# Low Dimensional Topology Notes July 7, 2006 

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## 1 Gabai

[The morning session will be for Etnyre and the afternoon for Fintushel-Stern.]
So, am I on? Can you hear me?
[No.]
Hello? Hello? Well, the light's on. So, welcome back? Does anyone have any questions about anything?
[You were going from the core, and you wanted to homotope out. Were you building a flow?]
Last time, given the manifold, we found a proper map from $S \times[0, \infty)$ with the properties that [unintelligible]and each section homologically seperates. These maps were simplicial hyperbolic interpolations and these maps were $\epsilon$-homotopies. Think of this as a football field. Here's Milnor, and he hands off to Kirby, who runs a jagged path to the twenty yard line and hands off to Eliashberg, who without going back runs to the thirty and hands off to Peter Ozsvath. So if you just follow the football, that's a proper path to $\infty$.

Before I just continue the proof of the taming criterion, I'd like to make a few remarks about ends.

Ends of three-manifolds have two types. One, I drew the core $C$. Now focus on the boundary component. Either the fundamental group of the boundary injects into the core, or not. In this case

Exercise 1 If $\pi_{1}(\delta C) \rightarrow \pi_{1}(C)$ is injective, then $C \rightarrow M-\circ C$ is a homotopy equivalence. Further, if $S$ homologically seperates $C$ from $\mathscr{E}$ and the genus is the same as $\delta C$ the inclusion is a homotopy equivalence.

These are the types for which Bonahan proved the tame ends conjecture.

The other case is when the care doesn't $\pi_{1}$ injeect. So this could be if $C$ were a handlebody. if you have a surface which homologically seperates the end from the core, then $\pi_{1}(S) \rightarrow \pi_{1}(M)$ is not injective.

So you had a sequence of surfaces exiting the manifold. In the case where a simple loop died, it was the boundary of the core compressing inside the core.

Maybe to demystify the ends of three-manifolds a little more, take a look at the following picture. Here's a three-manifold with finitely generated fundamental group. The Scott core looks like this picture. These have incompressible boundary. They inject at the level of the fundamental group into the core. Then you add on one-handles, thickened arcs. A priori, a manifold with finitely generated fundamental group could look like that. But the reality is that you can understand manifolds by, if you're given an end, it's an end of another manifold with a friendly looking fundamental group. It's the end of a manifold whose $\pi_{1}$ is a free product of surface groups and a free group.

Proposition 1 Let $\mathscr{E}$ be an end of an open 3 -manifold $M^{\prime \prime}$ with $\pi_{1}\left(M^{\prime \prime}\right)$ finitely generated. Then $\mathscr{E}$ is the end of another 3 -manifold $M^{\prime}$ with $\pi_{1}\left(M^{\prime}\right)=F * \pi_{1}\left(S_{1}\right) * \cdots * \pi_{1}\left(S_{n}\right)$.

The proof is by staring at this picture. Embedded in the end is this manifold $M$. Since these surfaces inject at the level af the fundamental group, the fundamental group of $M$ injects at the level of the full manifold.

In a covering space, you get ends of the sort $S \times \mathbb{R}$. I hope that helps clarify how to think of ends of three-manifolds.

Let's remind ourselves:

Lemma 1 (Stallings)
Let $X, Y$ be irreducible 3-manifolds and $T \subset Y$ embedded and $\pi_{1}$ injective. Let $f: X \rightarrow Y$ with $R=f^{-1}(T)$. Then suppose the preimage had a compressing disk $D$. Then $f \cong g$ via a homotopy supported near $D$ such that $g^{-1}(T)$ is $R$ compressed along $D$.

If $R_{1}$ is a two-sphere component of $R$ then $f \cong g$ via a homotopy supported near the three-ball bounded by $R_{1}$ such that $g^{-1}(T)=R-R_{1}$.

You start here, where you are before, and as you move in this direction you send the homotopy of sending this guy inside. You just sort of look at it.

