Low Dimensional Topology Notes
June 26, 2006

Gabriel C. Drummond-Cole

June 26, 2006

1 Khovanov

It's time to begin. I'd like to welcome you here on behalf of half of the organizers. The other half is still in the air. I have a couple of announcements. First thing is that the problem sessions for today are cancelled because you guys don't know enough yet. The other thing is that there will be an organizational meeting today at 9:40 for the research program. There's a TA meeting at 7:45 tonight. Both of those are here. That's all the announcement. This is our first lecturer, M. Khovanov from Columbia.

I'd like to thank the organizers. I'll give six lectures. I plan to spend the first two discussing braid actions on categories. Most of the time will be spent on one particular action. I want to discuss invariants of braid cobordisms. The last four I'll talk about link homology.

I'm going to do some preparation that will be algebraic but will become more topological as we go on.

We'll start with path rings. Let Γ be an oriented graph, with finitely many vertices and edges. It may have loops. The path ring $\mathbb{Z}\Gamma$ is the free Abelian group of with all paths as a basis. So for the graph $\bullet \longrightarrow \bullet \longrightarrow \bullet$ there are six paths, one of length two, two of length one and three of length zero. Multiplication is by concatenation of paths. Most products in this ring are zero because they don't compose. We have the paths a, b, c, α, β, and $\alpha\beta$. Then we have $(a), (b), (c)$ with $(a)^2 = 0, (b)^2 = 0, (c)^2 = 0$ and then $a(\alpha) = \alpha$ and $\alpha(a) = 0$ is 0.

Exercise 1 *Note that the sum of the zero length paths over all vertices is the unit.*

$$a \xrightarrow{\alpha} b \xrightarrow{\beta} c$$

Note that the path ring on a loop α is $\mathbb{Z}[\alpha]$.

I'd like to say a few words about modules over path rings. I'll talk about $mod - \mathbb{Z}\Gamma$, the category of right $\mathbb{Z}\Gamma$-modules. Then $M = \oplus_{\text{vertices}} M(a)$. Now if you multiply by α you get a map from $M(a)$ to $M(b)$.

Exercise 2 *What are morphisms in this language?*

Exercise 3 *Describe the analogous structure for left modules.*

Another thing you can do is, people like to put a field instead of $\mathbb{Z}$, and then this algebra has homological dimension one so every module is projective.

Something about, say $k\Gamma$ is a hereditary ring (homological dimension one)

We want also to quotient the path ring by relations. If there is more than one path between vertices, you can quotient by the relation $\gamma_1 + \lambda\gamma_2 = 0$. This is too general. The case we care about is zigzag rings A_n :

$$1 \rightleftarrows 2 \rightleftarrows \bullet \rightleftarrows \bullet \rightleftarrows n$$

And then the relations we want to add are $(i|i+1|i+2) = 0, (i|i-1|i-2|) = 0$, and $(i|i+1|i) = (i|i-1|i)$. We can write this as $\delta_1^2 = 0, \delta_2^2 = 0$, and $\delta_1\delta_2 = \delta_2\delta_1$ where δ_1, δ_2 are arrows to the right, left, respctively. So A_n modules are the same as bicomplexes, complexes with two commuting differentials. Note also that $A_n \cong A_n^{op}$.

Now $\mathbb{Z}\Gamma$ is graded by path length. The relations quotiented in $A_n = \mathbb{Z}\Gamma/\sim$ is also graded since the relations are homogeneous.

Any length three path is zero. A subpath which contains two steps to the right or left it's zero, and $\delta_1\delta_2\delta_1 = \delta_2\delta_1^2 = 0$. So A_n is small. It's a free Abelian group on $(1), (2), \ldots, (n)$, the paths $(i|i+1)$ and $(i|i-1)$ and the paths $X_i = i|i-1|i$.

So from A_n we will get a braid group action by means of the Temperley-Lieb algebra.

Take $P_i = A_n(i)$, which is spanned by paths that end at i. When i is in the middle, there are four such paths, one each of lengths zero and two and two of lengths one: $(i), (i-1|i), (i+1|i)$, and X_i.

