Low Dimensional Topology Notes
July 5, 2006

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1 Gabai, hyperbolic geometry and three-manifold topology

[Ed.: Uh oh, a projector talk.]

[All right, I'd like to make a few announcements. The problem sessions will be for Gordon in the morning and Etnyre in the afternoon. The research organizers should get together at 12:45, anyone who is a research organizer or wants to be involved in deciding what's going on next week, you should come. If you are in a related area, we'd like to encourage you to help with the problem sessions. It's a pleasure to introduce David Gabai from Princeton.]

Well, thank you Peter, it's a great opportunity to be able to give these lectures. So, the goal of these lectures is to give you a pretty good understanding of the proof of the tame ends conjecture for hyperbolic three-manifolds, due to Ian Agol and Danny Caligari and myself. A hyperbolic manifold with finitely generated fundamental group is topologically tame. It's a fantastic interplay between classical three-manifold topology and hyperbolic geometry.

I want to start with some classical topology. The question of tameness goes back to Whitehead, in 1935. He found an open contractible three-manifold that was not $\mathbb{R}^3$. He was trying to prove the Poincaré conjecture, deleting a point from a homotopy sphere. So it's contractible, has trivial fundamental group, but it's not $\mathbb{R}^3$. You build it up as an increasing union of solid tori, embedded in one another as Whitehead links. So notice that this manifold is certainly contractible. Any compact set lives in one of the solid tori, and you can homotope that in the next solid torus. On the other hand, why isn't this $\mathbb{R}^3$?

Proposition 1 *If C is a smooth compact codimension zero submanifold of $\mathbb{R}^3$ then $\pi_1(\mathbb{R}^3 - C)$ is finitely generated.*

Let $B \subset \mathbb{R}^3$ be a three-ball containing C. Then $B - \circ C$ is a compact manifold with boundary, with finitely generated fundamental group. Then $\mathbb{R}^3 - C$ deformation retracts to this space.

However, if you remove the original solid torus from the Whitehead space, the resulting space has infinitely generated fundamental group. That's an okay exercise.

Definition 1 *Let M be a three-manifold, possibly with boundary. Then M is tame if there exits a compact manifold X and a proper embedding $i : M \hookrightarrow X$ with $X = \overline{i(M)}$. So $M \cong X - Y$ where $Y \subset \delta X$ and Y is compact. If $\delta M = \emptyset$ then $M \cong M_1 \cup S \times [0, \infty)$, a nice collar near the boundary.*

Theorem 1 *Tucker, 1974*
If M^3 is orientable, irreducible, then M is tame if and only if for every nice smoothly embedded compact submanifold (nice meaning that near the boundary, the submanifold looks like a product, then $\pi_1(M - Y)$ is finitely generated.

A smooth submanifold lives in a compact submanifold which the original manifold deformation retracts to, and the complement has finitely generated fundamental group.

The difficulty is showing that having finitely generated fundamental group for the complement of nice compact sets is sufficient.

Let's look at the Fox-Artin manifold. Here $FA = \mathbb{R}^3 - \circ N(K)$ where K is this properly embedded ray in $\mathbb{R}^3$. Then $\delta FA \cong \mathbb{R}^2$, and $\circ FA = \mathbb{R}^3$. So is $(FA, \delta FA)$ tame? Is it the same as the standard half space?

Exercise 1 *Is this manifold tame? The hint is to use the Tucker criterion.*

Here's another example of another non-tame manifold. What this manifold is, it's an open manifold, which is the increasing union of solid handlebodies of genus two. It's homotopy equivalent to a standard handlebody.

Its construction is similar to that of the Whitehead manifold, as $V_1 \subset V_2 \subset \cdots$ The inclusions are all homotopy equivalences. To get the handlebodies you glue these disks to one another. This manifold has fundamental group $\mathbb{Z} * \mathbb{Z}$.

Here's some facts.

1. $\tilde{M} = \mathbb{R}^3$. That's a nice exercise.
2. The preimage of the curve γ drawn here in $\tilde{M}$ is the infinite unlink, but deleting the curve gives
3. $\pi_1(M - \gamma)$ is infinitely generated.

Mike Freedman in the early nineties had an approach to the tame ends conjecture. He expected the preimage to be knotted in the universal cover. We found this example in fall

1996. This was my introduction to the tame ends conjecture, and the argument we had was sort of a long evolution from this.

What I like about three-manifold topology is that it's something that you can really put your hands on. You can really construct these things, and it doesn't matter if this picture is upside down, you get these really concrete things.

There are uncountably many noncompact three-manifolds and they can be very complicated. How do you get a grip on these things?

Theorem 2 *Scott Core theorem*
If M is an irreducible 3-manifold, connected, with finitely generated π_1, then there exists a compact submanifold $C \hookrightarrow M$ whose inclusion is a homotopy equivalence. This is an algebraic fact, that finitely generated fundamental groups are finitely presented.

So the ends of M are in one to on correspondence with the boundary components of the core. That's a nice fact coming from this. So this, if you want a grip on the end of the manifold, you can look at the Scott core and see what's to one particular side.

Now I need to start talking about hyperbolic three manifolds.

Suppose you have N a complete hyperbolic three-manifold, if $\pi_1(N) = 1$ then $N \cong \mathbb{H}^3$. We know that the universal cover is Euclidean, from Cartan-Hadamard. If $\pi_1(N) = \mathbb{Z}$ then $N = \mathbb{H}^3/\langle s \rangle$, namely $\circ D^2 \times S^1$. We know the isometries of hyperbolic space, and just compute the quotient. If you look at the next simplest group, the free group on two generators, that was kind of unresolved until the end.

