Low Dimensional Topology Notes
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1 Khovanov

[The problem sessions today, this morning there will be a posthumous problem session for Szabo's lecture. This afternoon it will be for Cameron's lecture series. Here's... the final... Khovanov lecture... Oooooo!]

We're going to create a categorification of the HOMFLY-PT polynomial. Let R be a ring, M a right R-module and N a left-R-module. Then $M \otimes N$ is an Abelian group. We have to take a derived functor, meaning we convert M or N into a projective resolution, and then apply the tensor product to the projective resolution. The first time you see this is probably in the universal coefficients theorem, where $\mathbb{Z}_n = \mathbb{Z}_n \otimes \mathbb{Z}_n$, but you replace a $\mathbb{Z}_n$ with $\mathbb{Z} \to \mathbb{Z}$, and then you get $0 \to \mathbb{Z}_n \to \mathbb{Z}_n$, the extra factor in the universal coefficient theorem.

Because $\mathbb{Z}$ is hereditary you don't see anything beyond one more level, but in general you replace $0 \to M \to 0$ with $\ldots \to P_2 \to P_1 \to P_0 \to 0$, which is quasiisomorphic to M. It's exact in all terms except the last term, where $P_0/Im\ P_1 \cong M$.

So we have $P_2 \otimes N \to P_1 \otimes N \to P_0 \otimes N \to 0$, and then the ith homology group is called the ith derived tensor product of M and N. A standard theorem says that it doesn't matter which of M and N you projectivize and the choice of projectification. The tensor functor is only right exact.

Schematically we can represent $M \otimes N$ as

$$\begin{tikzcd} M[rr] & & N \\ M[R]_R \arrow[u] & & N[l]_R \arrow[ull] \end{tikzcd}$$

where you join along the actions. We'll tip these on their sides so we can deal with braid group actions:

$$M \qquad\qquad M[u]_R[dd][u]_R[d]_R$$

$$N[d]_R$$

$$N[u]_R[d]_R$$

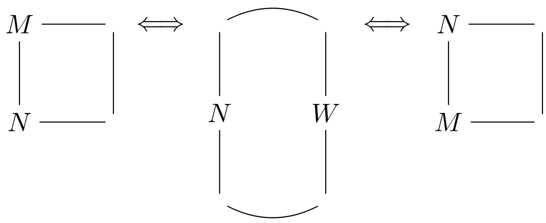
You change M into a complex of R-bimodules that are projective as right R-modules to perform this operation on bimodules.

Now we can close things off:

$$M[ur][dl] \qquad\qquad [ul][dr]$$

by looking at $M_R = M/[R, M] = M/(rm - mr)$, symmetrizing by setting the left and right actions equal to one another. These are called the R-coinvariants of M. You run into the same trouble because this functor is not exact, it's right exact. If you have an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$, it is taken to an exact sequence $(M_1)_R \to (M_2)_R \to (M_3)_R \to 0$ so you are missing the first 0. You have to do something to get this to be exact. I should give you a bit of philosophy. Typically quotient functors are right exact, while subobject functors are left exact. One example of a left-exact functor are R-invariants. Send $M \to M^R = \{m | rm = mr \forall r \in R\}$. A related way to think about this closure is as attaching M with the two R-actions to R with the two R actions, but reversed, so this is $M \otimes_{R \otimes R^{op}} R$. This is because R-bimodules are the same as left $R \otimes R^{op}$ modules.

Any module has the biresolution. To be thrifty we can take a smaller resolution. Let me note that this interpretation of the closure respects the composition of the bimodules. That is, $HH(M \dot{\otimes} N) \cong HH(N \dot{\otimes} M)$. There's the mysterious picture in the middle where it becomes W.



We want to look at $HH_*(R, M)]M \overset{L}{\otimes}_{R\otimes R^{op}} R$. To do this we resolve R into a complex of projective $R \otimes R^{op}$-modules.

So say $R = \mathbb{Q}[x]$. Then this is $0 \to \mathbb{Q}[x] \otimes \mathbb{Q}[x] \to \mathbb{Q}[x] \otimes \mathbb{Q}[x] \to \mathbb{Q}[x] \to 0$. The differential here first takes $1 \otimes 1$ to $1 \otimes x - x \otimes 1$, and the next one is multiplication.

When you tensor this map with M you get $0 \to M \to M \to 0$ with the map $m \mapsto mx - xm$. So $HH_0(M) = M_R$ and $HH_1(M) = M^R$, the coinvariants and invariants. This is a coincidence. Usually this would not be so nice, M^R is defined as $HH^0(M)$ but this will not always be $HH_1(M)$.

What about for R equal to polynomials in many variables $\mathbb{Q}[x_1, \dots, x_n]$? We can just take resolutions, one for each variable x_i, and tensor them together. We can take $0 \to R \otimes R \to R \otimes R \to 0$ where $1 \otimes 1$ goes to $x_i \otimes 1 - 1 \otimes x_i$, and tensor n copies of this together. The result is a resolution of length n with $n+1$ terms. Every term is a direct sum of copies of $R \otimes R$. At each level it's $(R \otimes R)^{\otimes \binom{n}{k}}$. You can think of this resolution as $R \otimes R \otimes \wedge(x_1, \dots, x_n)$. So for HH we need to tensor with M giving 2^n copies of M in various degrees and the boundary will take $m \to x_i m - m x_i$. In this case we will have $HH_0(M) = M_R$, the coinvariants, and all the way up to $HH_n(M) = M^R$ the invariants.

To get something stronger how can we specialize? Now let $R_i \subset R$ be the subring of polynomials invariant under the permutation of x_i and x_{i+1}. As an R_i-module, R is free of rank 2, with $R = R_i \cdot 1 \oplus R_i x_i$.

So take $B_i = R \otimes_{R_i} R$, this is projective as a left and as a right R-module. So for instance $B_i \otimes B_i$ is $R \otimes_{R_i} R \otimes_{R_i} R = B_i \oplus B_i\{2\}$. So this is $2 \cdot 2 = 2 + 2$.

We'll keep track of grading, giving each x_i degree two. So R_i is graded and B_i is graded.