Proposition 2 If $N$ is irreducible with $\pi_{1}(N)$ finitely generated and $\mathscr{E}$ is its end and $F$ : $S \times[0, \infty) \rightarrow N-C$ is proper, maps into $\mathscr{E}$ and is $\pi_{1}$ injective with $F(S \times\{0\})$ homologically seperating. Then $F$ is properly homotopic to a homeomorphism, i.e., $\mathscr{E}$ is tame.

A priori, in this complicated end, you want to find, homotope this map without messing it up near $S \times\{0\}$. How do you prove this? Lot $Y \subset Z=N-K$ be an embedded seperating
surface seperating $F(S \times m)$ from $\mathscr{E}$. Compress $Y$ as much as possible to obtain a new $Y_{1}$, so it might pass through this stuff out here. You can take $Y$ as the boundary of a given compact set. That's another thing. How do you think about open manifolds. Assume it's second countable. Then you can take it as a union of compact things. So any open manifold is built up as a union of bigger and bigger compact things. Just like you can think of the Whitehead manifold as a union of things.

So the claim is that we can assume we have the same picture to the right of $S \times\{m\}$ and we that $Y$ injects in $\pi_{1}(M)$. So in the preimage of this compressing disk you'll see a lot of disks from $S \times m$. If you look at a particular disk, the preimage of a compressing disk, it bounds a disk in $S_{m}$, Then you use $\pi_{2}=0$ to push the disk off the compressing disk. So after a homotopy of $S$ you can push it off the compressing disk and then compress.

I need one more projector. So the conclusion of what I just did was, I can assume that $Y$ is $\pi_{1}$ injective in the complement of $K$ and lies to the far side of $S \times\{m\}$. Now let's look at the preimage of $Y$ under $F$. Actually, well, the preimage, look at the part that lives in $S \times[m, \infty)$, and it's a little fact that if you have a surface cross an interval then the only compact $\pi_{1}$ injective surfaces in $S \times[m, \infty)$ are of the form $S \times \star$ up to isotopy. So once we have eliminated all the compressing things, the two-spheres, if there was nothing left, you apply the map $F$. So in the image, the image, it would have to cross the surface. The conclusion is that the preimage is a finite number of sections. This is a bad picture, and here is the good picture. Here's $X$, and $S \times\{m\}$ is whatever, wherever it goes. So, but notice that focussing on a single one of these, it's a surface of some genus mapping under $F$ to this surface. Since the image is homologically seperating, the degree of tis map is one. So if it's degree one, the genus could be bigger, in which case there's a kernel in $\left(\pi_{1}\right)_{*}$ or they're the same genus in which case it's isotopic to a homeomorphism.

Exercise 2 I've used this several times. If $f: S \rightarrow T$ is a degree one map between closed surfaces then the genus of $T$ is at most the genus of $S$, with equality if and only if $f \cong a$ homeomorphism.

Okay, so now we've made a huge amount of progress. We know, if the boundary of the core is genus $g$ then we can find a sequence of genus $g$ surfaces exiting the manifold. So $F(s \times i)$ is embedding and homologically seperates.

If we could show that the piece of the end between $T_{i}$ and $T_{i}+1$ were a product, we could put the products together and get a product structure on the whole end.

We do know that the image of $T_{i}$ is a finite number of surfaces isotopic to sections. So after more of the Stallings compressions and homotopies, we could arrange it so the preimage looked something like this. So the preimage of $T_{i}$ could be some surfaces. In a couple of dimensions lower the picture is something like, you have and end, and here's $T_{i}$, and the premiage might be here.

So, right, so, by Stallings you can assume that the preimage of the union of the boundaries is incompressible. Eventually you find a bit where both boundary components are used, at
which time you can say, from a theorem of Waldhausen, say you have a map $Q \rightarrow P$ of degree one and $\pi_{1}$ injective with the restriction to the boundary a homeomorphism. Then the conclusion is that the map $Q \rightarrow P$ is homotopic to a homeomorphism relative to the boundary.

If you look at this theorem, I don't have time to talk about it, but Stallings plays a big role.
Step three is that the original $F$ is homotopic to a homeomorphism. We proved the end is tame but showing this to be homotopic to a homeomorphism is another little step that we leave to you.