Theorem 3 *Marden conjecture–Agol, Calegari-Gabai*
If N is a complete hyperbolic 3-manifold and $\pi_1(N)$ is finitely generated then N is topologically tame.

Marden showed this for N geometrically finite. Thurston proved it for the algebraic limit of Fuchsian groups. Bonahan proved it for π_1 freely indecomposable, Souto proved it for another class of things. This uses some of the basic understanding of three-manifold topology, this builds on the work of Canary, Minsky, Kleineidam, Evans, Oshika, Brock, Bromberg, Long, and various others.

The work of Thurston includes an introduction of pleated surfaces. Bonahan has this long and hard paper, I don't understand French very well but there's a lot of good things near the beginning that I was able to understand. Souto uses a bunch of complicated machinery but includes a nice criterion for tameness, working hard in other parts of the mathematical world were these other people on three-manifolds, Brin, Thickston, Myers, a closed irreducible manifold with infinite fundamental group (now thanks to Perelman) its cover is $\mathbb{R}^3$.

Let me mention clearly, if N is a complete hyperbolic three-manifold with finitely generated fundamental group then it's determined up to isometry by the topological type, the conformal boundary of its geometrically finite ends and the ending laminations of its infinite ends. This is kind of fantastic. This is one application.

Here are some references, Agol, "Tameness of hyperbolic three-manifolds," on the ArXiv, and Calegari-Gabai, "Shrinkwrapping and the taming of hyperbolic three-manifolds," on the ArXiv or JAMS, Freedman-Gabai, "Covering a nontaming knot by the unlink" (how not to prove tameness), Soma "Existence of polygonal wrappings in hyperbolic 3-manifolds," Choi, "The PL methods for hyperbolic 3-manifolds to prove tameness" (ArXiv), Bowdich, "Notes on tameness" (Bowditch homepage).

So now I want to go over the outline of the proof. We're going to need a characterization of tameness. There's the Tucker criterion, but I don't know how to apply it. Here's the tameness criterion we'll use. To get to the heart of the matter and simplify notation, I'm going to assume the manifold has a single end. It's $\mathbb{H}^3$ moduli a group of isometries. I'm going also to assume that it doesn't have any parabolic elements. These cusps might force you to think of things as a relative manifold. With some patience you can sort it out, but the basic ideas are in the parabolic free part.

Here's the Scott core, and here's the end. It's tame if I can find a sequence of surfaces $S_1, S_2, \ldots$, mapped into the end, exiting the manifold, with certain properties:

1. Their genus is at most the genus of the boundary component of the Scott core.
2. they are $Cat(-1)$, they have curvature at most -1,
3. S_i homologically separates the core from the end, so that they generate H^2 of the end. A ray exiting the manifold from the core will hit each of these (algebraically) once.

Juan Souto proved the harder theorem that a hyperbolic manifold is tame if it satisfies only the first and third of these. It uses Bonahan, Canary, Gabai, and a lot of hard machinery. But if you include the second condition there's a much easier proof.

Ultimately, if you have these surfaces, the end is topologically tame. Ultimately, these surfaces could be embedded homologically separating surfaces, possibly isotopic to the boundary, a postiori. The conclusion is that this manifold compacifies to a compact manifold, and in particular there's a product structure near the end, it's homeomorphic to the surface cross $[0, \infty)$. It turns out by algebraic topology that to homologically separate, the genus will be the same as that of the boundary component.

So what is $Cat(-1)$? There are basically three types in a hyperbolic 3-manifold:

1. A minimal surface in a hyperbolic three-manifold has mean curvature 0 but its intrinsic curvature is less than or equal to the curvature of the three-manifold.
2. A pleated surface, which is geodesically embedded, but bent along some geodesics.
3. simplicial hyperbolic surfaces, triangulated surfaces S mapping $f : S \to N$ so that each 2-simplex is totall geodesic, and has cone angle at least 2π at each vertex.

I need to tell you something about hyperbolic geometry. First I want to talk about the thick-thin decomposition. There's a Margulis number, and you look at the places where you can put an embedded ϵ-ball around a point. That's the thick part $N_{[\epsilon,\infty)}$. Then you look below the injectivity radius and get thin parts $N_{(0,\epsilon]}$, which are solid geodesic tubes about short geodesic (Margulis tubes). Spots where the injectivity radius goes to zero are in these Margulis tubes.

It's important to keep places where the injectivity radius is small separate because of the complexity.

Lemma 1 *Bounded diameter lemma*
Let S be a $Cat(-1)$ surface in N such that essential curves of length $\leq \delta$ in S are essential in N. Then there exists a C depending on $\xi(S)$ and δ such that the diameter in N of S is at most C (modulo Margulis tubes). So if $x, y \in S - N_{(0,\epsilon]}$ then there exists a path of bounded length from x to y, bounded by this C.

The fact is, if you have a Margulis tube, you might have a tiny geodesic, you have a long thin annulus, but at some point it enters the Margulis tube, which contains most of the length.

The proof of the bounded diameter lemma is very nice. Assume S is a closed simplicial hyperbolic surface and $\delta \leq \epsilon$. If the injectivity radius in S of x is less than $\delta/2$, then the injectivity radius is small, so x is in the thin part.

Since S is intrinsically less than -1 curved, if D is a $\delta/2$ disk in S, then the area of that disk is greater than $\pi(\delta/2)^2$. Then Gauss-Bonnet implies that the area f the surface is at most $2\pi(\xi(S))$. So

Exercise 2 *Put these together to see that $S - N_{(2,\epsilon]}$ can be covered by a finite number of $\delta/2$-balls.*

I need to tell you more about hyperbolic manifolds. I need to tell you about the ends. Here, this is a geometrically finite end. These are extremely nice. It's topologically a product and the cross-sectional area of the sections grows exponentially with the distance of the core. If you think of T a hyperbolic surface, and take $\mathbb{H}^3$ modulo the same group of isometries and you get something that looks like $T \times \mathbb{R}$ topologically. But geometrically finite ends don't satisfy the tameness criterion. You can't have these separating surfaces because they'd get too big, they're in the thick part.

