# Low Dimensional Topology Notes
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## 1 Gabai, hyperbolic geometry and three-manifold topology

[The problem sessions will be for Gabai and Gordon. Okay, lecture number two.]

It's great to be back here. Today my goal is to give you a good exposition of the taming criterion. That was a condition for a hyperbolic 3-manifold to have tame ends. I'm assuming that there is only one end and no parabolics. The criterion says that we have a sequence of surfaces $S_1, S_2, \ldots$ exiting the manifold with genus at most the genus of the boundary component of the core.

The second condition is that these surfaces are $Cat(-1)$, and the third is that these homologically separate; $[S_i] = [\delta_{\mathscr{E}} C] \in H_2(N - \dot{C})$.

Here are some algebraic topology exercises worthy of being on your quals.

1. $S_i \to N - \circ C$ and $\delta_{\mathscr{E}} C \to N - \dot{C}$ are $H_1$-surjective (and actually injective)
2. the genus of $S_i$ is equal to the genus of $\delta_{\mathscr{E}} C$.
3. $S_i$ is $\pi_1$-injective into $N - \dot{C}$, at least on simple closed curves.
4. $S_i \to N - \dot{C}$ is $H_1$-injective.

This all ignores condition two.

To prove this taming criterion, I need to explain the theory of simplicial hyperbolic surfaces and surface interpolation (Thurston, Bonahon, Canary, Canary-Minsky).

**Definition 1** *A simplicial hyperbolic surface is a $f : S \to N$ where $S$ is a triangulated surface, and $f$ restricted to a simplex is a totally geodesic immersion. The cone angles at each vertex are at least $2\pi$. This makes the intrinsic curvature at least $-1$.*

**Definition 2** *$N$ is useful if there exists a unique vertex $v$ and some edge is an honest geodesic in the manifold.*

Her's a fact. If $N$ is a closed simplicial hyperbolic surface, there's a closed geodesic of boundod length. Since you have a surface of negative curvature. An embedded geodesic disk has area depending on the radius. So an embedded disk has a limit on the radius. Then you can see a path, which is homotopically nontrivial.

Another. A short curve which is an $S$-geodesic away from the core of $N$ can be homotoped into a $N$-geodesic by a homotopy away from the core, or into a Margulis tube. A further bounded homotopy modulo Margulis tubes transforms $S$ to a useful simplicial hyperbolic surface. If you have something of length $r$ and you project it a distance of $d$, that's at least $r \cosh d$. Either it will become extremely short or move to a geodesic in bounded time.

**Corollary 1** *After a bounded homotopy these can be homotoped to useful simplicial hyperbolic surfaces.*

*Now you should think from now on that the surfaces are these useful hyerbolic surfaces.*

Let me outline how to prove the taming criterion. First we want to understand the idea of surface interpolation. The geometrically finite ends are nice, they have product ends with exponential expanding sections. What we want to show is that the geometrically infinite manifolds satisfy the taming criterion, which will prove tameness.

So first we want surface interpolation. This will give us a way to homotope the $S_i$ toward the core. This will let us find a proper map $S \times [0, \infty) \to N - C$, which will be proper, $\pi_1$ injective on simple loops, and $G(S \times *)$ will homologically separate.

Step three will show that such an embedding gives a product structure.

I want to explain to you how to do this interpolation. I'm going to describe to you three moves on simplicial hyperbolic surfaces. They are the quad move, which take two triangles meeting along an edge to the "opposite" triangles in the tetrahedron spanned by the vertices.

One is sliding a vertex along the preferred edge. The third move is to change the preferred edge. Lifting to the universal cover, the convex hull is a totally geodesic triangle.

**Proposition 1** *Canary*
*The in-between surfaces in these homotopies are all simplicial hyperbolic surfaces.*

They may pass through non-useful ones, but at the endpoints they're useful.

Why are the in-between surfaces simplicial hyperbolic? One fact is that each of the surfaces is convex busting, so that all the cone angles are at least $2\pi$. The only issue is at vertices. So convex busting means that at any busting, for every little neighborhood of $x$ that geodesic disk hits a point of the surface away from $x$. The bad picture is the picture of a plane tangent to a convex thing.

**Exercise 1** *Prove that each vertex is convex busting.*

The second thing is that if it's convex busting then the cone angle is at least $2\pi$. Here's the proof. Look infinitessimally at a neighborhood of your vertex. You have a finite number of triangles. Look at the tangent bundle. At the level of the tangent bundle the directions, if you look at the unit tangent directions, if you see a triangle of angle $\theta$, then you'll see a a geodesic arc in the tangent bundle of length $\alpha$. So this will give rise to a piecewise geodesic $\alpha$ whose total length in the unit tangent bundle is the cone angle. Note that being not convex busting means that you can find a geodesic in $S^2$ disjoint from $\alpha$.

**Exercise 2** *If $\alpha$ is a piecewise geodesic of length less than $2\pi$ thne there is a geodesic on $S^2$ disjoint from $\alpha$.*

**Theorem 1** *Harer*
*Any two one-vertex triangulations of a surface can be connected by isotopy and the quadrilateral move.*

Therefore, if all the $S_i$ are $\pi_1$-injective, then each $S_n$ can be homotoped to $S_1$ through simplicial hyprebolic surfaces. The topological change in triangulation from one surface to the other, so you slowly change the triangulations one to the other, and eventuall, the moves are realized so that they have the same triangulatian. You make the preferred edges the same as one another. At that point you've done it, I mean, right, when they have the same preferred edge and the same triangulation, you have two surfaces where the one-skeleton agrees on the nose, then they agree and you're done.