So the next thing is to talk about shrinkwrapping. What is shrinkwrapping?
It's a technique discovered by Danny Calegari and myself to discover Cat $(-1)$ surfaces in hyperbolic $M^{3}$. We found it in the smooth category and Teruhiko Soma found it in the PL category, with a much simpler proof. Let me give you the statement.

Let $N$ be a hyperbolic three-manifold and $\Delta$ a collection of geodesics. Then the embedded surface is 2-incompressible relative to $\Delta$ if each essential compressing disk hits the geodesics at least twice. We'll have handlebodies and they will compress all the way down. But in some sense they're essential because they're homotopically nontrivial in the complement of the geodesics.

Theorem 1 Let $S \subset N_{\text {hyp }}$ closed and $\Delta$ a locally finite set of geodesics such that $S$ seperates $\Delta$, with $S \cap \Delta=\emptyset$ and $S 2$-incompressible relative to $\Delta$.

Then we can find a homotopy $S \times[0,1] \rightarrow N$ with $\left.F\right|_{S \times 0}=S, F_{S \times 1}$ a simplicial hyperbolic that hits, and $F(S \times[0,1)) \cap \Delta=\emptyset$.

Soma found this great proof of shrinkwrapping in the PL categary and also did something else very helpful with a clever use of covering spaces.

Now, we need a more general version where we replace $N$ by $\hat{N}$ a branched cover. Locally $h a t N$ is either hyperbolic or is an infinite branched cover of a hyperbolic ball branched over a geodesic.

So, the argument, let me show you some lemmas on route toward proving shrinkwrapping. Here is lemma one.

Lemma 2 Existence, uniqueness, and continuity of $\Delta$ geodesics. Suppose you have a path $\alpha_{0}$ in $\hat{N}$ avoiding the geodesics, then you can homotope this to a minimizing piecewise geodesic path $\alpha_{1}$ homotopic to $\alpha_{0}$ such that if $f: I \times I \rightarrow \hat{N}$ is the hamotopic then it is away from $\Delta$ until time 1 and the function $\alpha_{0} \leadsto \alpha_{1}$ varies continuously.

If you're in the schoolyard and you have a rope twined through the monkeybars, if you pull the rope tight, then it won't cross the monkey bars, the geondesics.

A geodesic pulled tight is called an $\alpha_{1}$. The original might wind around it twice, but somehow, so, anyway, this is sort of the picture. If the original curve is a loop you can still homotope it down to a $\Delta$-geodesic. If the germ starts on the geodesic you can still pull it tight.
[Do you care if you have singularities once it's pulled tight?]
The thing at the end is a piecewise geodesic. It keeps some information in the background. You know how to push it off. So how do you prove this?

Look at the original metric. Now find a continuous family of metrics on the complement of $\Delta$ with the metrics converging to $g_{0}$ and with negative sectional curvature. If you have a Riemannian metric with negative sectional curvature then between any two points there is a unique geodesic and they vary continuously.

So here's a way to think about this, in two dimensions, where you have a surface and a point. If you just put up a cusp for the point, then that metric is negatively curved and agrees with the original one outside the neighborhood. As $t$ goes to zero, you use smaller and smaller cusps.

So, well, I guess I'm out of time. On Monday we'll complete the proof of the shrinkwrapping and try to get more of an understanding of the tame ends theorem.
[When you take the two ends, how do you know you get a length minimizing geodesic?]
You have, well, metrics of negative sectional curvature, there's a unique geodesic of shortest length. If you had some guy in here, if you perturb him a tiny bit, you have a path which is just a little longer.
[You've talked about finitely generated fundamental group.]
Actually for shrinkwrapping you don't need finite generation of the fundamental group. I'm definitely using local finiteness. Thank you very much.

## 2 Fintushel-Stern

[I'd like to make an announcement. There was vandalism last night so they're going to be stricter about the nametags.]

It's really a pleasure to be here in Park City. I'm Fintushel-Stern. I'm one half of that, I'm Ron Stern. We're going to give six lectures. I thought I'd explain how we're splitting it up. I'll talk on Friday, Tuesday, and Friday, and Ron will be talking on Monday, Wednesday, and Thursday.