Geometrically infinite ends are everything else. Look at a manifold that fibers over the circle with fiber S then $\hat{N}$ is the infinite cyclic covering space, which is topologically $S \times \mathbb{R}$, and the end, the cross sections will have bounded area, not exponentially increasing area.

Here's a characterization of these by Francis Bonahon. An end is geometrically infinite if there exists a sequence of closed geodesics exiting the end. They can be assumed to be simple.

These are called finite and infinite because you can understand the geometry through the finite part, or can't.

I'm just throwing stuff out, but we're going to be focussing on bits of technology in the future lectures. To give a hint of proving the tame ends conjecture, let me show you a hint of a proof of

Theorem 4 *Dick Canary*
If $\mathscr{E}$ is an end of N and $\mathscr{E}$ is topologically tame, then either $\mathscr{E}$ is geometrically finite or $\mathscr{E}$ satisfies the taming criterion (it's geometrically tame is how it was originally stated)

This involves the full technology of Francis Bonahon. But let's look at a special case, $N = S \times \mathbb{R}$. We have these simple curves exiting the manifold. We can put these surfaces between the geodesics. We want to show that this satisfies the taming criterion, we want to be able to find the surfaces exiting the manifold, $Cat(-1)$ and separating. They satisfy conditions one and three but not condition two. To make them satisfy condition two you shrinkwrap, pull them tight with respect to the geodesics it's supposed to lie between. The shrinkwrapping gives you a surface which has curvature ≤ -1, and bounded diameter, so that like $S_{10,000}$ starts by lying between two geodesics, if you draw a path from one to the next it has algebraic intersection number one, and then bounded diameter lemma says we'll be only a bounded distance from the shrinkwrapped surface.

We all know about shrinkwrapping from the grocery store, we have this bag that is like a two-sphere, shrunk down to get as close as possible but not cross the turkey. The curvature, it's intrinsically -1 (0 in Euclidean space with positive curvature at the elbows) but the things we're shrinkwrapping to are geodesics, they don't have positive curvature, so most of it is minimal and the rest is bent along geodesics, so they have intrinsic curvature ≤ -1. My goal tomorrow is to go back and understand the taming criterion, and then on Friday the PL-shrinkwrapping, which is what Taruka-Soma did.

Thank you very much.

2 Gordon

Before I start, I have an annoucement to make. John has said he'll give a replay of his talk at 5:30 in deference to the World Cup.

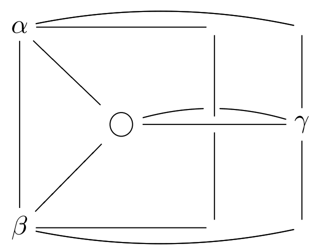
So the idea is to use rational tangle filling to construct interesting non-hyperbolic Dehn fillings on simple manifolds. So like I say, there's a nice machine here, it's easier to mess around with tangles than to mess around with Dehn filling.

So for $\mathscr{T}$ a marked tangle and S a boundary component then we can attach a rational tangle with the same marking to that tangle. Let me remind you that denotes a double branched covering, and if $p : \tilde{\mathscr{T}} \to \mathscr{T}$ is a double branched cover, then $p^{-1}(S)$ is branched over four points, so is a torus component of $\delta\tilde{\mathscr{T}}$. Then you can attach a rational tangle to get $\mathscr{T} \cup_S \mathscr{R}(p/q) = \mathscr{T}(p/q)$. Then $\widetilde{\mathscr{T}(p/q)} = \tilde{\mathscr{T}}(p/q)$. This is saying the double branched cover of a rational tangle filling is the same as the Dehn filling of the double branched cover.

Let me remind you of the special case, sometimes, suppose there's only one boundary component, filling it in gives a knot or link. Then the filled tangle $\mathscr{T}(p/q)$, that gives the unknot if and only if the double branched cover $\widetilde{\mathscr{T}(p/q)}$ is S^3, which implies $\tilde{\mathscr{T}}$ is the exterior of a knot in S^3. Maybe this is a point at which to say, this method to make Dehn fillings by double branched fillings doesn't give you anything. These always have a $\mathbb{Z}_2$ action, they're what is called strongly invertible. This won't give all of the interesting things. By construction you'll only get things with symmetries.

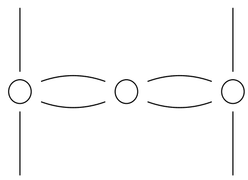
Filling in with a rational tangle giving you a two-bridge knot means that the Dehn filling will be a lens space.

Here's an example due to Eudave-Muñoz. Let α, β, γ be rational tangles with various restrictions. Imagine, what's the definition of this tangle?



What if I do 1/0 filling on this, I get, generically, $\mathscr{B}(1/0) = \mathscr{M}(-1/\alpha, -1/\beta) \cup_\delta \mathscr{M}(1/2, \gamma)$. Let $M = \tilde{\mathscr{B}}$. Remember the double branched cover of one of the Montesino guys, look on the left hand half, you branch over two guys you get an annulus. Provided these aren't integers you get a Seifert fibered space, so $M(1/0) = D^2(q_1, q_2) \cup_\delta D^2(2, q_3)$ as long as $1/\alpha, 1/\beta, \gamma$ are not integers. So this already has one interesting Dehn filling.