Consider a one-vertex triangulation where an edge maps to a nullhomotopic curve. Then you can transform the original triangulation to one where one of the edges changes to something nullhomotopic by quad moves. If you do this with an essential curve, eventually you are doing something not geometric. Eventually you're realizing this homotopy, the contraction of some loop to nothing.

So you start with some surface and do these transfarmations and eventually realize a nullhomotopy. Since this is $\pi_1$ injective outside the core, at some point you have to hit the core. So notice that, again, this proves step one.

That was that there was a compact set so that every one of the surfaces could be homotoped into that set without hitting the core.

Suppose these surfaces have curves which are homotopically trivial. Then you could do homotopy interpolations until you hit the core. But bounded diameter says that every compact set sits in a bounded distance modulo Margulis tubes (from a point?)

Anyway, if you do all these hyperbolic geometry exercises to prove step one, you'll really learn some hyperbolic geometry.

I want to show how you can use these homotopies to produce a proper map of $S \times I$ into the manifold.

So now, we're just starting anew. We have these surfaces which can be homotoped into a compact set containing the core. So how do we construct the proper map?

**Theorem 2** *Finiteness of Simplicial hyperbolic surfaces (Thurston, Souto)*
*Fix $\epsilon, \delta, K$. Look at all the hyperbolic surfaces that hit that compact set. Then the claim is that up to $\epsilon$-homotopy, there are only finitely many such surfaces. The $\delta$ condition is that $\delta$-short curves are essential. Given that and the compact set there are only finitely many homotopy classes.*

What does $\delta$ do for us? The idea we've used several times is that you'll either get something embedded or a short curve. If you have a short loop then, look at $K$. At any point, it has a little ball. The injectivity radius of the manifold is bounded below when restricted to $X$. Since this is $\pi_1$ injective on short loops, then you can't have loops that are too short in the surface (which correspond to short loops in the manifold) or the curve is homotopically trivial in the surface. It's living in a region with a lower bound on injectivity radius. So you can find a triangulation with the proper number of vertices, and the diameter is bounded. You do a tiny homotopy to the original surface and do a tiny homotopy and get small triangles and fewer vertices.

A little compactness argument then proves the theorem. There are only finitely many combinatorial types of the corresponding triangulation. The vertices are all lying in a compact set. A compactness argument then implies the bound.

If I know a vertex moves a bit, then the corresponding edge moves just a little bit.

I mean to say that there are a finite number of surfaces such that any other surface is within an $\epsilon/10$ homotopy of one of them.

The goal was to produce a proper map of $S \times \mathbb{R}$ into the manifold. The way to do that is to put together teh vorious surface interpolations. We can bring this to lie near the core. Now on the other hand let's just consider other surfaces going off to infinity. These give a sense of distance in the manifold. We have random surfaces, and now assume that you have this path. So you can ask the first time it hits $Y_3$, that's $S_3$, and so on. So you have the original set of surfaces. You look ath the first time these hit $Y_1$. This lets us assume that all the surfaces are, we pass to a subsequence and any two are $\epsilon$- homotopic to one another. I pass to a subsequence so that $Y_1$ is in the same class. So now I look at the orignial set of surfaces and then a another subsuquence that has baatem index.

Along all of these surfaces there is an infinite subset where they all live in the same homotopy class when they hit $Y_2$. Now I look at the homotopy class here, this guy faollowed a path from this time. Since this was the first hit for these guys forever more.

I'm not sure how well I explained this point but it's probably elementary.

Everything I say today is pretty foundational. That proves step two.

So in the proof of the tameness criterion, we've reached the second step. The map is proper by construction, and the first of these maps satisfied the other two properties.

Now by pure topology, we can show the volume proposition, whihc is that if you have a proper map which is $\pi_1$ injective and homologically seperate at each level, then it's a homoemorphism onto its image.

Unfortunately I'm running out of time, but at least let me tell you this basic lemma. Here's a picture,

**Lemma 1** *If $K$ sits in $N$ and no component of $K$ lies in a $B^2$.*

*Then if $T \subset N - K$ is a closed surface, $\pi_1$-injective on $N - X$, and there is a map $f : M^3 \to N - K$, with $f^{-1}(T) = R$ with $D$ a compressing disk for $R$. Then $f \cong g$ such that $g^{-1}(T) = R$ compressed along $D$ via homotopy $E : M \times I \to N - K$ supported near $D$. Somehow, using that you can homotope the map to realize that compression.*

**Exercise 3** *Prove this.*

Using this lemma we'll see how to deform our proper map into an embedding. That's all I have time for for today.

# 2 Giroux

The title is symplectic mapping classes and fillings. It's a work in process with Paul Biran. This is a talk in high dimensional symplectic topology. For me the manifolds are at least four dimensional. The object I am interested in is what I call the symplectic mapping class group. Consider a symplectic manifold of dimension $2n$, which I call $(F, n)$. I can look at the group of symplectomorphisms, and be interested in the subgroup consisting of the symplectomorphisms that are the identity near the boundary and near infinity.

I want to look at the components of this thing. I want to discuss how to discuss non-trivial elements. I want to discuss whether it is finitely generated. For high dimensional manifolds, in the smooth case these are infinitely generated. And what is the image of this group within the smooth mapping class group? Can something be isotopic to the identity but not by symplectomorphisms.

This has been studied by Seidel. Consider for $F$ the unit ball contangent bundle of $S^n$. Inside of this you can construct a sypmlectomorphism which is the identity near the boundary called the symplectic Dehn twist. You compose the time $\pi$ map of the geodesic flow of $S^n$. Then you apply the differential of the antipodal map. These sypmlectomorphisms coincide on $\delta F$.