The goal is to classify manifolds by some means. The goal is
a. Classify smooth simply connected 4 -manifolds
b. Classify symplectic four-manifolds ( $\omega \in \wedge^{2}, \omega \wedge \omega \neq 0, d \omega=0$ )
c. Complex manifolds (up to diffeomorphism)

I think of studying four-manifolds as being like a black box. You can't open them in the obvious way. You shake it, look at it, an occasionally you get it opened. Then inside it's just another pair of glasses, you look through them and realize that you were looking at the wrong black box. There are a lot of approaches you can take.

You could look with high dimensional methods. Every topological manifold in high $n$ has only finitely many smooth structures. There exist infinitely many topological manifolds that have homotopy type $\mathbb{C} \mathbb{P}^{n}$.

But in dimension four, most if not all $X$ have infinitely many smooth structures. And there exist only two topological manifolds homotopy equivalent to $\mathbb{C P}^{2}$, the other one being the non-smoothable Chern manifold.

Since not everything has a symplectic or complex structure, you can't necessarily get everything from those approaches.

We're going to use the approach of low dimensional topology. We're going to come up with a list of four-manifolds and some methods motivated by dimensions one, two and three.

So that's the vantage point we'll take.
Let's get some algebraic topology out of the way. I want it to be smooth, oriented, and closed, to begin with. One of the interesting problems you can discuss is that every finitely presented $G$ is the fundamental group of some 4-manifold. We don't want to get into group theory. So we're going to mostly study simply connected manifolds.

So then we have $H_{2}(X) \cong H^{2}(X)$ which is some collection of $\mathbb{Z}$. We're going to work with cohomology, which is typically much richer, it's a ring. We have the cup product form or intersection form. Given a couple of elements in homology $x, y \in H_{2}(X)$ you get a pairing $Q(x, y) \in \mathbb{Z}$. Given any $x \in H_{2}(X)$ it can be embedded by a smoothly embedded surface sitting inside of $X$. Suppose I give any $\alpha \in H^{2}\left(X^{n}\right)$. Then $P D(\alpha) \in H_{n-2}$ can be represented by an $(n-2)$ embedded manifold.

So any element $x$ can be represented by an embedded surface. So we have $x$ represented by a surface $\Sigma_{x}$ and $y$ represented by $\Sigma_{y}$, and you have them coming withh an orientation, and then you perturb them to transversally intersect and count $Q(x, y)=\#_{\text {sign }} \Sigma_{x} \cap \Sigma_{y}$. So you count the signed intersection from how the two of these intersect.

Basic facts about $Q$ are that $Q$ is symmetric and bilinear, unimodular, meaning that if we choose a basis $\left.x_{1}, \ldots, x_{n}\right) \in H_{2}(X)$ then the determinant has matrix $\pm 1$.

Now we're certainly going to look at Poincaré duality and how things intersect themselves and so on, but let me extract some stuff. A two dimensional surface in $X^{4}$ has a normal bundle, a tubular neighborhood of $\Sigma^{2} \hookrightarrow X^{4}$. This is a 2-disk bundle over $\Sigma$, so it has an Euler class. The Euler class of the tubular neighborhood $e\left(N\left(\Sigma^{2} \hookrightarrow X^{4}\right)\right)=Q(\Sigma, \Sigma)$. This
is pretty easy to see if you know what the Euler class is.
I don't want to keep carrying around the $Q$ so I'll write $x \cdot y$. So the study of symmetric bilinear forms has been around for a long time. It's still a problem to classify symmetric bilinear forms.

From $Q$ I'll extract some information. You can certainly get the Euler characteristic, which is just $r k H_{2}+2$. You can also get the the signature $\sigma(X)$. This is a symmetric integral matrix with determinant $\pm 1$, so diagonalizable over the reals. You can count the number of positive and negative entries in the diagonal. The signature is the difference between the number of positive entries and the number of negative entries on the diagonal.