So we could also do $\mathscr{B}(0)$. I get $\alpha + 1, \beta + 1$, and γ arranged in the form like the closure of the Montesinos knot:



So you have to do some inversion and you get $\mathscr{B}(0) = K[\frac{-1}{\alpha+1}, \frac{-1}{\beta+1}, \frac{-1}{\gamma}]$. So then $\widetilde{K[\frac{-1}{\alpha+1}, \frac{-1}{\beta+1}, \frac{-1}{\gamma}]}$ for the parameters not integers gives $S^2(p_1/q_1, p_2/q_2, p_3/q_3)$.

Okay, let's do one more, what about $\mathscr{B}(1)$?

Then I get a Montesinos knots or links, again, made up of three rational tangles, and again you get $K[\frac{-1}{\alpha-1}, \frac{-1}{\beta-1}, \frac{-1}{\gamma+1}]$, away from particular values. In terms of Dehn filling on M, we can say $M(0)$ and $M(1)$ are both Seifert fibered spaces with base S^2 and at most three exceptional fibers. These are then also interesting Dehn fillings.

Now, let's try $\mathscr{B}(1/2)$. Now you start to see the cleverness of this process. You can swing this part around the back and make it loop around, and so you get, can you do this with powerpoint? There you can sort of swing γ around. You have γ with two symmetric half-twists, and this is rather interesting. We want this to be the unknot, but generically it won't be because on this side, you're going to have a Seifert fibered space over the disk, and over here too. Sometimes, though, it really will be S^3. You can determine for exactly which values of α, β, and γ you get the sphere. So one of theme has to be a solid torus. It's just a matter of arithmetic to find out which values make this the unknot. There are infinitely many nontrivial solutions.

Using this description, this was Eudave-Muñoz, you see,

Theorem 5 *There exists an infinite family $\mathbb{E}$ of triples (α, β, γ) such that $\mathscr{B}(\alpha, \beta, \gamma)(1/2)$ is the unknot if and only if $(\alpha, \beta, \gamma) \in \mathscr{E}$.*

There are actually two infinite families parameterized by three integers.

If I were a tough instructor I'd put this on the homework.

Let me rephrase this now. I'll give an example in a minute. If you are in one of these triples, you see, then $\tilde{B}(\alpha, \beta, \gamma)$ has a filling that gives you S^3 so this is the exterior of a knot $E(\alpha, \beta, \gamma)$ in S^3. I'll call this a Eudave-Muñoz knot. Let me say that these are hyperbolic for all but a few small values of α, β, γ. Generically that's the situation.

At this point, so notice that here we're parameterizing slopes using the marking on the tangle. With respect to the $\mathscr{B}$-marking, that's the guy who gives you S^3, so that gives you the meridian $1/0$. Then the guy $1/0$, who gives you the toroidal filling, notice that $\Delta(1/0, 1/2) = 2$, so that this will correspond to $m/2$ for some m.

Corollary 1 *$E(\alpha, \beta, \gamma)$ has a half-integral toroidal Dehn surgery*

Theorem 6 *Gordon-Luecke 2004*
If K is hyperbolic in S^3 and $K(m/\ell)$ is toroidal, $\ell \geq 2$ then $\ell = 2$ and $K = E(\alpha, \beta, \gamma)$.

This is an important feature of the subject. It gives you hope that you might be able to classify all of the examples. It turns out with some luck that what you find that can work really is the only show in town.

Okay, let me give you an example $\mathscr{A} = \mathscr{B}(-2, 3, -2/3)$. Putting one half in should give the unknot. So I can untwist these the -2 and the three mainly cancel. It's just a matter of playing around with the blackboard, if I do it too quickly I'll get the trefoil and feel very silly. Maybe I should quit while I'm ahead. Okay, so now I have the unknot.

So $K = E(-2, 4, -2/3)$ which is in S^3. There's the trivial filling and then a half-integral filling with $K(m/2)$ toroidal. If I stick to the framings given from the tangle, well, far $K(0)$ and $K(1)$, one of each of the parameters becomes an integer. So you get lens spaces $L(18, 5)$

and $-L(19,7)$. In some sense this is the smallest known triple that works. This guy has three more interesting fillings. This is on your homework, which is just being typed out. $\mathscr{A}(1/3) = K[1/3, -1/2, -2/5]$. I should be calling these $M(1/0), M(1), M(0), M(1/2)$. This is M_K. It has another filling $M(1/3)$ which is a Seifert fibered space $S^2(2,3,5)$ whcih has finite fundamental group. Then $\mathscr{A}(2/3)$ and $\mathscr{A}(2/5)$ are of the form $D^2(2,2) \cup_\delta D^2(2,3)$. This is a very interesting knot. It has seven nonhyperbolic Dehn fillings. It has the half-integral one, the two lens space ones, and the three others which are a Seifert fibered space and a union of Seifert fibered spaces. In fact, this knot $E(-2,3,-2/3)$ is $-(-2,3,7)$-pretzel knot. You have to show this is hyperbolic but that's not a big deal. You can do this using fairly elementary facts about Dehn fillings.

Let me just say something about, well, you can construct the knots directly, with a sequence of surgeries, but they get complicated very quickly. But it's nice that these tangles are easy to work with, and without worrying about what the manifold is, you know that you have an interesting example.