There is a bimodule map $B_i \to R$ taking $a \otimes b$ to ab. This is a complex of bimodules, $C_i = 0 \to B_i \to R \to 0$. There's another $C_i' = 0 \to R \to B \to 0$ which takes 1 to $(x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})$. Just shift the degree of B_i down by two to make the complex differential have degree zero. This is just like what we did for $U_i = P_i \otimes {}_iP$ before.

Theorem 1 *R. Rouquier*
$C_i \otimes C_i' \cong R \cong C_i' \otimes C_i$
$C_i \otimes C_{i+1} \otimes C_i \cong C_{i+1} \otimes C_i \otimes C_{i+1}$
$C_i \otimes C_j \cong C_j \otimes C_i$ *for* $|i - j| > 1$.

The degree of x_i is two because $R = H^*_G$, flags of $G = GL(n, \mathbb{C})$. Then H^*_B of a point, where B are the upper triangular matrices, are $H^*_{(C^*)^n}$ of a point. This is just a step from the homology of $\mathbb{CP}^\infty$, and then there are n of them.

We have a braid group action, but what about braid cobordisms? For adding a positive crossing we could use

$$\begin{tikzcd} & 0 \arrow[r] & R \arrow[r] \arrow[d, "1"'] & 0 \\ 0 \arrow[r] & R \otimes_{R_i} R \arrow[r] & R \arrow[r] & 0 \end{tikzcd}$$

but for the negative one, because $R \to R \otimes_{R_i} R$ is injective, the vertical map must be zero:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d] & 0 & \\ 0 \arrow[r] & R \arrow[r] & R \otimes_{R_i} R \arrow[r] & 0 \end{tikzcd}$$

So any time you have a negative crossing you will get a zero, which is sad.

So we have various theories. With A_n we can make it to braid cobordisms (and get a categorification of the Burau representation). With H^n we can get tangle cobordisms and the Jones polynomial. With the R we can get the HOMFLY polynomial. So this is less degenerate in some sense but also more degenerate in this other sense.

Now we can construct a functor from braids to complexes. Assign the two complexes I've just generated via B to $\sigma_i^{\pm 1}$. This is a complex of graded R-bimodules.

Now we can take the Hochschild homology $HH(R, B^j(\sigma))$; then the differential becomes a map of Hochschild homology groups. So every term is bigraded. We have the Hochschild homology grading and in addition, the ring R has a grading so the bimodules have a grading as well, perpendicular to the Hochschild homology grading.

So I will get a bigraded vector space $HH(F^j(\sigma)) \to HH(F^{j+1}(\sigma)) \to$, with each term bigraded. The differential preserves the bigrading. We get a complex of bigraded vector spaces. So now we can take homology again since $HH(\partial)^2 = 0$ since HH is functorial. So $HHH(\sigma) = H(HH(F(\sigma)), HH(\sigma))$. This is now a triply graded vector space.

Why do this?

Theorem 2 *$H(\sigma)$ depends only on $\hat{\sigma}$ (up to a grading shift, which can be gotten rid of, due to H. Wu) and has Euler characteristic equal to the HOMFLY polynomial of the closure $\hat{\sigma}$. This is a modification of Rozansky, Khovanov*

The HOMFLY is uniquely determined by the conditions $aP(L_+) - a^{-1}P(L_-) - (q - q^{-1})P(L_0) = 0$ and the value of $P(\text{unknot})$ which we take as $\frac{a-a^{-1}}{q-q^{-1}}$.

This theory has quite a few weak spots. You want this to be a functor, for a completely legitimate link homology from link cobordisms to some algebraic category The objects will be oriented links and then morphisms will be isotopy classes of surfaces with these links as boundary in $\mathbb{R}^3 \times I$. If you restrict to trivial cobordisms between trivial links, then you can get a two dimensional TQFT so we would need the HHH of the unknot to be a Frobenius algebra. It's a commutative associative algebra and coalgebra with a unit and a counit. Any Frobenius algebra is finite dimensional. But HHH(unknot) is $\mathbb{Q}[x] \otimes \wedge(y)$ which is infinite dimensional. But we want to reduce this to a finite dimensional thing by getting rid of this, that is the goal. We can get rid of one factor, $\mathbb{Q}[x] \otimes \wedge(y)$, but then multiple coefficient links are still infinite dimensional.

There is a conjectural map $HHH^m(L)$ to Oszvath-Szabo-Rasmussen link homology. This is directly related to the categarification of the Jones polynomial. We should be able to draw arrows to Ozsvath-Szabo and then to Seiberg-Witten and then to Donaldson invariants. Anyway, I hope this can be fixed.

You can collapse the theory and get the Alexander polynomial but it's too large, it diverges from the Ozsvath-Szabo-Rasmussen theory, probably.

[Can you do this over $\mathbb{Z}$?]

This you can do over $\mathbb{Z}$. To move to quantum $\mathfrak{sl}_n$ invariants you have to use a suitable field.

Are there any more questions? Let's thank the speaker.

2 Gordon

Can you hear me? Let me begin by making a comment. We were talking about nonhyperbolic surgeries. If you get a lens space it has to be an integral surgery. Conjecturally if you get a Seifert fibered space it has to be integral. Regarding the cabling conjecture, there are several results, but one of them is the following. If X is a knot and $K(\alpha)$ is reducible then α is integral. In fact $K(\alpha)$ has a lens space summand. I should have mentioned that for completeness.

This isn't supposed to happen anyway unless the knot is a cable knot. It's natural to generalize from Dehn surgery to Dehn filling.

Let M be a 3-manifold and let T_0 be a torus component of boundary M You can attach solid tori along that torus, and those are parameterized by simple closed curves α (slopes) on T_0. Then $M(\alpha)$ is α-Dehn filling on M where you identify T_0 with the boundary of the solid torus in such a way that α is the boundary of a meridian disk. This is moving from the medical metaphor to the dental metaphor, surgery to drilling. When M is the exterior of a knot, M_K this is what we were talking about before.

If M is simple (it does not contain essential spheres, disks, annuli, or tori, meaning it has a hyperbolic structure by geometrization), then most Dehn fillings are simple. One wants to try to understand the exceptions.

In this case you can't look at the exceptional cases in quite the same way, since

Theorem 3 *(Myers)*
Any manifold N is $M(\alpha)$ for some simple three-manifold M.