This is left-projective, because they are direct summands, $A_n = \oplus A_n(i)$.

Likewise we can define ${}_iP$ as $(i)A_n$, and this is a right projective module. There is a similar decomposition. If you take ${}_iP \otimes_{A_n} P_j$ you get an Abelian group spanned by paths that start at i and end at j. Most of the time this is zero, that is if $i - j| > 1$. It's $\mathbb{Z}(i|j)$ if $j = i \pm 1$. If $i = j$ there are two paths, the zero path and X_i so the tensor product is $\mathbb{Z}(i) \oplus \mathbb{Z}X_i$.

Introduce $U_i = P_i \otimes_{\mathbb{Z}i} P$. This is an A_n-bimodule. If you tensor $U_i \otimes_{A_n} U_i$. So you get $P_i \otimes_i P \otimes_{A_n} P_i \otimes_i P. \cong (P_i \otimes_i P) \oplus (P_i \otimes_i P)$ where these are for (i) and X_i. So this is $U_i \oplus U_i$.

Then $U_i \otimes U_{i+1} \otimes U_i$ decomposes into six terms and what's left will be U_i. Similarly for $i-1$. When you tensor U_i and U_j when they are far away from one another (> 1) you get zero.

These relations are the Temporley-Lieb algebra with some additional relations.

This algebra is generated by diagrams u_i where there are $n-1$ vertical lines and cups and caps in positions $i, i+1$. You say a circle gives multiplication by $q+q^{-1}$. The less interesting picture is to set $q=1$ and multiply by 2. So $u_i^2 = 2u_i$. If you multiply $u_i u_{i+1} u_i$ you get u_i isotopically. You have $u_i u_j = u_j u_i$ for i, j far from one another, but you don't get that these are equal to zero.

This is just an algebra with generators and relations, but it has this nice geometric picture. Now you are taking tensor products instead of multiplications, and the equalities are isomorphisms. So we're lifting the equations of the Temperley-Lieb algebra.

The braid group has generators σ_i which go to $0 \to U_i = P_i \otimes_i P \to A_n \to 0$. This middle map β_i takes $a \otimes b$ to $ab \in A_n$. I'll talk about this at length.

Let A be any ring. Denote by $Kom(A)$ the category of complexes of A-modules. We take the quotient category, (the principle is that you have to modify morphisms) with the same objects, but where $f = g$ in the quotient category if the difference is nullhomotopic. What this means for complexes of modules is:

$$\begin{tikzcd}
{} \arrow[r] & M^i \arrow[r, "d"] \arrow[dl, "h"'] \arrow[d, "t"] & M^{i+1} \arrow[r] \arrow[dl, "h"'] & M^{i+1} \arrow[r] \arrow[dl, "h"'] & {} \\
N^{i-1} \arrow[r, "d"] & N^i \arrow[r] & N^{i+1} \arrow[r] & N^{i+1} \arrow[r] & {}
\end{tikzcd}$$

In this diagram, we have t is nullhomotopic if there exists h with $t = dh + hd$

Exercise 4 *Nullhomotopic morphisms form an ideal in $Kom(A)$.*

So $C(A)$ is the quotient category. $Hom_{C(A)}(M,N) = Hom_{Kom(A)}(M,N)/\sim$.

Exercise 5 *For any K, $0 \to K \xrightarrow{1} K \to 0$ and $0 \to 0 \to 0 \to 0$ are isomorphic in $C(A)$.*

Let $[1]$ be the left shift operator, namely let $M[1]^i = M^{i+1}$ and $d_{M[1]} = d_M$.