What can happen in terms of, recall that the figure eight knot, clearly the simplest hyperbolic knot no matter how you define simple. It has ten exceptional surgeries. There was the meridian which gives S^3, the 0 surgery which gives a torus bundle, and then $\pm 1, \pm 2, \pm 3$ which are Seifert fibered spaces over the sphere with three exceptional fibers. The quotient by the involution will be a tangle, and you see these exceptional filling slopes. In ± 4 you see a Klein bottle. If you take the so-called twist knots, then all of these guys, they all come from the Whitehead link. All of these fillings $0, 1, 2, 3, 4$, these push back to the Whitehead link. These surgeries on one of the components give you something nonhyperbolic. The figure eight, in this family, is the only one that is amphichiral. So it gets all the negative ones. Then the $(-2,3,7)$-pretzel has seven exceptional slopes. Again, in the $\mathscr{B}$-marking they are $1/2, 0, 1, 1/0, 1/3, 2/3, 2/5$. I just mentioned here, these are the only, there exist infinitely many hyperbolic knots with six exceptional slopes and I might mention again, the 0 and ± 4 surgeries on the figure eight, and the $1/0, 2/3$ and $2/5$ surgeries on the $(-2,3,7)$-pretzel are toroidal.

Then let me make the observation, the figure eight and the $(-2,3,7)$-pretzel are the only ones known with more than six exceptional surgeries, and also the only ones known with three toroidal surgeries.

It would be nice to prove that they really are the only two with toroidal surfaces, stuff like that. I'll describe with a little more generality what I'm heading toward. Then I'll head back and use the $\mathscr{B}$ to construct other manifolds, not knot exteriors. That's next time.

3 Etnyre

It looks like I competed a little better than I thought with the soccer game, maybe I shouldn't have said I'd talk twice.

If you recall, yesterday we talked about contact structures, foliations, and Darboux's theorem tells you that those are all the same locally. Any deformation of a contact structures, we saw, comes from a family of diffeomorphisms. I also told you to look at the level sets of a function. This is a foliation on the solid torus eventually, but if I try to draw it right off the bat you won't understand it, so I'll do it in steps.

The exercise was to find the level sets of $f(x, y, z) = \alpha(x^2 + y^2)e^z$.

Let's start in $\mathbb{R}^2$ with a line at $x = 1$ and then a set of asymptotic lines, or curves, translated up and down. Then spin this around the z-axis. The line becomes a cylinder, the curves inside the asymptote give you cups, and outside give you annuli. If I mod out by $z \to z + 1$ I get a foliation on $\mathbb{R}^2 \times S^1$. There, well, on the disk a leaf is the boundary torus, and then the leaves are these $\mathbb{R}^2$ that spiral out toward the boundary. This is called the Reeb foliation of $S^1 \times D^2$. A meridianal disk gets a singular foliation, but you have singularities, like a dartboard. You can say, in two dimensions, a singular foliation is the flow lines of the vector field.

So now I want to give an interesting contact structure on the solid torus. Start in $\mathbb{R}^3$ and we'll look at $\cos\ rdz + r\sin\ rd\theta$. What does this do? It looks similar to what we saw the other day. When $r = 0$ this is just dz. When you get to $\pi/2$ you will get rotated half way, and when you get out to π you're flat again. This is what happens, this infinite twisting, on any line perpendicular to the plane.

So now let's look at $z \mapsto z + 1$ and restrict to $\{(x, y, z)|\sqrt{(x^2 + y^2)} \leq \pi\}$. This gives me a contact structure on the solid torus. On D^2 cross a point I get a singular foliation. I'll get singularities at every point on the boundary and at the origin. Typically you don't expect to see that many singularities. If you bump D^2 a little bit, you get a singularity in the central and a spiral out of it, but the boundary is no longer singular, it's a closed leaf of the foliation.

[What is this perturbation?]

I push it up slightly. Put your finger at the origin and push it up a little bit.

[Some discussion of how this works.]

This picture is called an overtwisted disk, and $(D^2 \times S^1, \xi)$ is called a Lutz tube. Surprisingly enough if you have an overtwisted disk you have a Lutz tube.

A remark. It's very easy to construct foliations with Reeb components and contact structures with Lutz tubes on any closed manifold. If we have time at the end, we'll go back and do this as an exercise.

A contact structure without overtwisted disks (or Lutz tubes) is called tight. If it has overtwisted disks it's called overtwisted. Without Reeb components it's called Reebless.

[Should we think of examples one and two as related?]

We will see as we move along that they are very related. Foliations without Reeb components and contact structures without Lutz tubes are close to one another.

[How do you tell that things are tight?]

That sounds hard, doesn't it? In general it could take you a long time to check every embedded disk. I'll give you a criterion.

So another thing about overtwisted contact structures. Eliashberg classified them, and every manifold has infinitely many. In fact, what Eliashberg did was he showed that overtwisted contact structures (up to isotopy) are in one to one correspondence with homotopy classes of plane fields.

Exercise 3 *Show that any closed three-manifold has infinitely many homotopy classes of plane fields. A hint is, think Pontjagin-Thom construction, and if you don't know what that is, derive it.*

If you saw Yasha's talk, overtwisted contact structures are flexible, and tight contact structures are rigid. If you think you can do something with them, then you probably can't. Well, it depends who you are, but that's been my experience.

In some sense overtwisted contact structures are well-understood, and for our purposes we won't be very interested in them.

So let's give some facts about tight contact structures. The first fact is that not all manifolds have tight contact structures. The Poincaré homology sphere with reversed orientation, for instance, has no tight contact structure. This is due to Etnyre-Honda. There were a few other examples, there's an infinite family, I think all have finite fundamental group. They're all Seifert fibered spaces.

Exercise 4 *Give a hyperbolic manifold with no contact structure. Alternately, prove that no such thing exists*

Other facts that kind of show you the connection between tight contact structures and foliations with no Reeb components. Recall an oriented 2-plane field ξ has an Euler class, the obstruction to having a cross section $e(\xi) \in H^2(M, \mathbb{Z})$.