Of course, the Euler characteristic is then $b_{+}+b_{-}+2$ and $\sigma(X)=b_{+}-b_{-}$. There's also what we call the type of the form. It could be the case that the self intersection of every element is even, if $x \cdot x$ is always $0 \bmod 2$. This we call even, and all others we call odd.

The basic deep theorem, which we're not going to talk about.

Theorem 2 Freedman, Donaldson
Every intersection form can be the intersection form of a topological manifold. If the form is odd there are two manifolds; if the form is even there is one manifold.

The downside is that we don't know how to classify these forms.
What Donaldson does, we don't need to deal with some of this, we don't need to classify all of them. Given smooth simply-connected $X^{4}$, the Euler characteristic, the signature, and the type determine $X$ up to homeomorphism.

There's a manifold with intersection form $E_{8}+E_{8}$, they're not smoothable. There's also one with form $\Gamma_{16}$. They're not homeomorphic but I don't care because one of them isn't smooth.

If we're concentrating on simply connected manifolds, that might be our fatal error, that might be why we can't open the box.

There's a theorem putting up on the board saying that most 4-manifolds have infinitely many smooth structures. This is how I'll tell that I've changed the smooth structure but not the homeomorphism type.

So up to homeomorphism we know our four-manifolds.
Now we want to understand these from the vantage of low-dimensional topology. Let's start getting some examples of nice 4-manifolds.

- We have $S^{4}$.
- Via dimension reduction, $S^{1} \times M^{3}$.
- $\Sigma_{h} \times \Sigma_{g}$ where we'd say $g\left(\Sigma_{h}\right)=h$. So this is a trivial bundle. So we could look
at arbitrary surface bundles over surfaces. It's a technique of basically dimension reduction.
- We can also think of 4 as complexified dimension two. In dimension two we have $\mathbb{R}^{2} \mathbb{P}^{2}$, which are lines through the origin in $\mathbb{R}^{3}$, so we can look at $\mathbb{C P}^{2}$, which are complex lines through the origin in $\mathbb{C}^{3}$.

So what are some methods? One thing we can do, well, here are some operations on $X^{4}$.

- We can change orientation from $X$ to $-X$.
- We can take connected sums, $X_{1}^{4}, X_{2}^{4} \rightarrow X_{1} \# X_{2}$. I remove a disk neighborhood of a point in each one. I remove the disks and glue together with the identity map. I want to take an oriented connected sum so I would like one to be $S^{3}$ and the other to be $-S^{3}$. In principle we'd like to study modulo this operation, so irreducible, meaning that if $X=X_{1} \# X_{2}$ then one of the two manifolds is a homotopy 4-sphere.
[Do we have the unique factorization?]
We do, that's exercise five. Wait, well, let's, exercise five is to discuss this.
[Is $S^{1} \times S^{3}$ irreducible?]
Yes, well, uh. Maybe it should be called prime?
- "Dehn surgery." This is to remove a neighborhood of $S^{1}$ and sew it back in nontrivially, taking the meridian $m$ to $a m+b \ell$. This is sometimes called $a / b$-Rolfsen surgery.
You remove $S^{1} \times\left(S^{1} \times D^{2}\right)$ and sew it back in differently, so that $m$ goes to, say, $b_{1} \ell_{1}+b_{2} \ell_{2}+a m$. There are sort of three parameters here, not just two. So to do this I need a torus with trivial normal bundle. $T^{2} \hookrightarrow X^{3}$ with $\left[T^{2}\right]^{2}=0$ you can do "generalized $\log$ transform." i.e., $S^{1}$ times Dehn surgery.
- Branched covers. What, you don't like branched covers? Well, you don't have to listen to this part. You want branched covers of codimension two objects, so suppose $B^{2} \hookrightarrow X^{4}$. Then the complement has to have, look at $[B] \in H_{2}(X)$. Suppose it's not a primitive class, $[B]=n[A] \in H_{2}(X)$ and $n$ is the largest such $n$. Now suppose that, well, this means $H_{1}(X \backslash N(B \hookrightarrow X)) \cong \mathbb{Z}_{n}$. This assumes $X$ is simply connected. If $d \mid n$ then I can take a $d$-fold cover of $X \backslash N(B)$, and then sew in, this is a bundle of Euler class $n$. Oh, I've made a mistake. $e(N(B \hookrightarrow X))=n^{2}[A]^{2}$. So what happens to the Euler class of the bundle? Sew in the Euler class $(n / d) n[A]^{2}$ branched over $B$.
This is an interesting construction. Let's do one more.
- Sewing together knot complements. By this I mean, take $K_{1}, K_{2} \hookrightarrow S^{3}$. I can remove them and sew together the complements. This is an operation that sometimes gives you very interesting three-manifolds. Let's see what happens in dimension four. We could suppose we had two four-manifolds, with the surfaces $\Sigma_{i}$ inside them, and sew together their complements. To do that, I need to be able to identify their boundaries. The boundaries have to be sewable, so that the genus of the two are the same. Each
boundary has Euler class of the intersection numbers. In order to glue we need to know that $\left[\Sigma_{g}\right]^{2}=-\left[\Sigma_{g}^{\prime}\right]^{2}$ as well.
We can also cross everything with a circle. I can take out something and sew in a knot complement. So I want to suppose I have $S^{1} \times S^{1} \times D^{2}$. Suppose inside the manifold I have a knot, gluing a knot complement replaces a solid torus with a knot complement, I just want to cross this with a circle. So replace $S^{1} \times S^{1} \times D^{2}$ with $S^{3} \backslash K$. So suppose I have a $T^{2} \hookrightarrow X^{4}$ with trivial normal bundle. Then I can replace it with $S^{1} \times S^{3} \backslash K$.