**Theorem 3** *Seidel*
*The mapping class of this map $\tau$ is of infinite order and in the case $n = 2$ generates the whole group $MCG(F, \omega)$. In dimensions two and six this has dimension 2 in $MCG(F)$.*

The kind of diffeomorphism I'll be interested in is slightly different. Call it a fibered Dehn twist. Let $X$ be a closed manifold with a circle action and an invariant contact form. Then $X$ is a circle bundle over the quotient. You have an annulus bundle over the quotient by $K$ when you cross with the interval. Take a Dehn twist on each annulus fiber.

$\sigma_u(t,p) = (t, p + u(t))$ where $u(0) = 0,\ u(1) = 2\pi$, and $u' \leq 0$. The mapping class of $\sigma_i$ does not depend on $u$. The square of the Dehn twist $\tau$ is the twist $u$. The symplectic Dehn twist is the composition of two maps, which commute. Fiberwise you will get, if you say the sphere is a circle bundle, then in the complement of this zero section you get this Dehn twist.

Before stating the results I want to prove I need to say the kind of symplectic manifold I am interested in. We call a campact symplectic manifold with boundary a Weinstein homain, so that there is a primitive $\lambda$ of $\omega$ and a positive proper Morse function $\varphi$, such that the one-form $\lambda$ induces a positive contact form on each level set of $\varphi$ away from critical points, and $\delta F$ is a level set.

**Theorem 4** *Lefschetz*
*The critical ponits have index at most end.*

We call it subcritical if all the critical points have index at most $n-1$. Now given a contact manifold, we call it (subcritically) Weinstein fillable if it is the boundary of a (subcritical) Weinstein domain.

This is one piece that I will need. The second thing I will need is the notion of symplectic hyperplane sections. Consider a closed sypmlectic manifold for which the cohomology class of $\omega$ lies in the integral lattice of the second cohomology group. We say that a codimension two symplectic submanifold is a hyperplane section $\Sigma$ of degree $k$ such that $[\Sigma]$ is Poincaré dual to $k[\omega]$, and the complement of a standard tubular neighborhood is a Weinstein domain, where the tubular neighborhood is a sypmlectic disk bundle.

See that the boundary of $F$ is a circle bundle so admits a free action. What is a typical example of this kind of hyperplane section. If you intersect a projective manifold with a hyperplane, you get a symplectic hyperplane section of degree one.

**Theorem 5** *Donaldson*
*for $k$ large enough there exist symplectic hyperplane sections of degree $k$ in $M$.*

Now, the result I want to discuss. Let $\pi_2$ be trivial, $M$ a closed integral symplectic manifold, $\Sigma$ a hyperplane section, $F$ the complement of a tubular neighborhood.

**Theorem 6** *If $\pi_2(M) = 0$ then the symlpectic mapping class of the fibered Dehn twist along $\delta F$ is non-trivial.*

**Theorem 7** *Now assume that $L$ is a Lagrangian submanifold which is simply connected, and assume that $M$ is monoton and its minimial Chern number is at least $(n+2)/2$. Then the fibered Dehn twist is again nontrivial in $MCG(F,\omega)$*

If you take the quadric in $\mathbb{CP}^{n+1}$, this is $M$ with the $z_0 = 0$ hyperplane section, and $M \backslash \Sigma \cong T^* S^n$ while the minimal Chern number is $N = n$ and $N \geq (n+2)/2$ provided $n \geq 2$.

I want to give a method to study things like whether this is topologically trivial.

[Do you know anything about their order?]

They are infinite order.

An open book on a clased manifold $V$ is a closed manifold $K$ of codimension two with trivial normal bundle, and a fibration $\theta : V \backslash K \to S^1$ which in a neighborhood $D^2 \times K$ of $K = \{0\} \times K$ is the normal angular coordinate.

You can build manifolds via open books. If $F$ is a compact manifold with boundary $K$ and $\phi$ is a self-diffeomorphism of $F$ which is the identity near $K$, then you can create a canonical open book for $MT(F, \pi) \cup_\delta D^2 \times K$.

The very basic observation I will use is that if the diffeomorphism is the identity then the manifold is the boundary of $F \times D^2$.

So this construction has a symplecticaal contact counterpart, that if you have, there is a relation between contact structures and open books on manifolds. If you have a contact structure on a closed manifold, you will say that this is supported ore carried by an open book $(K, \theta)$ if it is defined by a one-form $\alpha$ such that $\alpha$ induces a positive contact structure on $K$ and $d\alpha$ induces a positive symplectic form on each fiber.

To each contact structure you can associate many open books, but the part that's improtant to me is the construction of manifolds by open books via Weinstein domains $OB(F, \phi)$ where $\pi$ is a self-symplectomorphism of $F$ which is the identity near $K$.

The closed manifold $OB(F, \phi)$ admits a contact structure supported by the obvious open book and any closed contact manifold with a carrying open book is obtained in this way.

How will I use this? Let me give an example which is the main example I will need. Consider the following. Take an integral symplectic manifold and inside it you have a symplectic hyperplane section $\Sigma$ and the complement is a Weinstein domain. You also take the circle bundle $V \to M$ which has Euler class $[\omega]$. I think all of the signs are wrong in this. Up to sign this is correct.

So you take this circle bundle over $M$, and then the manifold $V$ admits a contact structure wihich is the kernel whose differential is $\pi^*(\omega)$.

The binding is the preimage of $\Sigma$ and $\theta$ is the map to $S^1$ which trivializes outside $\Sigma$. The pages are copies of the Weinstein domains downstairs, and the monodromy is the fibered Dehn twist along $\delta F$.