Theorem 7

1. *(Thurston)*
   If ξ is a Reebless foliation and Σ a surface in M then $|e(\xi)([\Sigma])| \leq -\xi(\Sigma)$ if $\Sigma \neq S^2$ and is zero if $\Sigma = S^2$. This is a lower bound on the genus of a surface representing a homology class.

2. *(Eliashberg)*
   If ξ is a tight contact structure then the same bound is true

So we're seeing a connection between tight contact structures and Reebless foliations. Taut foliations don't have Reeb components, and we'll see theorems that give a lot of those in the future.

Exercise 5 *Show the theorem implies that only finitely many elements in $H^2(M, \mathbb{Z})$ can be the Euler class of a tight contact structure for a Reebless foliation.*

[Is there a manifold with neither a positive nor negative contact structure?]

Yes, it behaves well with respect to connected sum. So the manifold I described before connect sum with itself with the opposite orientation has no such structure. That's a cheat. It's not known whether there is an irreducible one.

Do we have any tight contact structures at all? This was proved because he wanted to show that the standard contact structure is tight. For that I need to describe or recall what a symplectic manifold is.

So a symplectic 4-manifold is a four-manifold X and a two-form ω with $d\omega = 0$ and $\omega \wedge \omega$ nonvanishing. That's a symplectic four-manifold.

Now a three-manifold boundary M of X and ξ a contact structure on M, then ω dominates ξ if $\omega|_\xi > 0$. This is an oriented plane field, you can plug the plane field with the preferred order, into ξ.

If (M, ξ) is one component of a contact manifold (M', ξ'), (X, ω) dominates ξ', and X is compact, then we say (X, ω) is a weak semi-filling of (M, ξ). If M' is connected, then you call it a weak filling.

We're going to see later a thing called the strong filling. I'll try to put weak or strong in front every time. What's the point of all of this?

Theorem 8 *Gromov, Eliashberg*
If (M, ξ) is weakly semi-fillable, then ξ is tight.

There is no definition for tight or overtwisted above dimension three. No one knows what it should be.

Example 1 *Let (S^3, ξ) have ξ be* $\ker\ \alpha$ *if $\alpha = r_1^2 d\theta_1 + r_2^2 d\theta_2$. Let $\omega = d\alpha = 2r_1 dr_1 \wedge d\theta_1 + 2r_2 dr_2 \wedge d\theta_2$, then ω is a symplectic form on B^4.*

We want to show that this is a tight contact structure. Well, $\omega|_\xi = d\alpha|_\xi$, which is pretty obviously positive, since $\alpha \wedge d\alpha$ is positive. So (B^4, ω) weakly fills (S^3, ξ).

This theorem will help us a lot in the future. You might ask, is every tight contact structure weakly semi-fillable? The answer is no, there are tight contact structures that are not semi-fillable. So the tightness is really a three-dimensional phenomenon. The semifillable things, you would not be studying all of them.

This is the end of the basic terminology I wanted to discuss. We know something about contact structures and foliations, tight and overtwisted and so on. So now we're ready to move on to

3.1 Part II: Foliations into contact structures

Consider the really interesting foliation which is just going to be, on $S^1 \times S^2$, and on every point $\mathscr{S}_{(\theta,p)} = T_p(\{\theta\} \times S^2)$. You might think this is boring.

Theorem 9 *Eliashberg-Thurston*
Any C^2 foliation ξ on an oriented closed 3-manifold other than $(S^1 \times S^2, \mathscr{S})$, can be approximated by a positive and negative contact structure $\xi_\pm$.

Let me say some things about this. This $\mathscr{S}$ is unique because it can't be approximated.

There will be two stages in the proof where you can't go through in a C^1 or C^2 way. We may be able to do more but this is what we have, it's enough for now.

So let me comment on a few of the terms in here. We say ξ can be C^k-deformed into a contact structure if there is a C^k family ξ_t of plane fields such that $\xi_0 = \xi$ and ξ_t is contact for $t \neq 0$. This is a one-parameter family of contact structures that collapse onto the foliation. They're all contact except at $\xi = 0$.

You're thinking of the Grassman bundle of planes in the tangent plane, and this is a C^k-section of that.

Okay, and then I just defined something that was supposed to clarify the statement of the theorem but then I didn't say anything with the words in the theorem. So ξ can be C^k-approximated by a contact structure if in any C^k-neighborhood of ξ there is a contact structure.

Notice I'm not saying you can deform to the contact structure, you might have to jump. You can't move the ξ smoothly or continuously into a contact structure but you know there is one nearby.

Let's look at the following example. Consider T^3, and the coordinates x, y, z. Then for $n > 0$, $\alpha_n^t = dz + t\cos(2\pi nz)dx + \sin(2\pi n)dy$, so that's a form, and it's easy to prove that α_n^t is a contact form for $t \neq 0$. When $t = 0$ it's a foliation.

The kernels of these one-forms is the T^2s, the vertical one, and you deform these into contact structures. I have this n here so it looks like there's more than one, and in fact there are.

It's a theorem of Kanda and Giroux independently that $\xi_n = \ker\ \alpha_n^t$ are distinct and all tight contact structures are equivalent to one of these.

I have one foliation, and I get a bunch of different contact structures near it. The nearby contact structures might be the same, the theorem doesn't tell you that.

[Is there an example of a foliation that can be approximated but not deformed?]

You might be able to say that a C^2-foliation can be C^2-deformed to a contact structure.

I think that's a good place to stop, and we'll come back to this theorem tomorrow.