Then let the cone $C(f)$ for $f : M \to N$ be $M[1] \oplus N$. with differential $D = -d_M + f + d_N$.

$$\begin{tikzcd}
& M^i \arrow[r, "-d_M"] \arrow[dr, "f"] & M^{i+1} \arrow[r, "-d_M"] \arrow[dr, "f"] & M^{i+2} \arrow[dr, "f"] & \\
N^{i-1} \arrow[rr, "d_N"] & & N^i \arrow[r, "d_N"] & N^{i+1} \arrow[r, "d_N"'] & N^{i+2}
\end{tikzcd}$$

For N a bimodule and M a module we have $N \otimes_A M$ a module, so $N\otimes$ is a functor from $A - mod$ to $A - mod$. If N and M are complexes of bimodules, modules, resepectively, then

$N \otimes M$ is a complex of modules via

$$\begin{tikzcd} N^{i+1} \otimes M^i \arrow[r] & {} \\ N^i \otimes M^i \arrow[u, "d_N"] \arrow[r, "d_M"'] & N^i \otimes M^{i+1} \arrow[u] \end{tikzcd} ,$$

just taking the total differential.

Theorem 1 *There are isomorphisms* $R_i \otimes_{A_n} R_{i+1} \otimes R_i \cong R_{i+1} \otimes R_i \otimes R_{i+1}$.
$R_i \otimes R_j \cong R_j \otimes R_i$ *for* $|i-j| > 1.6$, *and*
$R_i \otimes R'_i \cong A_n \cong R'_i \otimes R_i$.

Exercise 6 *We need* R'_i *to be the complex* $0 \to A_n \xrightarrow{\gamma_i} U_i \to 0$ *with* $\gamma_i(1) = (i-1|i) \otimes (i|i-1) + (i+1|i) \otimes (i|i+1) + X_i \otimes (i) + (i) \otimes X_i$ *representing the inverse of* σ_i.

The theorem fails in $Kom(A - bimodules)$ *but holds in the homotopy category of complexes of bimodules.*
This action is faithful.

The tensor product can be viewed at a functor $C(A_n) \to C(A_n)$, and so equivalences in the braid group become isomorphisms of functors.

This is striking, why would we have an action and why would it be faithful? This is related to tricomplexes. We have A_n with δ_1, δ_2, and when we deal with complexes we have the additional d. So we have tricomplexes up to homotopy and in this category we have a faithful action of the braid group.

So far beyond this, in four-complexes, I have seen nothing interesting. Bicomplexes correspond to spectral sequences and tricomplexes to the braid group action. I don't know about four.

2 John Morgan

One announcement, we found a speaker for this afternoon, and now I present John Morgan from Columbia, who will speak on $3 - manifolds$.

Later during this conference I'll be giving talks on Ricci flow. This is not the first in this series. Rather I'm giving an introduction to what you'll be hearing about these weeks. It's a general introduction to three-manifolds.

Someone said to me, this is the first time we're having a low dimensional topology conference in twenty years. It's actually been twelve years, and we've dropped from dimension four to dimension three, mainly.

I'm going to start with the basic terms and notions in the subject. Then I'll talk about some of the examples, which leads into geometric three-manifolds, and also some auxillary notions like foliations, contact structures, and then some about invariants of three-manifolds, and the various sources of them, gauge theory, combinatorial.

So M^3 will be closed, oriented, and connected. The first approach is to look for surfaces inside them to cut the manifold into pieces.

The first question is about embeddings $S^2 \subset M^3$. We say a two-sphere is trivial if it separates into two pieces, one of which is a three-ball.

If it's not trivial, it may be separating or nonseparating. In the separating case you can cut and get two manifolds with S^2 boundary which you cap off with B^3. Then there's a unique way to do this assuming everything is oriented, called connect sum. In a nonseparating two-sphere, you get a connected sum decomposition into the capped off cut manifold and $S^1 \times S^2$.

You can say that M is prime if every separating S^2 bounds a 3-ball. Every manifold contains only a finite number of disjoint etc. two-spheres and so it turns out that prime decompositions are unique up to ordering.

Exercise 7 *What corresponds to* 1? *It's* S^3.

What are some examples? Before I get to that, surfaces of higher genus are also things to cut along. For now I only want to cut along two-spheres, but we'll come back. We've already seen S^3. There's a whole list of examples related to this, take finite freely acting $\Gamma \subset SO(4)$. Then S^3/Γ will have $\pi_1 M = \Gamma$ which includes $L^3(p,q)$ where you divide out by $\begin{pmatrix} \zeta & 0 \\ 0 & \bar{\zeta}^q \end{pmatrix}$ where ζ is a primitive p root of unity. You also have the Poincaré dodecahedral space has Γ the group of symmetries of a regular dodecahedron in $\mathbb{R}^3$. This was the counterexample to the original Poincaré conjecture which was at the level of homology.