So you can't understand all possible pairs M, α. So the goal here might be to classify the triples M, α, β, where M is simple and $M(\alpha), M(\beta)$ are not with $\alpha \neq \beta$.

The point is, you see, when you have a knot in S^3, the case of knots, is the case when one of the fillings, $M(\beta) \cong S^3$. We already had the meridianal filling. This is why there was some hope of classifying the nontrivial fillings. Any three-manifold can show up.

Okay, so in some sense this will be our sort of context. I'll go back to talk about knots in S^3. I want to go back to tangles and talk about a calculus.

For our purposes a tangle will be a pair (B, A), where B is S^3 minus a disjoint union of some open three-balls, and A is a properly embedded one-manifold where A meets each component of the boundary of B in four points.

The trivial tangle is three arcs in the three-ball, that's D^2 with two points cross an interval. Maybe you have S^3 with three punctures, and then you just have same arcs. They meet each boundary component in four points. So we just have a bunch of things like this picture. They can be knotted up.

In general there can be some simple closed curves as well. Okay. So this is a typical nontrivial tangle. Okay, so there's one class of tangles, sometimes when we're talking about marked tangles, a marked tangle has a specific marking on the boundary, an identification of each boundary component, $(S, S \cap A)$ with the standard sphere with four marked points Q. So in marked tangles we have to preserve this specific identification. So here they're equivalent if they're isotopic with the boundary fixed. What I want to get to now is the definition of a rational tangle. This makes a lot of the work on Dehn filling very easy. Let's consider the following three operations on marked tangles in the three ball.

1. there's the addition of a horizontal twist. You add a positive half-twist in the positive direction. I call this h.
2. there's a corresponding operation in the vertical direction v. There's all kinds of sign convention. This is not the usual one.
3. reflection in the northwest-southeast plane, so just reflection. These are three operations on marked tangles in the ball.

Let $a_1, \ldots, a_k$ be a sequence of integers, with all but the first one nonzero. Then I can define the rational tangle $\mathscr{R}(a_1, \ldots, a_k)$ to be, well, I start with a fairly trivial marked tangle, and apply the twists according to these integers. I start with $\mathscr{R}(1/0)$ which by definition is the marked tangle with two vertical lines. So you start with our guy, and then reflect and put in a_1 horizontal twists, where the sign of the twists are by the sign of a_1, and you continue doing this. So this is $(h^{a_1}r)(h^{a_2}r)\dots(h^{a_k})\mathscr{R}(1/0)$ which is $h^{a_1}v^{a_2}\dots v^{a_k}\mathscr{R}(1/0)$ for k even or $h^{a_1}v^{a_2}\dots h^{a_k}\mathscr{R}(0/1)$ for k odd.

These are called rational because

Theorem 4 *Conway*
$\mathscr{R}(a_1,\dots,a_k)=\mathscr{R}(a'_1,\dots,a'_{k'})$ as marked tangles if and only if the two sequences are equivalent as continued fractions.

Let me say, according to the conventions I've established, as an example, the single crossing is 1 and in the opposite direction it's -1. This is $\mathscr{R}(3)$ and this is $\mathscr{R}(-1/5)$.

So for $3/2$ you put in two vertical twistsand then one positive horizontal twists. Look at two thirds, that's $1-1/3$, so we can put in three vertical negative twist and one positive horizontal twist. You can also write $2/3=[0,1,2]$, so with this expression, we put in two horizontal twists, and then a positive vertical twist, so we get that.

The proof of Conway's theorem, you just show how two continued fractions can be equal and then show that they correspond to Reidemeister sequences.

[Gordon-Kirby sign convention altercation omitted for propriety.]

Look at $4/11=[1,-2,2,3]$, sa we start off with three positive vertical twists, two horizontal twists, two negative vertical twists, and one more horizontal twist.

Now, where is the relation to Dehn filling? The relation comes from taking the double branched cover. So take $\tau=(B,A)$. There will be a unique homomomorphism $H_1(B-A)\to\mathbb{Z}_2$ which sends a meridian of each component of A to the nonzero gnerator.

If we call $X=\overline{B-N(A)}$ there will be a double covering $p:\tilde{X}\to X$ corresponding to this homomorphism. The meridian doesn't lift but its square lifts, you get a double covering, and complete it, extending the projection map to a branched cover. Let me go over here. This is $\tilde{\tau}\to B$. This is branched over the one-manifold A.

I now want to apply a technique I learned from Khovanov, and of course I've cleverly drawn it upside-down.

Okay guys, a bit of audience participation there. If we start in the other direction, and look at the rotation, by the quotient I get the solid cylinder, and I get this three-ball with two marked arc. The green and blue circles in the picture correspond to the meridian and longitude of the solid torus.

So if $\mathscr{R}$ is a rational tangle, $\tilde{R}\cong S^1\times D^2$.

Now we can talk about slopes on the four punctured sphere and relate them to slopes on the torus. So a slope on S^2,Q is an isotopy class relative to the boundary of an embedded arc τ with $\delta\tau\subset Q$. So when you look at the double branched cover $T^2\to S^2$ branched over

Q you get $\tilde{\tau}$, a lift of τ, a simple closed curve in T^2. Now μ, λ, inside S^2, Q lift to $\tilde{\mu}, \tilde{\lambda}$, and you orient them so that $\tilde{\mu} \cdot \tilde{\lambda} = 1$. Then slopes on S^2, Q correspond to slopes on T^2, which correspond to $\mathbb{Q} \cup \{1/0\}$. Then the meridian of the solid torus $\tilde{R}(p/q)$ has slope p/q.

Exercise 1 *On the homework there are some hints about how to prove that. It's certainly true for* $1/0$. *The meridian of the corresponding solid torus, it's the meridian, so* $1/0$. *With the usual sign convention it's* $-p/q$.

This gives a nice correspondence between tangle filling and Dehn filling. So for τ a tangle, S a component of the boundary of τ, and p the covering $\tilde{\tau} \to \tau$. Then $p^{-1}(S) = T \cong T^2$. Then define $\tau(p/q) = \tau \cup_S \mathscr{R}(p/q)$. Then $\widetilde{\tau(p/q)} = \tilde{\tau}(p/q)$. This is Dehn filling on the right. If I take a tangle filling and then the double branched cover, that's the same as taking the double branched cover and then doing Dehn filling.