4 Auckly, Gromov-Witten "=" Reshetikhin-Turaev

Most of this is going to be expository so I can get you up to speed on the second part.

We have a preprint that's not up on the arXiv, because we need to get the history right. If you want this version, sign up on the pad going around.

Let's start with X a six dimensional symplectic manifold then $\beta \in H_2(X)$ and $g \in \mathbb{Z}_{\geq 0}$, and we'll consider a gadget $\bar{\mathscr{M}}_g(X,\beta)$. This is the stack of genus g J-holomorphic stable curves in class β.

This $\bar{\mathscr{M}}_g(X,\beta)$ should be thought of as maps $u : \Sigma \to X$ with $g(\Sigma) = g$, $u_*[\Sigma] = \beta$, and $\bar{\delta} u = 0$, modding out by the equivalence that $(\Sigma, u) \sim (\Sigma', u')$ if there is a holomorphic $\varphi : \Sigma \to \Sigma'$ so that the diagram commutes:

$$\begin{tikzcd} \Sigma \arrow[r, "u"] \arrow[d, "\varphi"'] & X \\ \Sigma' \arrow[ur, "u'"'] & \end{tikzcd}$$

Theorem 10 *There exists* $[\bar{\mathscr{M}}, (X,\beta)]^{Vir} = H_{top}(\bar{\mathscr{M}}_g(X,\beta)$. $top = 0$ *if* $c_1(TX) = 0$ *and this should be over* $\mathbb{Q}$.

So for $g = 0$ if we have $\Sigma = \mathbb{CP}^1$ and $X = \mathbb{CP}^1$, where $\beta = 2[\mathbb{CP}^1]$. Then the map u could be z goes to z^2. Then we can take φ to be $z \mapsto -z$.

We can parameterize the set of all degree two maps. A degree two map has two critical values. A pair of points in $\mathbb{CP}^1$ [unintelligible]

So the moduli space $\bar{\mathscr{M}}_0(\mathbb{CP}^1; 2[\mathbb{CP}^1) \cong \mathbb{CP}^2 = Sym^2(\mathbb{CP}^1)$. You go back and forth from coefficients of polynomials to roots.

You have to add in nodal curves with double roots. This is really an orbifold and the correct thing over here, the example is that $[\bar{\mathscr{M}}_0(\mathbb{CP}^1, 2)]^{Vir} = 1/2[\mathbb{CP}^2]$.

Definition 2 *The full Gromov-Witten free energy*

$$\bar{F}_X^{GW}(t, Y) = \sum_{g=0}^{\infty} \sum \beta \in H_2 Y^{2g-2} e^{-} \langle t, \beta \rangle \int_{[\bar{\mathscr{M}}^{Vir}} 1,$$

here t is the second cohomology class of the symplectic form.

. In the special case that the dimension is six, the spaces are all zero dimensional.

You whirl things around a bit and get a formula

$$\hat{F}_{X_{S^3}}^{GW}(t, y) = \frac{t}{24} - \frac{1}{12} ln\ t + \zeta(3) y^{-2} + 3t^2 y^{-2}/4 + t^3 y^{-2}/12 - t^2 y^{-2} ln\ t -$$

$$\sum_{h=4,6,\ldots} \frac{2}{h(h-1)(h-2)}(2\pi)^{2-h}\zeta(h-2)(it)^h y^{-2}$$
$$+\sum_{h=2,4,\ldots} \frac{1}{6n}(2\pi)^{-h}\zeta(h)(it)^h +$$
$$\sum_{g=2}^{\infty} \frac{B_{2g}}{g(2g-2)} \sum_{h=0,2,\ldots} \binom{2g+h-3}{h}(2\pi)^{2g-2-h}\zeta(2g+h-2)(it)^h y^{2g-2}$$
$$+\sum_{g=2}^{\infty} \frac{B_{2g}}{g(2g-2)}(it)^{2-2g}y^{2g-2} - \sum_{g=2}^{\infty} \frac{B_{2g}}{g(2g-2)}\zeta(2g-2)y^{2g-2}.$$

Here ζ is the Riemann zeta function and B are the Bernouli numbers.

So let's look at $U_q(\mathfrak{sl}_N(\mathbb{C})|_{q=e^{i\pi/(k+N)}}$, which is a particular Hopf alebra. You look at a representation of these guys. $f : V \to W$ looks like an f in a box with an arrow coming in labelled by V and an arrow at looking like W.

You also have the $X_{V,W} : V \otimes W \to W \otimes V$ which you represent schematically by the crossing.

This has properties you want to it satisfy, like the quantum Yang Baxter equation:

$$(1_W \otimes X_{U,V}) \circ (X_{U,W} \otimes 1_V) \circ (1_U \otimes X_{V,W}) = (X_{V,W} \otimes 1_U) \circ (1_V \otimes X_{U,W}) \circ (X_{U,V}, 1_W).$$

This also has $\cap_V : V^* \otimes V \to \mathbb{C}$ which satisfies axioms like a vertical line being equal to a cap and cup. Out of this gadgt you get an invariant, the colored Jones polynomial $J_{V_1,\ldots,V_c}(L)$ where L is a c-component link. This is the trace of the knot diagram. If you do the fundamental representation with $N = 2$ you get the Jones polynomial, and for $N = N$ you get the N-specialization of the HOMFLY, $p(q^N, q)$. You also need to take linear combinations a_V to be invariant under the Kirby move.

Now

Definition 3 *The Reshetikhin-Turaev invariant is* $\tau_K^{SL_N(\mathbb{C}}(M)$, *which is defined to be*

$$\sum_{V_1,\ldots,V_c \text{ simple}} a_{V_1,\ldots,V_c} J_{V_1,\ldots,V_c}(L).$$

The Reshitikhin-Turaev free energy is $F_M^{CS}(N, X) = ln\ \tau_K^{SL_N(\mathbb{C})}(M)$. *Here* $X = 2\pi/(k+N)$.