These are round, or elliptic, because the metric on S^3 descends to the quotients with constant curvature $+1$.

You can then ask about flat examples, $\mathbb{R}^3$ with the standard Euclidean metric. Without thinking about rotations, consider translations, we can take a lattice and divide out to get $\mathbb{R}^3/L \cong T^3$, which carries a flat metric, locally isometric to $\mathbb{R}^3$. Every one of these is covered finitely by a flat three-torus.

Both in the elliptic and flat case all examples are known, it's easy to classify them classically.

The next example is hyperbolic, negatively curved three-manifolds. Why is it called hyperbolic? The growth of volume is exponential in r as you grow r.

Take a curve of radius i. Over the complexes, take the form $-x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1$ and $x_0 > 0$. So we get the upper sheet of a hyperboloid. When you restrict the quadratic form to the tangent space at any point, you get something positive definite.

You get an induced metric, a Riemannian metric on hyperbolic space, with isometry group $SO^+(3,1)$. This is not compact and in fact has another description as $SL(2,\mathbb{C})/\{\pm I\} = PSL_2(C)$.

We make hyperbolic manifolds in the same way, taking discrete $\Gamma \subset PSL(2,\mathbb{C})$, cocompact and torsion-free, and then $\mathbb{H}^3/\Gamma = M^3_{hyp}$. This is the third purely homogeneous class of manifolds. It's not known how to classify the discrete cocompact subgroups.

However, there's a beautiful theorem called Mostow rigidity, which says if M^3 is a compact topological 3-manifold then it has a unique hyperbolic structure up to automorphism if it has one. This makes the link spectrum of closed geodesics, the volume, etc., into topological invariants. To point out the state of knowledge, we don't know the lowest volume hyperbolic manifold. Well, we know it but it's not a theorem yet.

Let me give you some necessary conditions for the manifold to be hyperbolic.

First, it has to be prime (otherwise the S^2 would lift to the universal cover). Second, it has to have infinite fundamental group. Third, more interestingly, every solvable subgroup of $\pi_1(M)$ is $\mathbb{Z}$ or trivial. The fundamental group of M contains a free group of rank 2. Any two elements α, β with nontrivial commutator leads to an N such that α^N, β^N generate a free group.

Closely related objects that play an important role are hyperbolic manifolds of finite volume. Consider Γ as before, discrete and torsion free but with cocompact replaced by "co-finite volume." Cocompact means the quotient is compact, cofinite volume means that quotient has finite volume. Really the statement is in terms of classical Lie theory, but then you're quotienting by a compact group.

The end of the cofinite volume group that you divide by to get the punctured torus have exponentially decreasing radius. The cross-section you get in the three-dimensional space has torus cross sections. The areas of the tori decrease exponentially fast. You can have a finite number of cusps, not just one. This is what finite volume hyperbolic manifolds look like. The conditions I gave you on the fundamental group, you no longer have the statement about Abelian groups because you have these tori. These are still Mostow rigid. The fundamental groups of the tori give you $\mathbb{Z} \times \mathbb{Z}$. So now the statement is that that you have $0, \mathbb{Z}$, and subgroups coming from these cusps.

So from the hyperbolic manfiolds you can get other manifolds which are not hyperbolic but are accounted for by the hyperbolic ones. You can perform the topological operation of attaching along a cusp by your favorite diffeomorphism of T^2. That is not hyperbolic, because it's compact and contains the fundamental group of T^2, which is $\mathbb{Z} \oplus \mathbb{Z}$. It's not a geometric manifold, as one can show, but it comes from geometric manifolds, in some way.

I mentioned that there are other geometric examples. They're mainly trivial, related to surface theory, but let me just put them on the board for completeness. The three examples we've gone over are constant curvature. They're homogeneous and there are isometries which take every direction to every other direction.