Let me give some examples. If we cap off $\mathscr{R}(p/q)$ above and below, this is called the rational two-bridge knot or link $K[p/q]$. If you take this and take the double branched cover. You attach the solid torus along a p, q curve, onto a solid torus. So you just get the lens space $L(p, q)$. It's probably $L(-p, q)$.

Let's get some more interesting examples. Let's go back to Seifert fibered spaces M, where F is the (orientable) base surface. And suppose that the exceptional fibers are of multiplicities $q_1, \ldots, q_n$ and neighborhoods $V_1, \ldots, V_n$, then $M_0 = M - int \amalg V_i$ which is a circle bundle over F_0. Then you can choose a basis (c_i, t) for δV_i, where $c_i = \delta_i F_0$, where F_0 is F minus n open disks.

Exercise 2 *The meridian of* V_i *is* $q_i c_i + p_i t$, *and* t *is a* (p_i', q_i)*-curve on* V_i *where* $p_i' p_i \equiv 1 \mod q_i$.

So $M = F(p_1/q_1, \ldots, p_n/q_n)$.

So now we're getting to interesting examples of a double branched covering. The double branched cover of the four lines connecting an exterior sphere to an interior one in a trivial way is $T^2 \times I / A^2_\lambda \times S^1_\mu$. So now when you have two interior spheres connected like this, you identify the two annuli along an arc. So it's a pair of pants cross S^1 in the double branched cover.

A Montesinos tangle $M(p_1/q_1, p_2/q_2)$ is the tangle that comes from attaching two rational tangles of the appropriate index horizontally. Then $\tilde{M}(p_1/q_1, p_2/q_2)$ is a disk $D^2(p_1/q_1, p_2/q_2)$. Similarly, you can attach more than two of these together.

For example, let's take this Montesinos tangle. This is $M(1/2, 1/3)$, so the double branched covering $\tilde{M}$ is a Seifert fibered space over the disk with two exceptional fibers of multiplicities $1/2, 1/3$. So it's the exterior of the trefoil.

Let me just finish with an exercise to get you used to this stuff. This is the basic setup, it's not very exciting yet, maybe one more bit of notation here. Obviously you can define

Montesinos knots or links, in the same way that you capped off the rational tangle to get a rational knot, by definition, you take the corresponding Montesinos tangle and then cap it off as a marked tangle. Let me let you ponder one example. I'm going to define a tangle in the three-ball. $\mathscr{B}(\alpha,\beta,\gamma)$ for $\alpha,\beta,\gamma\in\mathbb{Q}$. Then here's the picture. That has one boundary component, and this is a rather magical, due to Eudave-Muñoz. It's the source of many examples.

Exercise 3 $\mathscr{B}(-2,3,-2/3)(1/2)$ *isss the unknot. Can you generalize this to other* α,β,γ. *If you fill in and get the unknot, then the branched cover is the three sphere, so without drawing a knot in* S^3, *but by focusing on tangles downstairs can find interesting Dehn surgeries.*

3 Etnyre

This is John Etnyre. He'll be talking about contact geometry, but I think it's a very important course, bringing together things in the first and second halves.

I want to start with talking about how contact geometry has come into play in low dimensional topology. A lot of that has happened in the last three or four years, there had been hints for a long time. So contact geometry is a key tool in the following recent results (I'll define things shortly, but this is motivation)

1. Kronheimer-Mrowka's proof of property P for nontrivial knots; nontrivial surgery on a nontrivial knot gives you something with nontrivial fundamental group. There's a lot that goes into this, but some of it uses some contact geometry
2. Ozsvath-Szabo's proof that the unknot, trefoil, and figure eight are determined by surgery. I mean that if you do rational surgery on any knot, and it comes out the same as for one of those knots, then it had to be that knot.
3. Ozsvath-Szabo's proof that Heegard Floer invariants detect the Thurston norm of a three-manifold and genus of a knot.

I could go on and list quite a bit more. For example, Heegaard Floer determines fibered knots. Let me not belabor the point.

I want to give a sketch of the contact geometric part of the three of these. This is a sketch, I hope to go over the definitions later.

Start with an irreducible three-manifold M, and take a surface Σ in this manifold, oriented, such that, with minimal genus among surfaces homologous to it.

1. Gabai gives a taut foliation $\mathscr{F}$ that contains Σ as a leaf. Roughly you fill the three-manifold with surfaces stacked together.

2. Eliashberg and Thurston perturb this foliation into contact structures, positive and negative, $\xi_\pm$ that are C^0 close to $\mathscr{F}$.

3. They also give you a symplectic structure on $M \times [-\epsilon, \epsilon]$ that fills $(M, \xi_+) \amalg (M, \xi_-)$.

4. Eliashberg and independently Etnyre give a closed symplectic manifold X that $M \times [-\epsilon, \epsilon]$ embeds in. You have $M \times [0, 1]$ with these bowed out boundaries, and you can cap this off with symplectic caps. You use for this

4a. Giroux's relation between open books and contact structures.

4b. Constructions of Eliashberg and Weinstein (contact surgery and symplectic handle attachment in dimension four.) that allow you to construct symplectic structures by attaching one and two handles.

5. Use Seiberg-Witten or Heegaard-Floer theory to conclude something about M and Σ based on the existence of X.

Let me give a concrete example about how part five works out, since I won't be going over it. These have strong nonvanishing results for symplectic manifolds. For instance, the Heegaard Floer invariants of X are nonzero since X is symplectic, so $HF^+(M, \mathfrak{s}_{\mathscr{F}})$ is nonzero, where this is the $Spin^c$-structure that comes from $\mathscr{F}$. It is known that $|\langle c_1(\mathfrak{s}_{\mathscr{F}}), [\Sigma]\langle| \leq 2g - 2$, but since Σ is a leaf of $\mathscr{F}$, we have $|\langle c_1(\mathfrak{s}_{\mathscr{F}}), [\Sigma]\langle| = 2g - 2$. This doesn't work for Σ equal to S_2.