Now, these are a basic collection of operations. How do they alter the Euler characteristic, signature, and type?
[Is there an analogue of Heegaard splittings?]
Yes, we don't have time to talk about it here.
I'm going to confuse you a little bit. I want to rewrite these three variables as $c=3 \sigma+2 e \in$ $\mathbb{Z}, \chi=\frac{e+\sigma}{4} \in \mathbb{Z}[1 / 2]$. It's an easy exercise that this is always in $\mathbb{Z}[1 / 2]$.

- Changing orientations, here the operations are a little perverse. So here $c(-X)=$ $48 \chi(X)-5 c(X)$ and $\chi(-X)=5 \chi(X)-\frac{c(X)}{2}$. The type is unchanged.
- Connected sum is a stupid operation, I'm going to leave it as an exercise to see what that does.
- For $d$-fold branched covers, $B \hookrightarrow X^{4}$, then $c(\tilde{X})=d c(X)-2(d-1) e(B)-\frac{(d-1)(d+1)}{d}[B]^{2}$. So it's almost multiplicative. $\chi(\tilde{X})=d \chi(X)-(d-1) e(B)-\frac{(d-1)(d+1)}{3 d}[B]^{2}$. Also almost multiplicative. The type might change.
- for the generalized log transform, it's very easy that $c\left(X_{\text {log transform }}\right)=c(X)$, and $e$ doesn't change either. The type might change.
- for the fibered sum over $\Sigma_{g}$, we have $c$ equal to $c\left(X_{1}\right)+c\left(X_{2}\right)+8(g-1)$ and $\chi\left(X_{1} \#_{\Sigma_{g}} X_{2}\right)=$ $\chi\left(X_{1}\right)+\chi\left(X_{2}\right)+(g-1)$.

This duh type construction of mimicking the lower dimensions gives us a very rich class of 4-manifolds.

## 3 Etnyre

Let me remind you what we've been doing for the last couple of days.

## Theorem 3 Eliashberg-Thurston

asy $C^{2}$ foliation $\xi$ on an oriented three manifold $M$ other than $\zeta$ on $S^{1} \times S^{2}$, can be $C^{0}$ approximated by a positive and negative contact structure.

We're considering mainly positive contact structures. Step one was to perturb $\xi$ to a confoliation such that every point in $M$ was connected to a point in $M$ where $\xi^{\prime}$ is contact by a path tangent to $\xi^{\prime}$. Step two is to perturb $\xi^{\prime}$ into a contact structure.