So there are mixed manifolds, like $S^2 \times \mathbb{R}$. You can divide out by a lattice and get $S^2 \times S^1$ or divide out by a dihedral group $\mathbb{RP}^3 \# \mathbb{RP}^3$. I'll let you figure that out. This is the only non-prime example that comes from geometry.

There are also manifolds from $\mathbb{H}^2 \times \mathbb{R}$, and so you get $\Sigma_g \times S^1$.

We don't need to worry about the flat two-dimensional one, because that would yield $\mathbb{R}^3$, the flat one, which has extra isometries; we've already dealt with that.

Now there's three more. There's Nil, which comes from upper triangular matrices with ones on the diagonal. This is a fibering of S^1 over T^2. There is Solv, which comes from $\mathbb{R}^2 \rtimes \mathbb{R}^*$. It might not be obvious that there are such three-manifolds; [example]. All of these are finitely covered by torus bundles over the circle.

The last examples are based on the universal cover of $PSL_2(\mathbb{R})$, which are all finitely covered by S^1 fibrations over Σ_g. You could take hyperbolic manifolds of finite volume here, and in the $\mathbb{H}^2 \times \mathbb{R}$ geometry. These will have $T^2 \times [0, \infty)$. The metric will be different. You get one circle fixed and the other shrinking exponentially. It's an exercise to see that $\pi_1(T^2) \hookrightarrow \pi_1(M_{geo})$.

So I have three kinds of finite volume examples with incompressible tori along which I can cut and glue. I can mix and match the pieces as I see fit. We've upped the ante, cutting along tori as well as two-spheres. That's the only way to understand these geometrically.

The $\mathbb{H}^2 \times \mathbb{R}$ have separating incompressible tori, and so do the other mixed hyperbolics, so you don't need a complete decomposition.

Theorem 2 *Thurston Geometrization Conjecture (1981 or 1982).*
Every prime compact oriented connected 3*-manifold can be cut apart along incompressible tori and Klein bottles to give geometric pieces (finite volume metrics from one of these eight types).*

Incompressible means that π_1 *of the torus injects into* $\pi_1(M)$*. Cutting along a Klein bottle, you can cut on the nontrivial* $I-$*bundle over the Klein bottle. This has boundary a torus. Cutting along a Klein bottle means removing something that looks like this.*

You can make this unique by making a minimal cutting up to isotopy.

That's how we think about three-manifolds. Almost all of them are hyperbolic, and the others are trivial counterexamples, well understood.

Let me go on to the other ways that people understand manifolds. Let $\Sigma_g \hookrightarrow M^3$. Say it's incompressible if $\pi_1(\Sigma_g) \hookrightarrow \pi_1(M)$. It was realized that these were much easier to understand. You can cut down until you get all to balls. They're called sufficiently large, sufficiently large to have an incompressible surface. One is enough because with boundary you always have incompressible surfaces.

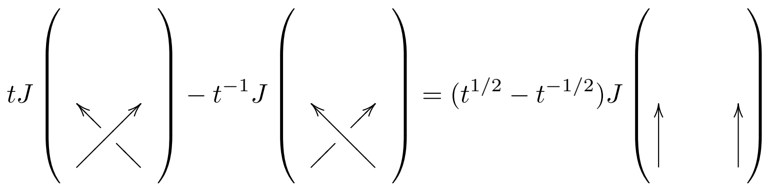
Thurston showed that the Geometrization conjecture holds for sufficiently large 3-manifolds.

So Thurston could characterize which manifolds with torus cusps were hyperbolic; it basically reverses the conditions I wrote. So these are called sufficiently large or H aken manifolds. These are not finite fundamental groups, though, so this doesn't touch the Poincaré conjecture.

Knots. You'll hear a lot about knots. Classically K is $S^1 \hookrightarrow S^3$ up to ambient isotopy. You consider the complement $S^3 \backslash K$. Papakiriakopolous showed that if $\pi_1(S^3 \backslash K) \cong \mathbb{Z}$ then K is trivial. You can glue a solid torus back in with a different diffeomorphism, via a rational number. This is called Dehn surgery. Every three-manifold is obtained by Dehn surgery on a link in S^3. This is a four-manifold statement, actually.