Consider $E$ the Dermitian line bundle associated with $\pi$ and let $s$ be a section vanishing transversally along $\Sigma$.

Clearly $s/|s|$ trivializes $\pi$ over $M\backslash\Sigma$, and $d\alpha$ induces a sypmlectic form on each section $us(M\backslash\Sigma)$. Here $u \in S^1$.

Then $K$, the preimage of $\Sigma$ is a contact submanifold and $\nu(K)/\pi^*\nu\Sigma = \pi^*(e|_\Sigma)$ is trivial since $\pi^*(E)$ is trivial. In a split neighborhood of $k$, the circle action takes $(p, z)$ to $(u \cdot p, uv)$ so is generated b4 $\partial_\theta + \eta$ where $\eta$ is the vector field of $\alpha|_K$.

Then vector field $\partial_\theta + f(r)\eta$ gives a fibered Dehn twist as the monodromy.

**Theorem 8** *In either of the folllowing cases, $\pi : V \to M$ the contact circle bundle with Euler class $[\Sigma]$ has no subcritical Weinstein fillings:*

1. $\pi_2(M) = 0$
2. *$M$ is monotone, contains a simpley connected Lagrangian, and its minimal Chern number is at least $n + 2/2$.*

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Here is a start of a proof. Let $W$ be a Weinstein filling and $\bar{L} = \{p, q : p = \pi(q)\}$. Yor appropriate choic of contact form on $V$ teh manifold $\bar{L}$ is Lagrangian. On the other hand, $\pi_2(W, V) = 0$ so $\pi_2(M, \times W, \bar{L}) = \pi_2(M \times V, \bar{L}) = \pi_2(M \times M, \Delta = \pi_2(M) = 9$.

But if $W$ is subcritical, Gromov, Biran-Chelibak yields a holomorphic disk with boundary on $\bar{L}$.

[I missed the discussion of case two]

The general situation I want to look at is the following. You start with a Weinstein domain, and then its contact boundary. I denote by $F$ the completion of the domain $F_0$ with $K \times [0, \infty)$ with $\lambda$ extended with $e^t\alpha$. In this way I get a so-called complete Weinstein manifold. This is endowed, again, with a sypmlectic form which is obtained by gluing the sypmplectic forms together. You have a notion of an exact sypmlectic diffeomorphism where the pullback of $\lambda$, if $\phi^*\lambda - \lambda$ is the differential of a compactly supported function. Any exact symplectomorphism of $F$ induces a contactomorphism of $K$.

A very basic observation is that the restriction map $Diff(F, \lambda/d) \to Diff(K, \xi)$ is a Serre fibration so that there is a long exact sequence $\pi_k Diff_C(P, \omega) \to \pi_k(Diff(F, \lambda/d) \to \pi_k Diff(K, \xi) \to \pi_{k-!} Diff_C(F, \omega)$.

You can take any loop to look at its image. I don't know any examples that are not from circle actions.

# 3 Gordon

Recall, $M$ is a hyperbolic 3-manifold, $T_0$ is a torus component of the boundary, and here hyperbolic means there's no essential $S^2$, $D^2$, $A^2$, $T^2$. We're trying to understand the exceptions to this general rule. Now $M(\alpha)$ is usually hyperbolic, and you want to look at the ones that aren't. The general program is to, well, say ${}_1$ and ${}_2$ are general classes of nonhyperbolic three-manifolds. For example you might choose $S^3$, or lens spaces, or manifolds containing essential tori, or connect sums, or Seifert fibered spaces, or $S^2 \times S^1$. The idea, then, would be, to show that if $M$ is hyperbolic and $M(\alpha_i) \in {}_i$, then $\Delta(\alpha_1, \alpha_2) \leq \Delta_0 = \Delta_0({}_1, {}_2)$. For instance, we did this for $S^3$ and lens space surgeries.

After doing this, you want to try to classify the triples $M; \alpha_1, \alpha_2$ that realize that $\Delta_0$.

Really you want to try to classify all hyperbolic manifolds that can do these things. But if $\Delta_0$ is high, as you get further from the bound it becomes less likely that you can do it, it starts to depend on whether you have a life or not.

Let me summarize what is known about the first part. I'll do this in the form of a table.

| | $S^2$ | $D^2$ | $A^2$ | $T^2$ | $S^3$ | $L(p,q)$ |
|---|---|---|---|---|---|---|
| $S^2$ | 1 | 0 | 2 | 3 | ? | 1 |
| $D^2$ | | 1 | 2 | 2 | $*$ | $*$ |
| $A^2$ | | | 5 | 5 | $*$ | $*$ |
| $T^2$ | | | | 8 | 2 | ? |
| $S^3$ | | | | | 0 | 1 |
| $L(p,q)$ | | | | | | 1 |

Here the $*$ entries are irrelevant. The only ones that are not known are the question marks. What is th4 $S^3$ and $S^2$ one about? That's the cabling conjecture. This should be $-\infty$. This is the cabling conjecture. That's an obvious sort of gap in our knowledge. If you have a toroidal filling and a lens space filling, the gap is either 3 or 4. There are examples that realize three and the gap is known to be at most four. Since 4 isn't a Fibonacci number the answer is almost certainly no, but that's not really a proof.

So one thing left out of this table is, what about Seifert fibered spaces of type $S^2(q_1, q_2, q_3)$? Not much is known about them.

This is the obvious gap in our knowledge. If I have time I'll try to make a few remarks about how you prove these things. There are only two entries left to be filled in here.