We're going to spend the next five lectures getting familiar with this terminology. I want to explain part two pretty well, the perturbation into the contact structure, and then the symplectic form on $M \times [-\epsilon, \epsilon]$ and the construction of the caps.

but for now,

3.1 Part I, basic ideas and definitions

Throughout this lecture series M is an oriented 3-manifold. The orientations are important in contact geometry.

a plane field ξ on M, can locally be given as the kernel of a one-form α. If you think about it it's fairly obvious.

Exercise 4 *This is orientable if and only if α can be chosen globally.*

Here's an example. Let $\mathbb{R}^3$ be M and $\alpha_1 = dz$. Then ξ_1 is the span of ∂_x, ∂_y. This is a very boring plane field.

Here's a more exciting one. Let $\alpha_2 = dz - ydx$, $alpha_3 = dz + ydx$. Let ξ_i be $\ker \alpha_i$. As long as the y-coordinate is zero this is the xy plane. At $x = 0$ and $y = 1$, you can see that this is tilted forty degrees. This does a $90°$ left-handed twist as you go out to infinity. ξ_3 does a ninety degree right-handed twist.

Definition 1 *1. ξ is a foliation if $\alpha \wedge d\alpha \equiv 0$.*

2. ξ is a positive (negative) contact structure if $\alpha \wedge d\alpha$ is never zero and defines the given (opposite) orientation on M.

3. ξ is a positive (negative) confoliation if $\alpha \wedge d\alpha$ is nonnegative (nonpositive)

Exercise 5 *Show that this definition doesn't depend on α.*

Most people would write $\alpha \wedge d\alpha > 0$. This would mean it's a positive multiple of the volume form, just like in the confoliation definition.

Theorem 5 *(Frobenius)*
If a plane field ξ is closed under [,] (meaning that the bracket of two vector fields in the plane field is still in the plane field) then $M = \amalg S_\lambda$ where S_λ is a surface and $\xi_x = T_x S_\lambda$

Exercise 6 *Show $\alpha \wedge d\alpha \equiv 0$ is equivalent to ξ closed under Lie bracket.*

Back to our examples.

1. For $\alpha_1 = dz, \xi = \ker \alpha_1$, you get $\mathbb{R}^3 = \cup S_z$, the horizontal planes. Here $T_{(x,y,z)} S_z = \xi_{(x,y,z)}$.
2. For $M = S^1 \times \Sigma$ and $\xi = \ker d\theta$ then $\xi_{(\theta,p)} = \{\theta\} \times T_p\Sigma$.
3. For $\alpha_2 = dz - ydx$ we get $d\alpha_2 = dx \wedge dy$ so $\alpha_2 \wedge d\alpha_2 = dx \wedge dy \wedge dz$. So ξ_2 is a positive contact structure. Similarly, ξ_3 is a negative contact structure.
4. Think of $S^3 \subset \mathbb{C}^2$, and let $\xi_4 = \ker\ \alpha_4$ where α_4 is $r_1^2 d\theta_1 + r_2^2 d\theta_2$.

   Exercise 7 *Check that $\alpha_4 \wedge d\alpha_4$ is positive on S^3.*

Any questions about these examples?

The burning question is, how prevalent are these? Here's a nice fact that with any luck we'll be able to prove. All oriented 3-manifolds have foliations and positive and negative contact structures.

Let's try to understand the contact condition a little better. The foliation condition means that the planes are untwisted enough that you can find a surface locally tangent. If it's a contact structure it will be too twisted to do that anywhere.

Lemma 1 *Given a plane field ξ one can find local coordinates such that $\alpha = dz - a(x, y, z)dx$.*

Exercise 8 *Prove this.*

Let me give you a hint. Starting near a point, try to find a disk transverse to the plane field. Then you should get a line of intersection. Then find a vector field tangent to the plane field and transverse to the disk and integrate.

[Is it clear that if I have a positive contact structure that I have a negative one?]

It requires proof. It follows that there's a negative contact structure on the same manifold with the opposite orientation.

Lemma 2

1. *ξ is a positive (negative) contact structure if and only if $\frac{\partial a}{\partial y}$ is positive (negative)*
2. *ξ is a foliation if and only if $\frac{\partial a}{\partial y}$ is identically zero*

Exercise 9 *Prove this.*

This is a nice local lemma. You have a pretty nice form for a contact structure or a foliation locally.

Theorem 6 *Darboux,Pfaff*

1. *If ξ is a foliation then you can take $\alpha = dz$ locally.*
2. *if ξ is a positive (negative) contact structure then you can locally take $\alpha = dz - ydx$ ($\alpha = dz + ydz$).*

These are not local, Riemannian metrics can look different at a point, but contact structures can't.

There's one very important difference between foliations and contact structures. There are differences, one big difference is the following

Theorem 7 *(Gray)*
If ξ_t is a family of contact structures for $t \in [0,1]$ then, a suspenseful pause here, there exists a one-parameter family $\psi_t : M \to M$ of diffeomorphisms such that $(\psi_t)_*(\xi_t) = \xi_0$.

If you move through a family of contact structures, this happened on the level of the manifold, you just dragged the manifold around. On the other hand, this is not true for foliations. For example, let $\mathscr{F}_s$ be the foliation of T^2 by lines of slope s. Now let $\xi_s = \mathscr{F}_s \times S^1$ which is a foliation of T^3. When s is rational, you're foliating T^3 by tori, and when irrational by annuli.

Exercise 10 *Show there is no family of diffeomorphisms $\psi_s : T^3 \to T^3$ such that $(\psi_s)_*\xi_s = \xi_0$.*

I'm going to leave you to think of the following:

Exercise 11 *Consider the level sets of $f : \mathbb{R}^3 \to \mathbb{R}$ that sends $(x, y, z) \to \alpha(x^2 + y^2)e^z$, where α is 0 at 1, 1 at zero, and is monotonically decreasing. This will foliate $\mathbb{R}^3$ in a very interesting way which we will talk about tomorrow.*

[This theorem of Gray, do you need M not to have boundary?]

Then you construct a vector field whose flow gives you the family of functions. You have to set up some condition to not flow off the boundary. There are some restrictions, I don't know them off the top of my head.

[How is this a family of foliations?]