I have two minutes to talk about gauge theory invariants and combinatorial invariants. Gauge theory invariants use elliptical PDEs. Now three-manifolds is where all of mathematics meet; it used to be that this was separate from the rest of mathematics. Now it's the crossroads where all mathematics meets. These are associated to a three manifold or a four manifold with metric, you take the moduli space of solutions, and then some cohomological products are invariants of the three manifolds.

Combinatorial invariants, one way of considering the Jones invariant, you can resolve a planar projection of a knot or change the crossing. So you get a skein formula like



you probably saw a much more rarified version of this in Khovanov's talk. With that I finish my survey.

[In the geometrization conjecture, do you expect a compatibility along the tori?]

No.

[Is the geometrization conjecture true?]

Looks good to me.

[Do you have a comment about the paper by the two Chinese authors?]

I haven't seen it. Yao held a press conference in China stating that two Chinese professors completed a proof with help from Perelman, it made it sound like there's something new there. I haven't seen it.

[Why do you need to cut along a Klein bottle?]

Because you can. It's easy to see it has no geometry, you have to deal with it somehow.

[Can you tell if a surface is incompressible?]

There's Dehn's lemma and the loop theorem. If it's incompressible, you can find a loop on the surface and a disc in the complement whose boundary is the nontrivial loop.

3 Szabo

[Picture]

I'm going to talk about Heegaard Floer homology. This is joint with Peter Ozsvath. We're going to think about a 3-manifold Y^3 and associate with it invariants, such as $\widehat{HF}(Y)$, where the hat means one of the various versions. Y is an oriented closed three-manifold. We want at the very least for it to be an Abelian group, here finitely generated, with some extra structure. Maybe I should say where that's coming from. In four-manifolds you have the Donaldson invariants. You study what happens when you cut the four-manifold into two parts along some three-manifold. So this is about solving a PDE. If it has some three-manifold embedded, you could try to change the metric to make the part around the three-manifold very very long, like $Y \times [-T, T]$, a very long neck. Then you can study what happens as T goes to ∞. So you can write down $Y \times \mathbb{R}$ and study the solution space, you get a Floer homology group. Then came the Seiberg-Witten equations which have a similar structure. You can ask the question, can you get an interesting three-manifold invariant out of it? This leads to Floer homology called Monopole Floer homology. This is work done by Kronheimer and Mrowka. You're trying to understand what these equations really compute. Some are determined by fixing a metric and some other data, and then trying to solve a certain equation. But it's practically impossible to compute the solution space.

We are hoping to understand what these groups compute. We haven't done this, but instead we came up with a very similar structure. We forget the origin and try to construct the result. Halfway through we will be able to write down a chain complex where the generators are easy to understand. The boundary will be harder.

We'll start with the manifold and then write down a chain complex $\widehat{CF}(Y)$.

So before we start, what kinds of three-manifolds are there? There are S^3, the lens spaces $L(p, q)$ which are $(w, z) \in \mathbb{C}^2$ with $|w|^2 + |z|^2 = 1$ and then the quotient under the action $(w, z) \mapsto (\zeta w, \zeta^q z)$. For $p, q > 1$ and relatively prime this gives an order p action.

You can get other equations. Take $z_0^p + z_1^q + z_2^r = 0$ in $\mathbb{C}^3$. This gives a codimension two subset. You can take the unit sphere in $\mathbb{C}^3$ and take the intersection. So add the equation $|z_0|^2 + |z_1|^2 + |z_2|^2 = 1$. This is an exercise, this is a 3-manifold, the Brieskein sphere $\Sigma(p, q, r)$. These are nice and easy to understand.

What we should study is, suppose we have one of these three-manifolds. How is it built up from easier pieces? You can look at links, as in John Morgan's talk, and you can also look at Heegaard decompositions. How can we construct interesting three-manifolds by hand? Let's take a three-ball, with boundary S^2 and then add handles. Imagine it sitting in $\mathbb{R}^3$ and add disjoint neighborhoods of arcs with boundary on S^2. This (U) is called the handlebody of genus g, and has boundary the surface of genus g. You can take two handlebodies of the same genus and now I have two handlebodies U_1, U_0 with boundary Σ_g. So I can glue these to one another along a homeomorphism of Σ_g. Any decomposition like that of my three-manifolds into two handlebodies of genus g is called a Heegaard decomposition of genus g.