What about the second part of the program. Let me write the same classes. This table hasn't actually changed for a long time. I mean, this table, I'm afraid, there hasn't been very much progress on it. The only way to make progress is by proving or disproving the cabling conjecture. It might be possible to prove that $\Delta(T, L)$ really is three. There has been a fair amount of progress on the second part here. Now we want to identify the manifolds that realize the maximal values. Most examples for $S^2, S^2$, use tangles, but there is no conjecture, even.

| | $S^2$ | $D^2$ | $A^2$ | $T^2$ | $S^3$ | $L(p,q)$ |
|---|---|---|---|---|---|---|
| $S^2$ | ? | $\surd$ | $\surd$ | $\surd$ | ? | ? |
| $D^2$ | | ? | $\surd$ | $\surd$ | $*$ | $*$ |
| $A^2$ | | | $\surd$ | $\surd$ | $*$ | $*$ |
| $T^2$ | | | | ? | $\surd$ | ? |
| $S^3$ | | | | | $\surd$ | ? |
| $L(p,q)$ | | | | | | ? |

I was once giving a talk at Texas A&M and someone asked me about the lower half of the table.

So let me make some comments, what about $D, D$, where the two fillings have a distance 1 and both are boundary irreducible. If both of them are $S^1 \times D^2$, this is known, this was Gabai, Berge, and I talked about when you do Dehn surgery on a satellite knot. So this is known. It's not so bad as it looks.

Let me go to some of the others, here's another case we've really met, where one gives us $S^3$ and the other filling at a distance of 2 gives you a toroidal manifold. Those are the Eudave-Muñoz knots. That's an example of that kind of complete classification. The $S^3, L$ case is precisely the Berge conjecture. They should be precisely the Berge knots. Let me make a comment on one more entry. You can sometimes to better than just at the maximum value. The maximal value is 8 and there are two triples which give this maximal distance. One is the figure eight exterior and the other is the figure eight sister. In fact, this is now known completely for toroidal fillings down to distance four. This was recent work with Wu.

For knots in $S^3$, the figure eight and the $(-2, 3, 7)$ pretzel seemed special.

**Corollary 2** *$X$ a hyperbolic knot in $S^3$ has two distinct pairs of toroidal surgeries at distances $\geq 4$ if and only if $K$ is the figure eight or the $(-2, 3, 7)$ pretzel knot. The $(-2, 3, 7)$ had toroidal fillings at $(1/0), (2/3)$ and $(2/5)$ at distances $3, 4, 5$.*

Let me be more specific with respect to a couple of these entries, give more examples in this more general context.

So here, remember our famous tangle $\mathscr{B}_n(-2, n+2, 1/n)$ for $n \geq 2$. This is Eudave-Muñoz-Wu looking at these guys.

We did this in general, if you did $1/0$ filling you have this four punctured sphere which gives rise to a torus, so if you let $M_n = \tilde{B}_n$, then, of course you have to worry that these are nondegenerate, but they are, so that $M_n(1/0)$ is toroidal. It's a union, as always, of two Seifert fibered spaces with two exceptional fibers. There's a rather clever filling which makes use of these particular values. Let me draw the picture again. This time we're going to do the $1/3$ filling. You stare at that for a while and realize how clever this is. You have $n+2$ positive twists, horizontally, and then $n$ negative vertical twists, There's a band that connect them to one another. So whatever you get is independent of $n$. So, are you happy with this picture? I'll leave it, let's see what you get when $n$ is zero. So if you throw in $n = 0$ you get this picture.

Flip this guy around and you get this, so moving things around you get the trefoil connect sum the Hopf link. So here the two pieces are $L(3,1)$ and $\mathbb{RP}^3$. So $M_n(1/3)$ is reducible, and is always $L(3,1)\#\mathbb{RP}^3$.

**Theorem 9** *Kang, 2006*
*$M$ is hyperbolic, $M(\alpha_1)$ is reducible, and $M(\alpha_2)$ is toroidal with $\Delta(\alpha_1, \alpha_2) = 3$ if and only if $(M; \alpha_1, \alpha_2) \cong (M_n; 1/3, 1/0)$.*

So let's do an annular one. We'll need two boundary components. So remove $\gamma$ and call the resulting tangle $\mathscr{A}(\alpha, \beta)$, then $\mathscr{A}(\alpha, \beta, 1/2)$, forgetting about the markings, we can swing one arc through the other and get this, as an unmarked tangle. Now we see, if you turn it on its side you're going to see, you now get a disk branched over two points, and then $\alpha, \beta$ are not integers. One of the two is a solid torus, but this annulus is a $2, 1$ annulus, so it's not boundary parallel. The moral is that $\tilde{A}(\alpha, \beta)(1/2)$. We want to get another interesting filling, but if what we do now is take $\mathscr{A}_n$ to be $\mathscr{A}(1/n, -1/n)$. This looks like this picture. Now I'll do the $1/0$ filling. Now the tangle splits. There's an essential sphere, and if $N_n = \tilde{\mathscr{A}_n}$. Then $N_n(1/2)$ is annular and $N_n(1/0)$ is a connected sum of a solid torus and $\mathbb{RP}^3$. The table I erased established that this is the maximal distance.

**Theorem 10** *Lee, 2005, for $M$ hyperbolic, $M(\alpha_1)$ reducible and $M(\alpha_2)$ is annular with $\Delta(\alpha_1, \alpha_2) = 2$ then $(M; \alpha_1, \alpha_2) \cong (N_n; 1/0, 1/2)$ for $n \geq 3$.*

**Exercise 4** *Show that $N_2$ is not hyperbolic.*

Let me say a few words about how someone proves a theorem like this. The first thing is, you have the hyperbolic manifold $M$, and you assume that these contain $\hat{F}_i$, a surface that is essential or Heegaard, and you add the solud torus that does the Dehn filling, and you make $\hat{F}_i$, which intersects $V_i$ in some collection $n_i$ of meridian disks.