The point is that everything is happening at the level of the tangent space, you can't look at the level of the manifold.

[Is there any knowledge of what the connected components of the moduli space of foliations looks like?]

That's very delicate, I don't know about that. Thank you, see you tomorrow.

4 Eliashberg

[Welcome to the second Clay lecture of this PCMI. As those of you in here earlier today have realized, there's a problem with the refrigeration. This could have one of two possible outcomes. Either some time during the lecture, the winds from the north pole will begin to blow, in which case we will close the doors at the appropriate moment. The other option is that it becomes unbearably hot. If I see anyone collapse, we'll stop the proceedings and decide what to do. Halfway through I will interrupt to check with the audience. If it really gets unbearably hot, we'll come up with plan B. Hopefully it won't come to that. We've lowered the lights, and it's a powerpoint talk, so...]

[Welcome to the second of the series of lectures. Eliashberg is one of the leading symplectic topologists and geometers. He got the Veblen prize, and is a member of the National Academy of Sciences.]

This is my first ever attempt to use the high tech, I don't know how it will go. The title of this talk is "Rigid and Flexible Mathematics." This is a picture of mathematics. On one side there are rigid problems, on the other the flexible problems, and in the middle a gray area. In the rigid world you expect to have very few solutions of your problem, solving a differential equation, say. Maybe it's unique. On the flexible side you have a problem, and you can maybe approximate every function with a solution. I used to think that rigid problems are the important ones, because, in nature things should be unique. But not so long ago I found out the flexible problems actually also belong to very applied mathematics, for instance crystal growing. Systems may have an enormous number of stationary states, and understanding the corresponding equations can really help out.

My personal taste, I was interested in symplectic geometry from very long ago. When I first came to this area, it was completely clear what to expect. Is there anything nontrivial to expect? You can state one problem, and you can think maybe there were many solutions and maybe none at all. All the symplectic geometry developed in the balance between the rigid and flexible world. I'l stop with the philosophy and I'm going to go over some examples. Then maybe you'll see what I mean.

Given smooth functions $f, g : [0, 1] \to \mathbb{R}$, find a function F such that F and F' approximate f and g, namely $\max\{|F - f|, |F' - g|\} < \epsilon$. Can you solve this problem? Of course not. If $g \equiv 0$ then the variation of f cannot be bigger than ϵ or you will be in trouble. Let's see if anything meaningful can be said along this line.

Now let's be on the plane. Suppose now we have three functions of three variables $f,\ g_1,\ g_2$. Can one find a function F of two variables so that it and its partials $\frac{\partial F}{\partial x_1}$ and $\frac{\partial F}{\partial x_2}$ approximate the three functions in a possibly smaller neighborhood of the unit interval? The answer is negative for the same reason, any solution would restrict to a solution of the previous problem. But if you slightly modify this question you get a positive solution.

Oh, the green doesn't come out very good. The theorem states the following thing, suppose we have this function and two derivatives in the neighborhood of the interval. We can't approximate it in the neighborhood of the original interval. But you can always find an arbitrarily small function such that in the neighborhood of this graph, the approximation is possible. You can wiggle the interval a little and in the neighborhood of the new one, this can be done.

If you would like to make the same problem in the neighborhood of a point, it would be possible. There you can use the Taylor expansion, which approximates a function near a point. You can choose a variable constant of approximation and correspondingly pick a size of a neighborhood, find many small neighborhoods where our approximation is possible. So we divide this up into N pieces, in the neighborhood of $1/N, 2/N$ et cetera. So schematically these are like overlapping intervals. The distance to the squares is $1/N$, and the size δ is very big compared to this.

So problem is they do not agree. If you could have them somehow overlap, there would be no problem.

So now instead consider the domain in this picture. This is a neighborhood of our interval. The width here is $1/n$. So you have n pieces. It's a little less than $1/n$, so that there's room to cut little slices. Now δ is much bigger than $1/n$, so this is a tall and narrow thing. Now construct a function such that on the first shape, first ignore this gluing, on the first gluing take f_1, on this one f_2, et cetera. Then change them slightly like this in the middle. Because the size, the length is very big compared to the width, I can make these two sections arbitrarily close. And therefore this kind of bending does not really change the function and its derivative. Inside this domain, I can now inscribe the graph of the function I want. So now inside this domain I can get the function I said.

So now, if you're a little bit more accurate, I can do this holonomy approximation too in a relative form. I don't have to change it if it's already okay on the boundary. I don't have to change anything. It holds in a parametric form if I have a family of functions.

It holds in a higher dimensional case. I didn't use much in the interval. For instance, in a neighborhood of a square in three dimensional space I do this along every line in the square and then I do this kind of second argument exactly like this in the second direction. So now I will get something I failed to draw properly.

Now of course you can't understand what is here. This is a graph of the function sin $x + \sin y$. That's exactly what happens in the first approximation. You get something with high frequency in both directions. It doesn't look very good, but I got help with MAPLE.

Now, application. Let's start with this famous theorem. Oh, my hand. So, the problem of immersions of circles into the plane. This was a problem which, in 1937, was solved by Whitney. Can you classify immersions of circles in the plane, that means smooth curves with self-intersection but without any cusp. This is up to regular homotopy so during the homotopy you also require that this curve remain immersed.

For instance this picture is prohibited for regular homotopy. So, well, let's kind of see how this follows from our previous theorem. to an immersion you associate it's Gaussian map, you take a tangent vector and normalize it, you get a point on the circle. It's the tangent winding number, or is sometimes called the Maslov index. This the only topological invariant of an immersed curve. Two immersed curves are regularly homotopic if and only if they have the same winding number.

So how does this follow from the holonomic approximation lemma? Suppose we are given two immersions. What do we do? First we replace this immersion with an immersion of thin annuli. The problems of immersions of a circle and immersions of thin annuli are the same problem. Now any two maps of an annulus to $\mathbb{R}^2$ are homotopic if you don't require anything. Choose such a homotopy F_t.

Also the fact that the Gausseian maps are homotopic allows me to construct a homotopy map G_t for the derivatives F_0' and F_1' through nondegenerate linear maps.