Some of the easy examples are:
$S^3 = B^3$ glued to B^3 along S^2. It's easy to see that any such gluing of B^3 to itself is an S^3.

Exercise 8 *This isn't so hard. Take $L(p,q)$. Take the part where $|w|^2 \leq 1/2$. Then take the rest, $|w|^2 \geq 1/2$. The multiplication itself fixes this part. This passes to the quotient in the lens space. So $L(p,q)$ has a genus one decomposition.*

So there are easy examples I can easily construct. More surprising is that

Theorem 3 *(Singer)*
Every oriented closed 3-manifold admits a Heegaard decomposition

I can triangulate a 3-manifold and then take a neighborhood of the one-skeleton. This is a handlebody and its complement is too; the points in the middle of the three and two-cells give this to you. This is very high genus generally. That's one sketch of a proof.

Another way would be by means of Morse functions. The following picture will be useful later, e.g., for $Spin_{\mathbb{C}}$ structures. So I have a Morse function where at the critical points, the Hessian will have determinant zero. These have an index, the number of negative entries. An index zero critical point is a minimum and an index three is a maximum. There are also one-and two index critical points. Every three manifold, you can show, have self-indexing Morse functions. The maxes and mins will map to 0 and 3, and there will be points with index 1 and 2 as well. You have to show this, it's a little bit of work, but great. This idea of using Morse functions to understand the topology and geometry of a manifold was used by Smale to understand the higher dimensional Poincaré conjecture. You divide it into pieces, and watch what happens as you go through critical points. You look at $f^{-1}(-\infty, t]$. If I could understand how the manifold changes with t, I would understand what my manifold is. When there are no critical points, changing t slightly will not change the manifold, but going through the critical points gives a more drastic change. You get a bowl from the minimum. Going through the index one points gives handles. Then you build up a handlebody of some genus; call it U_0. What about the other side? When you go through an index two critical point, you can add 2-handles and then a 3-manifold. You can also look at the same picture from the opposite direction. Then the maximum will be an index 0 critical point and the 2-handles will be index 1 so it's a handlebody by the same argument.

That's a general result. I guess the idea is, you have that 3-manifold, and there are infinitely many ways to decompose it into two handlebodies. So choose a Heegaard decomposition and try to associate to it a chain complex. This will be realized $\widehat{CF}(Y)$. This will have to depend on the choice of decomposition, while we want it only to depend, eventually, on the manifold. Before we get this we need to enrich the Heegaard decomposition to a Heegaard diagram. You have $Y = U_0 \cup_{\Sigma_g} U_1$. It's possibly better to think about, well, the boundary of

U_0 is Σ_g and δU_1 is $-\Sigma_g$. So what I'm going to do, instead of thinking of the identification, inside the surface I can find simple closed curves that give sufficient information to build the handlebodies. I can associate some small curves, g of them on the boundary, which bound disks in the handlebody. I can choose them disjoint from one another. So this is what I need, disjoint simple closed bounding curves $\alpha_1, \ldots, \alpha_g$. This gives me U_0, and I also have $\beta_1, \ldots, \beta_g$ which correspond to U_2. So a Heegaard diagram is this data. I also want the α circles as homology classes in $H_1(\Sigma_g, \mathbb{Z})$ to be linearly independent, and the same [dropped mic]–sorry–for the β circles.

I don't have the luxury to prescribe the intersection numbers. I want to be able to reconstruct the Heegaard decomposition. So this [picture] is another genus three handlebody. The hole inside of it in $\mathbb{R}^3$ is a handlebody, and I can choose these three circles. I just give three circles on an abstract surface, you can attach a disk cross an interval at each one of the circles. This gives you a manifold with one boundary the surface and the other boundary S^2. So you fill this with a ball. So if I have a surface I can reconstruct the handlebody. But the circles themselves can intersect in a lot of ways. This [Morse] picture can also help us look at Heegaard diagrams. How can we find these? I need a self-indexing Morse function and a metric. Then I can look at $\pm\vec{\nabla}_f$. Then what happens is that any point other than a critical point can flow and you can see where you end up. Most points flow down to a minimum, but most flow down to an index one critical point. These form a circle. There are circles flowing from the boundary to the index one critical points. Again, it's easy to see that these circles satisfy the properties you need.