You whirl things around a little bit and get

$$\hat{F}_{S^3}^{CS}(N,x) = \frac{N(N-1)}{2} ln\ x + \frac{1-N}{2} ln\ (k+N) + \frac{N^2}{2} ln\ N - \frac{1}{2} ln\ N - \frac{3N^2}{4} - \frac{1}{12} ln\ N - \zeta 1(0)N + \zeta^i(-1)$$

$$-\sum_{h=4,6,\ldots} \frac{2}{h(h-1)(h-2)}(2\pi)^{2-h}\zeta(h-2)N^h x^{h-2} + \sum_{h=2,4,\ldots} \frac{1}{6h}(2\pi)^{-h}\zeta(h)N^h x^h - \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} N^{2-2g} +$$

$$\sum_{g=2}^{\infty}\sum_{h=2,4,\ldots}\binom{2g+h-3}{h}\frac{B_{2g}}{g(2g-2)}(2\pi)^{2-2g-h}\zeta(2g-2+h)N^h x^{2g-2-h}$$

Theorem 11 *Auckly, Koshkin*

$$Re(\hat{F}^{GW}_{X_{S^3}}(iNX,X)-F^{CS}_{S^3}(N,X))=\frac{5}{12}ln\ x-\zeta(3)x^{-2}-\frac{1}{2}ln(2\pi)-\zeta'(-1).$$

Some of the terms might not be exactly right. The final version that appears on the arXiv will be correct.

So $\tau_K^{SL_N(\mathbb{C})}(S^3)=N^{-1/2}(K+N)^{(1-N)/2}\prod_{j=1}^{N-1}[2\sin\frac{\pi j}{K+N})]^{N-j}$.

So start with $0\to\mathbb{Z}^k\xrightarrow{Q}\mathbb{Z}^{3+k}$ this gives a map from $\mathbb{R}^k\to\mathbb{R}^{3+k}$ which gives a map $T^k\times\mathbb{C}^{3+k}\to T^{3+k}\times\mathbb{C}^{3+k}\to\mathbb{C}^{3+k}$ by point multiplication.

[I missed a bunch. There are a lot of matrices. He isdoing an example for $L(p,-1)$ and $L(3,-1)$. The matrices for the specific case are 6×3 and 3×6. He's defining the fiber dual.]

$$Q=\begin{bmatrix}0&0&1\\1&2&1\\-2&-3&-3\\1&0&0\\0&1&0\\0&0&1\end{bmatrix};P=\begin{bmatrix}1&0&0&0&0&-1\\0&0&1&2&3&3\\1&1&1&1&1&1\end{bmatrix};$$

$$R=\begin{bmatrix}1&0&0\\1&-1&1\\0&1&0\\0&0&0\\0&0&0\\0&0&0\end{bmatrix};T=\begin{bmatrix}1&3&9\end{bmatrix}^T;\hat{\mu}=Q^T\begin{bmatrix}|z^1|^2\\\vdots\\|z^{3+k}|^2\end{bmatrix}$$

I can define a map $T^3\times X_{Q,t}$ ta $T^{3+X}\times X_{Q,t}$ by R, and then by point multiplication to $X_{Q,t}$. The action is well defined and preserves the condition that these things goto zero, and the simplectic form. The normalized momemnt map of R of an equivalence class of Z, $\hat{N}_R([Z])=R^T\begin{bmatrix}|z^1|^2\\\vdots\\|z^{3+k}|^2\end{bmatrix}$.

The moment polytope is $\hat{N}_R(X_{Q,t})$. Maybe I'm drawing this a little too sharp. You cut off the top here, and then cut off the top here, you get something that looks like this picture. You're taking the set of all z that satisfy Q. A little linear algebra shows this is $\{y+R^T\gamma|P^Ty+\gamma\in\mathbb{R}^{3+k}_{\geq 0}\}$. and here $P^T\gamma=t$.

These satisfy the condition $c_1 = 0$. If you look at the $\theta = dz_1 \wedge dz_2 \wedge dz_3$, then this thing extends to $X_{Q,t}$. It's a section of the third exterior power of the tangent bundle. So $\wedge^3 TX$ is trivial so $c_1(TX)$ is zero, which defines a special torus $T_X = \{\lambda \in T^3 | (L_\lambda)_*(\theta) = \theta\}$

The 3-web of X is projection to $(Lie\ T_X)^*$ of the one-skeleton of the moment polytope.

[Argument via pictures excised because I couldn't keep up.]

I'll finish by saying that this is work in progress with Karp, Koshkin, to compute and compare $\hat{F}^{GW}_{X_{Q_p}}$ and $F^{CS}_{L(p,-1)}$, to compute the normalization factors and that might help you discover more.

[So the grand plan is to associate to any three-manifold a six-manifold. Is it Kähler?]

The physicists say it should be Calabi-Yau. Maybe I should listen to the physicists and throw it in the garbage, but I'm going to look at it for a little bit. Calabi-Yaus have zero dimensional moduli space, but in general we can do insertions for larger dimensional moduli space.

You have a natural log of a sum of something with representations in translates of an affine Weyl alcove.

Based on the conjecture that these things were equal, they came up with a combinatorial way to calculate the Gromov-Witten invariants for toroidal Calabi-Yau threefolds.

[More questions? Let's thank the speaker again.]

I'll make three more announcements. We would like to advertise our REU at research universities, please take a poster. There's one paper copy, or email for the electronic copy.