So here's $M(\alpha_i)$, and you see the solid torus puncturing the surface in some number of meridian disks. Notice that $F_i$ is sitting in $M$, where that's just $\hat{F}_i \cap M$. So the boundary is a $n_i$ of simple closed curves of slope $\alpha_i$. Then you have this other surface. In this picture the boundary components on the other surface go around like this. The boundary components of, like, $F_1$, are like $\alpha_1$. Then there's this other slope $\alpha_2$, so you have $n_2$ parallel copies of these guys. So you notice that $\delta$ shows up, and the component of $\delta F_1$ intersects $\delta F_2$ in $\Delta(\alpha_1, \alpha_2)$ points. So now you know the intersections on $T_0$ and put these in general position, and get arcs in $F_1 \cap F_2$, which gives rise to graphs sitting in $\hat{F}_i$. So maybe $\hat{F}_1$ was a two-sphere. Then your knot in $S^3$ was puncturing this like two times. Your other surface might be a torus, say, and again, over there the core of the surgery curve will puncture this some number of times, and you get these arcs. So it's natural to think of these as graphs. Every boundary component in one hits every one on the other some certain number of times. Usually you can then argue that for a given type of surface, if this $\Delta$ becomes very large, then that's going to force more and more topologically parallel edges, which eventually contradicts the assumption that $M$ was hyperbolic. So a combinatorial analysis of the graphs shows that

$\Delta \leq \Delta_0$. So you tend to get these upper bounds. Maybe this is the first thing you get. The other thing is, when you're at the critical value, then you can usually show that one of these numbers of intersection has to be pretty small. Then you can start really using the combinatorics to get a clear topological picture, to build using the faces of these graphs, and you can accumulate information about these and eventually identify these things in enough detail to say what $M$ actually is.

One more thing, let me say, you have to get back to tangles somehow. For example, when $M(\alpha_1) = S^3$ and $M(\alpha_2)$ was toroidal, and $\Delta = 2$, which was the Eudave-Muñoz knots, how do you get to that point. Summarize Proust, you remember that? No Monty Python fans? Lost on you. So what happens, one first shows that $\Delta \leq 2$, and then shows that if $\Delta = 2$ then $n_2 = 2$. You have a specific graph on the torus, and you identify that $M(\alpha_2)$ is the union of two Seifert fibered spaces over the disk stuck together along $T$ just by studying the graphs and you find that the fibers of the two sides intersect once on the torus.

So you have two Seifert fibered spaces where the fibers intersect once. So drawing neighborhoods of the fibers, you got two copies of a Hopf link. If you drill out the cores, what's left is just a sort of product. But the cores intersect once. It's a Hopf link with an extra component. So $M(\alpha_2)$ minus neighborhoods of the exceptional fibers is a neighborhood of this link. So you again go back and look at these graphs and find that you can identify what the core of the knot looks like. It looks like this, and I'll be done pretty soon. It lies quite simply with respect to this picture. This makes sense because this knot intersects the annulus exactly once, so that the curve intersects the torus twice. Now we can conclude that $M$, the manifold you're interested in, is what you get by doing some Dehn surgery on these four components. Now what you find is that this link is strongly invertible, so there's an involution in which each of these five components is just rotated, and you find that the quotient is a very special tangle that John Conway calls the pentangle, where you have this very nice thing, with certain rational tangles filled in here, which tell you how to put in the exceptional fibers. So this manifold is of this form. Here these $\alpha, \beta, \gamma, \delta$ correspond to these exceptional fibers. The $\delta$ has to be $-1/2$. So this, with, and now you just flip this over here, and change the rational tangle by flipping this over here, and I hope you recognize this. This is $\mathscr{B}(\alpha', \beta', \gamma')$. So this has to come from a filling on these guys. You also have to know, there's a filling that gives you $S^3$, and you have to go back to the combinatorics and identify the filling, which is $1/2$, so that you have to solve the problem of what goes to the unknot. I'm sorry that this is extremely sketchy, but you look at intersections of surfaces and get bounds on the intersection number. You look more carefully and get small numbers of punctures, and it's Dehn surgery on a strongly invertible link, and roughly speaking, that's how the other two theorems are proved as well. I'm sorry I've taken too long, I'll stop now.

Strongly invertible means that each component of the link meets the fixed point set of a particular involution in two points.

# 4 Etnyre

**Theorem 11** *(Eliashberg-Thurston)*
*Any $C^2$-foliation $\xi$ on an oriented closed 3-manifold, ather than $\zeta$ on $S^1 \times S^2$, may be $C^0$-approximated by a positive and negative contact structure.*

The talks from yesterday, we kind of ended them with a discussion of this theorem. To recap, this says how to perturb foliations into contact structures. First of all, it needs to be $C^2$. Other than this strange foliation of $S^1 \times S^2$ where it's just the tangents to the $S^2$ direction, you can approximate them with contact structures. It would be great to replace $C^0$ with $C^2$ and approximated with deformed, and that might be true. There are no counterexamples.

So let's look at this weird foliation.

**Theorem 12** *Reeb stability for confoliations*
*Suppose a confoliation $\xi$ on $M$ admits an integral $S^2$. This means there's an $S^2$ embedded in $M$ such that $S^2$ is always tangent to the planes. That means for all $x \in S^2, T_xS^2 = \xi_x$. Then $(M, \xi)$ is diffeomorphic to $(S^1 \times S^2, \zeta)$.*

Even if you thought you started with a contact structure, you were wrong, it was integral on every slice, and it was a foliation.