So this is a Heegaard diagram, and now I want to explain this picture. It's supposed to be the genus two surface. I add a point at infinity to the blackboard by deleting disks and identifying the boundary (or attaching handles to the boundary). So I've drawn here α_1 and α_2. Here's β_2 where you have to use the identification. This gives us a three-manifold. This is from attaching disks cross intervals. The questian is what is this?

Exercise 9 *Show this is the Poincaré sphere.*

What are even easier looking Heegaard diagrams? Take the torus with one circle and then the circle which intersects it once. What is the three manifold we get this way? One of them will bound a disk inside the torus, a cross section. So this gives the solid torus. So S^3 has a solid torus inside. Outside is also a solid torus. Take this line and infinity to be one of the circles and the inside to be the other one. These all intersect this disk in just one point. The bottom line is that S^3 has a genus one Heegard decomposition. So any time I wrie down this picture I get S^3. This is good because we're understanding something but bad because, if I have two Heegaard diagrams, you can choose two points and form a connected sum. You get the connected sum of the two original three-manifolds. But using the special one we just constructed, if I have a Heegaard diagram, and change it by adding one to the genus, with the trivial pair of circles, I get a higher genus Heegaard diagram for Y. So I have no hope of finding a unique one. This is called stabilization.

Exercise 10 *I would like you to think about some of these homeworks. My TA Josh is ready to answer some or all of these question; it makes sense to attempt some of the easier ones.*

We looked at $L(p,q)$ and saw that they admit genus one Heegaard decompositions. Try to find a genus one Heegard diagram for them. That's not hard to guess. There are two curves in the torus. One of the curves can be extended, it's along the edge of the square, and then β will depend on p and q. It's something to work out.

We really now have $(\Sigma_g, \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ and we also have to fix a basepoint in the complement of all of these things. These are the inputs we have so far.

Now I would like to describe a chain complex $\widehat{CF}(\underline{\alpha}, \underline{\beta}, z)$. The group $\widehat{CF}$ is freely generated by collections of points $\underline{x} = (x_1, \ldots, x_g)$ in Σ_g, with some properties. The α and β curves likely intersect. I will look at pairs of points with some properties. I choose them so that every α circle and every β circle each contain exactly one point. So $\cup_\sigma \in S_g(\alpha_1 \cap \beta_{\sigma(1)}) \times (\alpha_1 \cap \beta_{\sigma_2}) \times \cdots \times (\alpha_g \cap \beta_{\sigma(g)})$.

I have in my example picture of $\Sigma(2,3,5)$ fifteen generators of the kind $(\alpha_1 \cap \beta_1) \times (\alpha_2 \cap \beta_2)$. There are six of the kind $(\alpha_2 \cap \beta_1) \times (\alpha_1 \cap \beta_2)$ so you get twenty-one total.

You can get Heegaard diagrams with empty $\underline{x}$ set; e.g. for the obvious genus one case you can get $S^1 \times S^2$. There's versions of Floer homology that work for every diagram; then the homology will be zero. If $b_1(Y) = 0$ then every α intersects a β.

[Can't you make the α and β linearly independent?]

Exercise 11 $H_1(Y) = H_1(\Sigma_g)/[\alpha_i] = 0, [\beta_i] = 0$. *So if* $H_1(Y) \neq 0$ *then there will be some dependence.*

I can twist a bunch of times at the end in this picture. Look at the family when you do n twists at the n. Compute $H_1(Y_n)$. When it's three $H_1(Y_n)$ is zera but when it's not three, it does something else. Do you have a guess for what manifold it could be? Of course. This is as far as topology will help us. The boundary map itself is a more complicated object that will involve holomorphic disks. I'll have to explain that next time.