We did step two yesterday. For step one we will do the following.

Theorem 4 1. If $(M, \xi)$ is a $C^{k}$ foliation and $\Gamma$ is a curve in a leaf of $\xi$ with nontrivial linear holonomy, (recall, that's a map defined near zero, if you took the derivative it was not equal to one) then $\xi$ can be $C^{k}$-approximated (actually deformed) into a contact structure near $\Gamma$. So linear holonomy is good, it allows us to make these nice deformations.
2. If $\Gamma$ is a curve in a leaf of $\xi$ with weakly attracting holonomy then $\xi$ can be $C^{0}$ approximated into a contact structure near $\Gamma$.

So this is the theorem we're after. I'll sketch two different proofs for the first part. The first proof is so much easier than the second.

So let $U=\Gamma \times[-1,1] \times[-1,1]$ with coordinates $x, y, z$. Let me draw this picture again. This is $\Gamma$ and then these are the $x, y$, and $z$ directions. I haven't actually claimed you can do this, because $\Gamma$ is a closed curve. You can take this closed curve and repeat the same proof as before. Here we can wriet $\xi=\operatorname{ker} \alpha$ where $\alpha=d z-a(x, z) d x$. What does the condition of having nontrivial linear holonomy tell us about $\alpha$ ? It does give us some information. I claim we can choose coordinates such that $\frac{\partial a}{\partial z}>c$ for some constant $c$. I don't think you can say this in every coordinate system. You write down a model situation and check that your situation agrees with the model.

Now that you have this let $h$ be a function which decreases from 1 to 0 and then stays at 0 after parameter 1. Let $\beta=h\left(y^{2}+z^{2}\right) d y$. Then $\alpha \wedge d \beta+\beta \wedge d \alpha=-a h^{\prime} 2 z(d x \wedge d y \wedge d z)+a_{z} h d x \wedge d y \wedge d z$. The second factor is strictly positive in a neighborhood. I claim the other quantity is at least zero. This is because $h^{\prime}$ is always negative, and $a$ shares the sign of $z$. It's a very easy exercise to show that al $\tilde{p h} a=\alpha+\epsilon \beta$ is contact for small $\epsilon$.

This shows clearly that you get a deformation, which you do gradually into a contact structure.

More generally we set up the coordinate system as before on the neighborhood $U$. Considur the annulus $A_{y_{0}}=\left\{(x, y, z) \mid y=y_{0}\right\}$. If $y=-1$ this might look like this. When $y_{0}=0$ you have the same picture. For all $y_{0}$ you more or less have the same picture.

In the region $R$ between $A_{-1}$ and $A_{-1 / 2}$, apply a diffeomorphism that is the identity on $\delta U \cap R$. On $A_{-1 / 2}$ all the tangents to the foliation are rotated clockwise.

What do I mean by that? Let me draw a picture. Before we had something that looked like this. Here are just a few leaves. If I just shift this down, the tangent is clockwise. So now if I make the other lines steeper they're always clockwise.

So the plane field is no longer continuous as I go from one side of $A_{-1 / 2}$ to the other side of $A_{-1 / 2}$. So as I go along things the $y$ axis. So I just push it counterclockwise and I eventually match up.

On each ray, $x_{0}, y, z_{0}$, as $y$ increases, rotate the plane counterclockwise to match up with what we had before. These rays are always tangent to the foliation, and as we move along we want to see what they do along the $y$-axis.

Does everybody believe that I can do this?
This hopefully gives you a picture of what is happening.

Exercise 3 Make this work. The following lemma is useful in the nontrivial linear holonomy case. There's a similar lemma in the other case.

Lemma 3 Let $v_{x}$ be a family of smooth functions from $[-1,1]$ to $[-1,1]$ such that $v_{x}(0)=$ 0 for all $x$ and $v_{x}$ is monotonically increasing for all $x$. Then there is a diffeomorphism $f:[-1,1] \rightarrow[-1,1] C^{\infty}$ close to the identiyt and tangent to the identity at $\pm 1$ satisfying $f^{\prime}(z) v_{x}(z)>v_{x}(f(z))$ for all $z \in(-1,1)$ and all $x$.