This is well-known in the context of foliations. I'll leave it as a challenging exercise:

**Exercise 5** *Try to show this if $\xi$ is a foliation*

Two hints. Show the region of $M$ foliated by $S^2$s is open and closed in $M$. Also pay attention for the rest of the lecture.

[That will never work.]

[Is that just the same Reeb or is it connected to Reeb foliations?]

It's just the same guy.

Okay. So now

**Theorem 13** *Any confoliation of $S^1 \times S^2$ that is $C^0$-close to $\zeta$ is diffeomorphic to $\zeta$.*

This means I want to be able to fiber by $S^1$ so that the planes are never tangent to the $S^1$ factor. That's probably a better way to state it.

So these two theorems kind of explain the unique nature of this foliation, and say why we have the exception, but they also point in the right direction to proving the theorem.

**Theorem 14** *Let $\xi$ be a confoliation on $B^3$ which is standard near $\delta B^3$, meaning it's given by* ker $dz$. *Then $\xi$ is a foliation and diffeomorphic to the standard foliation.*

This will follow easily from the Reeb stability theorem. This tells me how not to prove the approximation theorem. I could try to perturb in a neighborhood in a point, but that won't let me change it into a contact structure. It makes sense, you'd have to sort of twist and then untwist, and since it's a confoliation it would have to be either always positive or always negative, so you shouldn't be able to do that.

*Proof.* Suppose we're given $(B, \xi)$. Let's look at $S^1 \times S^2$. Let's embed a little ball, and this has a standard foliation. Then I replace this ball with the new confoliation which is standard near the boundary. Okay, so suppose we're given the confoliation $\xi$ stated in the theorem. Embed $B$ into $S^1 \times S^2$ so that $\zeta|_B$ is standard, and then replace the $\zeta$ with $\xi$. Then this is a confoliation of $S^1 \times S^2$ with an integral leaf. Then Reeb stability tells me that this is a new confoliation which is just $(S^1 \times S^2, \zeta)$.

[Can this be done by a diffeomorphism that is relative to the boundary?]

I don't know, I haven't thought about it. Other questions? Okay, now that we've seen how not to prove the theorem, let's see how to prove the theorem. The proof then involves two steps. What are the steps? Step one will probably be tomorrow. It will be to perturb $\xi$ into a confoliation $\xi'$ such that $\xi'$ is contact on a "sufficiently large" portion of $M$. Then step two is to perturb $\xi'$ into a contact structure. When I state the steps this way it sounds stupid, "start making it a contact structure" and then "make it a contact structure." But it will be helpful. We'll start with step two and that will help us to figure out what sufficiently large means in step one.

So what exactly do we need. How much actually has to be a contact structure before it's easy to make it a contact structure everywhere. Okay, so given a confoliation $\xi'$, let $H(\xi')$ be $\{x \in M$ such that $\xi_x$ is a contact structure, that is, $\alpha_x \wedge d\alpha_x > 0\}$. This is called the hot zone or contact zone $H(\xi)$. You'll see why the terminology in the moment. If you study contact structures, then foliations are cold and hard, and contact regions are warm and nicer.

Let $G(\xi') = \{x \in M$ such that there exists a path $\gamma$ in $M$ from $x$ to $y$ such that $y \in H(\xi')$ and $\gamma$ is tangent to $\xi'\}$.

**Theorem 15** *If $G(\xi') = M$ then we can $C^\infty$-deform $\xi'$ into a contact structure. So this is as nice as you could possibly hope for.*

*Proof.* There are actually two ways to prove this. There's an analytic way due to Altschuler and a topological way due to Eliashberg and Thurston. I want to give you both of them. I like them both and also I've been doing a bad job giving you proofs so far, and now I hope to make up for past transgressions.

So if we choose a Riemannian metric on $M$ and a one-form $\alpha$ such that the kernel of $\alpha$ is $\xi'$, and $|\alpha| = 1$. So a metric on the tangent space gives me a metric on the cotangent space. I can then just demand that the length be 1 and if it's not, divide by something. It's easy to arrange this. So that's the setup. What's the current way to make things nice nowadays? You look at heat flow. Look at the following equations:

$$\frac{\partial}{\partial t} = \star(\alpha \wedge df)$$

$$\beta(\cdot, 0) = \alpha(\cdot)$$

where $f = \star(\alpha \wedge d\beta + \beta \wedge dx$ and we try to solve for $\beta \in \Omega^1(M) \times \mathbb{R}_+$ (the $t$ factor).

I should have said, let's assume the confoliation is oriented. You can deal with the nonoriented case, but I'll just do this version of it.

This is a weakly parabolic equation. Altschuler proved that there exists a unique solution for $t \in [0, \infty)$. You can ask how does $f$ change with respect to time. It evolves by the equation

$$\frac{\partial f}{\partial t} = \Delta_\alpha f + \nabla_X f.$$

This $\Delta_\alpha$ is the Laplacian in the direction only of the tangent plane. Here $X$ is a time-dependent vector field, if you have a solution, you write down what $f$ has to evolve by, and that's what $X$ has to be.

I want to state some properties of $f$. A version of the maximum principle gives, notice that $f$ is a function on the manifold, but also depends on $t$. So if $f(p, 0) > 0$ then $f(p, t) > 0$ for all time. If $q$ is a point connected to a point $p$ in the hot zone by a patch tangent to $\xi'$, then $f(q, t) > 0$ for all $t > 0$. This is very common. Back in your first PDE course ever, you saw that heat flows infinitely fast to all points of the manifold. This is kind of the analogous statement for this setting.