So the parametric form of the holonomic approximation lemma then allows me to find a homotopy that connects F_0 and F_1, that is C^0-close, and the derivatives are close to the derivatives in G_t, so in particular nondegenerate because they are close. The homotopy was defined on kind of the formal neighborhood of the curve, which is okay because the property of being an immersion is invariant under that, and this is the whole proof.

Let us then go more into explaining what is going on. Okay, here's, I just want to, here's the whole philosophy of all the results of the holonomic approximation lemma. You have a curve, or a more high dimensional manifold. But let's take a curve.

Think that this curve is made out of long molecules. And they can overlap, they can sit on top of each other. At some moment you allow them to become loose, and they start to turn, and for instance they become perpendicular to one another, and start to move independently of one another, but still there is some kind of interaction force, and nearby molecules kind of move in a close way. Sorry, here's what we get. This is the line, and then this is molecules. They can fill out to become loose and then for instance they start to turn and they turn like this. And in fact they didn't really turn because the end of them have to be connected, in the process of turning, they are drawn in red and there is some kind of black connector drawn between them. This is the kind of process. Here is a line of two molecules. They kind of move in the space arbitrarily. But they stay kind of close. At any moment you can join them with a zig-zag, whatever they do. This will work in all dimensions.

The same argument shows you the following theorem is true. Take two functions on the small annulus. One function is $x_1^2 + x_2^2$ and the other is $-x_1^2 - x_2^2$. Can you find a deformation from one function to the other which in the process doesn't have any critical points? If you try to do it directly, it kind of looks impossible but you can apply this theorem. So there is the famous Smale theorem, which, this solves the immersion problem and so much more, but this famous Smale theorem, he proved that an immersion of S^2 to $\mathbb{R}^3$, all of them are regularly homotopic to the standard embedding. So what does that kind of prove?

It's the same proof. First you move from the problem of the sphere to one of annuli. Then one needs to use the Holonomic Approximation Lemma to decouple the homotopy of derivatives from the maps. Then the only thing left is to decide when the derivatives are homotopic as nonsingular things. Here you want to apply a piece of topology, the space of nondegenerate linear maps $\mathbb{R}^3 \to \mathbb{R}^3$ is homotopic to $SO(3)$, the group of rotations. There is an obstruction in the second fundamental group and the obstruction is actually zero. This piece is not so much visualizable. You can do it explicitly. There is the following counterintuitive corollary, which makes the Smale theorem so famous. One can take a sphere and invert it inside $\mathbb{R}$. Unfortunately I failed to interpolate it into this presentation so let me try to go back and show you some movie.

Okay, so, I don't know how to, now.

[Hit the space bar.]

What?

[Movie]

Of course, it's a beautiful picture but if you think for a second what is going on you see that it's exactly what I told you.

Now I move to, that was a beautiful laugh, everything was flexible and kind of possible. Now let's do something that is not possible. Again, the question which I will ask would be really on the borderline, so it would not be clear a priori that it's not possible. And the fact that it is impossible creates a lot of beautiful mathematics.

Okay, so let's start with this question, not question but just discuss a little bit of contact geometry. So given a function you can consider a curve in three space that is simultaneously the graph of the function and it's derivative. So that is $\Lambda_f = \{(q,p,z) | z = f(q), p = f'(q)\}$. So this graph satisfies this kind of differential equation, because dz=pdq, and then it's tangent to this nonintegrable plane field, which is an example of a contact structure. Here is a picture. Unfortunately, at the last moment I changed my notation from x, y to p, q but I forgot to change it on this picture. So y is p and x is q.

So they can rotate as I do this. And Legendrian curves are tangent to this one. A generic Legendrian curve does not need to be graphical, and its projection to the qz plane is called the front of the Legendrian curve, and it looks like this. I can think about a Legendrian curve as a multi-valued function. So each piece of this front, which is graphical, is the graph of some function. So here is the derivative of this function. Here's a great exercise if you are teaching calculus. This is the function, compute the derivative, and here is the derivative, compute the function. How do you compute the function? You integrate, count the area. So maybe it's not exactly true on this picture, but maybe it is.

You can say, whether it is embedded or not. Here the projection intersects, but the third coordinate is p, so unless two branches are tangent, at this point you don't have an intersection. And at the cusp point, it's tangent to the p axis. There's no actual singularity on the curve, only on the picture.

so anyway, so let's start with Rolle's theorem, which says a smooth function on the interval which is the same on the ends much have a vanishing derivative at a point in the interveal. How to say it in the contact geometry jargon is that the corresponding Legendrian curve which is the graph of all the partial derivatives cannot be contained in the upper half-space. Does the same hold true for any Legendrian curve, even if it's not graphical. The answer is clearly negative. Here is an example. The value at this end is lower. Let's try some other condition. Let's suppose we know the derivative in the pq-plane is not necessarily graphical but embedded. Then the answer is true. Because $z = \int pdq$, Since it's embedded it bounds some positive area, and [unintelligible].

Moreover the answer is positive for Legendrian curves which are Legendrian isotopic to graphical ones. Let's look at this picture. Look at two planes in 3-dimensional space, $q = 0$ and $q = 1$. Take the space of all Legendrian curves, not self-intersecting, with one end on one and one end on the other. Actually what I wrote here, I forgot one condition. Moreover I want, this space can contain many components, I want the component that contain graphical curves.

So and then, the front, if you have such a curve that is Legendrian isotopic to a graphical one, then the Rolle theorem also holds.

Here is kind of the picture, so you see, the front would have to have a form like this. In order for me to explain how this is proved, in fact I need to go to higher dimensional geometry. This looks like a trivial result about fronts. Well, you can try to prove it this way, look at all possible deformations. You just move this front, in such a way that during the whole of your deformation you are not allowed to have any tangency. It is possible at some point, some moment, some kind of zigzag like this is created. It corresponds in Legendrian projection to derivative direction. It's a kind of just [unintelligible]like this. It's a deformation which is allowed [unintelligible]

You can, it's possible to prove it this way, but it's extremely complicated. So let us do it the other way. Let's extend to n dimensional case. Suppose you have a function and the graph of its gradient. We consider, we have a q, p space the coordinates of p are our partial derivatives, and also in $2n+1$-dimensional space add one coordinate and also register the value of the function.

