groups, thankfully.

[Can we deduce this for the untwisted whitehead double?]

Yes, but not with what I've stated. I think it would be best to compute this for a knot. [unintelligible]I've done such a poor job explaining it. . . shambles

Corollary 1

$$\tau(WD_n(K)) = \begin{cases} 0 \text{ if } n \geq 2\tau(K) \\ 1 \text{ if } n < 2\tau(K) \end{cases}$$

This is [unintelligible]-Livingston for all but finitely many values of the framing parameter.

Corollary 2 *If $\tau(K) > 0$ then $WD_0(WD_0(WD_0 \cdots WD_0(K) \cdots))$ is not smoothly slice (is $g_4(Z) = 1$).*

But all of them, having trivial Alexander polynomial, are topologically slice. It's an open question whether the Whitehead double is smoothly slice if and only if the original knot is smoothly slice. Other work on this was done with smooth knot concordance. If you just do this process once, then the zero twisted Whitehead double with τ bigger than zero is not smoothly slice.

What made this really interesting was

Theorem 12 *(Ording,—) $\tau(K) \neq \frac{1}{2}s(K)$ where $s(K)$ is the combinatorial smooth concordance invariant defined by Rasmussen using Khovanov homology.*

It would be nice if you could compute this gauge-theoretic invariant combinatorially, that would be incredible, but maybe this isn't really depressing. So perhaps Khovanov homology is a really new thing, some new thing for four-manifolds. All the things we've come across since gauge theory came about. Most of the invariants have some conjecture saying that they contain the same information.

I could keep talking, but I think people don't want to walk out in the middle of the talk,. . . , some preconceived notion, it's not really the middle of the talk, more like, well,. . .

[Which of τ and $1/2s(K)$ is bigger?]

I don't think there is a strict one one way or another.

[How much can you do for other satellites?]

A lot of it, actually. There's some stuff that I have proved that I haven't written down. I can do things when the winding number is zero and some part when the winding number is not zero.

I can say that it's in an interval for arbitrary satellites. Certainly the interval is,. . . .

[Announcement: Jason Behrstock is giving a previously unscheduled talk at 1:00 PM tomorrow. The title is "The quasiisometric classification of three-manifold groups"]

For those of us who aren't TEXsuperheroes, the knot Floer homology of the Whitehead double is determined by the filtered chain homotopy type of the original knot.

I believe that given P, there is a formula for the knot Floer homology of $P_n(K)$ based only on n and K. It will involve the knot Floer homology of $P \subset S^3$ and the knot Floer homology of doing zero-surgery on the meridian, $P \subset S^1 \times S^2$. Those will be involved if you fix K and vary P. it may also involve maps on homology between them. It's sketchy how that works when the winding number is nonzero at this point.