The second hint for the exercise is, take $v_{x}(z)=a(x, z)$.
So what do we have now? The main idea is that if you can rotate these counterclockwise it's easy to get a contact structure. If we have this lemma that allows us to perturb into a contact structure, now that we know that holonomy is useful, what can we do with that fact? If we can arrange that all leaves of the foliation $\xi$ come arbitrarily close to curves with holonomy, then we're done. Hopefully this is clear. If we have holonomy, we can make it contact, and then flow to all the nearby leaves. We can arrange that every leaf comes close to a curve with holonomy. Here's where we need to bring in some foliation theory.

Let $\xi$ be a foliation on $M$. A minimal set of $\xi$ is a nonempty closed set that is the union of leaves and is minimal among such sets. Another way to say this is, a closed union of leaves that is the closure of any leaf in it in the set. That's another way to think of a minimal set.

Another exercise,

Exercise 4 show that any leaf in $\xi$ limits to a minimal set.

Thus we need to see that we can perturb $\xi$ so that every minimal set has curves with holonomy. Then we'll be able to take each leaf to be arbitrarily close to a curve with holonomy.

Let's see what minimal sets can be.

Theorem 5 In a $C^{2}$-foliation on a 3-manifold, every minimal set is

1. all of $M$ (such leaves are called minimal).

## 2. a closed leaf.

3. an exceptional minimal set, meaning the complement of the other two.

This is easy as stated, but exceptional minimal sets have a lot of nice structure. That's the content of the theorem.

## Theorem 6 Sacksteader

Exceptional minimal sets contain leaves with linear holonomy.

So we can actually turn a neighborhood of an exceptional minimal set into a confoliation which is locally a contact structure.

If $\xi$ is a minimal foliation then either there is holonomy or there isn't. If there's holonomy, there's a nice theorem of Ghys that says there's nontrivial linear holonomy, so we can perturb into a contact structure. When there's no nontrivial holonomy, there's a theorem of Tischler that says we can $C^{0}$ approximate $\xi$ by a fibration. So $M$ is a fibered manifold and the new foliation is the tangents to the fibers. Here's another case where $C^{0}$ comes up. If you look at the foliation of $T^{2}$ by irrational lines, it's not a fibration, but you can tilt it slightly and get a rational foliation where you have a fibration with circle base.

If your manifold is a fibering over $S^{1}$ then $M=\Sigma \times[0,1] / \sim$ where $(0, x) \sim(1, f(x))$ for some homeomorphism $f: \Sigma \rightarrow \Sigma$. So you can cut along a curve and then shear along that curve. In a disk, the picture is really obvious. If you glue by the identity map from one side to another, with horizontal lines, then things will be trivial, but there will be spiral leaves if you shift slightly. So we have two closed leaves, both with holonomy.

So we're done in this case too. So either there's holonomy, or there's none so that we can break things and get two closed fibers. This leaves us only with closed leaves. What do we do there? First, we can perturb $\xi$ so that there are only finitely many closed leaves. That's fairly believable. You can pick two of them and spiral toward one and away from the other. There can be some weird things, like closed leaves that are the interval cross a cantor set.

If the closed leaf has weakly attracting holonomy, we're done. We know we can perturb that to a contact structure.

We might have no holonomy at all. I mean it's only trivial, then there's, we said this Reeb stability theorem implies a neighborhood of $\Sigma$ is foliated by $\Sigma \times[-1,1]$. But once we've arranged this we only had finitely many closed leaves.

We have one more situation we have to deal with. Weakly attracting or repelling is fine. All the holonomy we have might be attracting on one side and repelling on the other.

In that case we want to replace one leaf with two, both of which have weakly attracting holonomy. If you thing about it, this can be done with a $C^{\infty}$ perturbation. I guess you'd like to see that there are only finitely many minimal sets, and you don't want to perturb an infinite number of times.

I was supposed to finish that twenty minutes ago, maybe I'll stop there and save the rest for next week.