Since $G(\xi') = M$, then $f(p, t) > 0$ for all $p \in M, t > 0$. So the real question is why do we care? Let's consider $\eta = \alpha + \epsilon\beta_1$, where $\beta_1(\cdot) = \beta(\cdot, 1)$. Then $d\eta = d\alpha + \epsilon d\beta_1$.

[How do you see that $f$ is positive?]

You get $\alpha \wedge d\alpha$, which is nonnegative.

Okay, so

$$\eta \wedge d\eta = \underbrace{\alpha \wedge d\alpha}_{\geq 0 \text{ as } \alpha \text{ is a confoliation}} + \underbrace{\epsilon(\alpha \wedge d\beta_1 + \beta_1 \wedge d\alpha)}_{*f \geq 0} + \underbrace{\epsilon^2 \beta_1 \wedge d\beta_1}_{\text{contains } \epsilon^2}$$

So for $\epsilon$ sufficiently small, $\eta$ is a contact form.

The Hodge star is just the natural duality between $k$-forms and $n-k$-forms on an $n$-manifold.

Okay. Now let's do this the topological way. First let's consider a region and let me just draw the picture here, and fill things in, we have this square region, and we have $-1$ to $1$ in the $x$ and $z$ directions and $0$ to $1$ in the $y$ direction. So this is $V$. Now suppose in $V$ we have $\alpha = dz - a(x, y, z)dx$. You could locally find coordinates like this. Another part of this exercise, was to show that this was a contact structure (positive) if $\frac{\partial a}{\partial y}$ was positive and a foliation if $\frac{\partial a}{\partial y} = 0$.

Assume $\xi'$ is a contact structure near $\{y = 1\}$. We want to kind of propagate it. How do we do that? It's actually quite simple. Since $\xi'$ is a confoliation we know that $\frac{\partial a}{\partial y} \geq 0$, and it's strictly positive near $y = 1$.

I think everybody knows the proof now. For a fixed $(x_0, y_0)$, what does $a(x_0, y_0, z_0)$ look like? It's flat, or increasing until it gets near 1 and then it's strictly increasing. So now you can make it always increasing by a small perturbation; replace $a(x_0, y_0, z_0)$ with a strictly increasing function without moving it at the boundary.

**Exercise 6** *Make sure you can do this in a family.*

It's not that hard to do. This tells us that we can certainly extend the contact zone. So now we need to make sure we can cover our manifold with regions like that. Pick $\gamma_1, \ldots, \gamma_n$ such that $\gamma_i$ are tangent to $\xi'$, the $\gamma_i$ start in the hot zone, and each $\gamma_i$ has a neighborhood as above, and finally, $V_i$ cover $M \backslash H(\xi')$.

**Exercise 7** *What happens when $V_1$ intersects $V_2$? Convince yourself you can perturb $\alpha$ with something arbitrarily small, I can fix $V_1$ without messing up $V_2$.*

Essentially what will happen is that you can introduce some bumpy regions. It takes a little bit of work, but it's not too hard.

As in a lot of topological proofs, you have this annoying thing where you have to show that fixing one part doesn't mess up the other part.

So now wo know in step one of the proof that we need to arrange that $G(\xi') = M$, that is, every point can be connected to the hot zone by a path tangent to $\xi'$. So that's what sufficiently large means. So now we start step one.

Since this part of the proof has provided a really nice deformation, in this step we'll have to do very violent things to our foliation.

We need a new idea from foliation theory, and that's the following.

**Definition 3** *The holonomy of a foliation is the following. Let $\gamma$ be a closed curve tangent to the foliation $\xi$. Let $A = (-\epsilon, \epsilon) \times \gamma$ be an annulus $\pitchfork$ to $\xi$ such that $\{0\} \times \gamma = \gamma$. Then $\xi$ induces a foliation on $A$.*

*So you get a line field on the annulus. You think of the intersection as being a line field in $A$. Just take a vector that points along the lines everywhere, solve an ODE, draw lines along the trajectories and that's what you get.*

*Pick a point $p$ in $\gamma$ and use the foliation on $A$ to define a map on $I = (-\epsilon, \epsilon) \times \{p\}$.*

*What do I mean by this? If I follow along the foliation, I can flow along the annulus. I might flow off the annulus, but if I'm close enough to* 0 *in the* $(-\epsilon, \epsilon)$ *coordinate you'll flow back around the circle of the annulus, then you will flow back to the arc. To go into all of the details, this would be the germ of a map at zero, or even more precisely the conjugacy class of the germ. This will give a map, or something like a map,* $\phi_\gamma : I \to I$, *which is well-defined near zero.*

*Okay. I guess we should finish up here in just a minute or so. Now* $\phi_\gamma$ *is called the holonomy*

This turns out only to depend on the homotopy class of $\gamma$. It does depend on the orientation.

[What about $p$?]

If you tilt this, you change the conjugacy class. As you're going to see tomorrow, we'll pick a particular annulus and work with that, we don't need well-definition. But I told you to pay close attention and maybe you've been paying too close attention. I've been talking about holonomy like this because it will help with proving the Reeb stability theorem.

**Definition 4** *The holonomy is called nontrivial if* $\phi_\gamma \neq id$.
$\gamma$ *has nontrivial linear holonomy if* $\phi'_\gamma \neq 1$.
*Holonomy is attracting if* $|\phi_\gamma(x)| < |x|$.
*Holonomy is repelling if* $|\phi_\gamma(x)| > |x|$.
*Holonomy is called weakly attracting if* $|\phi_\gamma(x)| < |x|$ *for* $x$ *arbitrarily close to zero.*
*Holonomy is called weakly repelling if* $|\phi_\gamma(x)| > |x|$ *for* $x$ *arbitrarily close to zero.*