Then again, like before, we can consider this differential form pdq. And this graph, the simultaneous graph of its function and its derivative is again tangent to the contact structure, a manifold which satisfies this property again is called Legendrian.

The form pdq restricted to this Legendrian manifold is just df. And this one-form, the differential of this, ω, then this form will vanish on some submanifolds, and submanifolds that satisfy this property are called Lagrangian. And f is called a generating function for L_f. Let me go back, and explain again, for the one-dimensional case, we had the projection for p, q and there it was an arbitrary curve. In the two dimensional case, it would already be a very constrained surface. We'll talk in a bit more about that.

In general Legendrian and Lagrangian submanifolds are not of course necessarily graphical. They correspond not to functions but to fronts, which can be viewed as multivalued function. Here is the fundamental new idea. Sometimes you can generate with functions which are not defined on our original space but on the product with a high dimensional auxillary manifold.

So this is, in fact, it turns out, that in our problem is one dimensional, it's about a curve on the plane. It's extremely complicated combinatorics. But then you restate it as a problem about functions on a high dimensional manifold. There it becomes much more transparent. So first kind of simple lemma. Suppose you have a function of q and an extra variable v which lies in a high dimensional space. Let's define again a submanifold of $\mathbb{R}^{2n}$ and $\mathbb{R}^{2n+1}$ by these equations.

$$L_f = \begin{cases} p = \frac{df}{dq}(q, v) \\ 0 = \frac{df}{dv}(q, v) \end{cases}$$

$$\Lambda_f = \begin{cases} q = \frac{df}{dq}(q, v) \\ z = f(q, v) \\ 0 = \frac{df}{dv}(q, v) \end{cases}$$

[Ed.: These formulas, esp. for Λ_f, do not seem to be correct; in particular shouldn't it have some p-dependence?–but they were copied down from the slides.]

Then the claim is that the L_F is Legendrian and the corresponding Λ_F is Lagrangian. For topologists this is known as a Cerf diagram. What, this corresponding front of this thing. You have this space $\mathbb{R}_q$, and you just think of the function as a family of functions on V parameterized by $\mathbb{R}_q$, and look at the critical values.

So again, we call this F a generating function (or family) for the Legendrian and Lagrangian manifold. Suppose V is a vector space which is split as a sum of two vector spaces, and that our function at the infinity of V is a quadratic nondgenerate form. Then we say that our function is a quadratic generating function, quadratic at infinity. The usefulness of this can be explained by this lemma.

If we know the front F can be generated by a function quadratic at infinity, not any function, First of all, critical points of the function are in one to one correspondence with intersections of this Lagrangian manifold with $p = 0$, this is always true. If F is quadratic at infinity, then the front contains the graph of a continuous piecewise function. Let's go back to this picture.

So you see, you have this front, this front contains in it the graph of a piecwise smooth and continuous function, unlike this front, which does not contain such a thing. Okay, so how to prove this? one direction is just trivial. You just, if you take a critical point, then not just the derivative on the fiber is zero but the local derivative is zero. That's just [unintelligible]. But, uh, to prove the other direction requires a little more. Consider the space of embeddings of this negative subspace of $\mathbb{R}^n$ such that at infinity they just look like flat space. For each of these consider the minimax value $\varphi(q) = \min_\lambda \max_v f(q, \lambda v)$. Suppose you have a quadratic function. A quadratic function has one critical point. As long as you look like a quadratic function near infinity. Then there is a canonical way to pick one critical value.

What you do, you consider all the subspaces hanging down, and you pull it down as much as possible, and that's the minimax value, the critical point. It has this fantastic property that if you have a family of functions, then the minimax changes continuously, it cannot jump. That's the whole proof. You construct the generating function, get the minimax value, You get the graph inscribed in [unintelligible]

Here's the theorem that, if you have, say a Legendrian manifold, which is Legendrian isotopic to the zero section, or a Lagrangian manifold that is Lagrangian isotopic to the zero section then they can always be defined by a generating function, and hence it would prove the theorem that we hoped.

So a few remarks. First, any Lagrangian isotopy of the zero section lifts to a Legendrian isotopy of the zero section in the space plus one. By definition of Lagrangian, you just add a [unintelligible]. So, but, what is kind of more interesting is that the Legendrian claim just in turn follows from the Lagrangian. This is a general thing if you are in symplectic geometry. You cannot be contact geometers or symplectic geometers. You cannot be a specialist in even dimensional things or only odd dimensional things. Of course, there are three dimensional topologists and four dimensional topologists, but even they sometimes somehow talk to one another.

Here there is continuous interaction. Every symplectic statement in dimension $2n$ usually implies a lot about contact geometry in dimension one less, and vice versa. It's important then. It turns out that there are things called Legendrian and Lagrangian that reduce the question to [unintelligible]if you know it in dimension one more.

So what it does is the following thing. The claim is the following thing. Suppose we have the odd dimensional space, and you have a Legondrian manifold, you can always embed it as an affine space in one dimension bigger symplectic space and find a Lagrangian manifold there whose intersection with this space is a Legendrian manifold. So the generating function restricted to the subspace, will become a generating function for the Legendrian. I don't have time to explain this. It's very elementary geometry.

But now, I need to somehow explain how to find this generating function. To do this I need to discuss some Hamiltonian dynamics. In fact symplectic geometry was born for mechanics. Hence this is a mechanical terminology.

Let's think about $\mathbb{R}^n$ as a configuration space for some mechanical system and the symplectic manifold $\mathbb{R}^{2n}$ as the phase space for the system. So the q are coordinates and the p are momentum. So the coordinates any function of these two variables generates a dynamics given by the Hamiltonian equations $\dot{p} = -\frac{\partial H}{\partial q}$ and $\dot{q} = \frac{\partial H}{\partial p}$. Here dot means the time derivative. So the corresponding vector field $-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}$ is called a Hamiltonian vector field. It's useful when you think about this as a complex space. Then you take the Hamiltonian vector field to just be $i\nabla H$. In particular, this vector field is tangent to this H. An important basic fact about the flow of Hamiltonian vector fields is that it preserves, in $\mathbb{R}^{2n}$, if you have, then any isotopy that preserves the symplectic form is generated by a Hamiltonian that could probably be time-dependent. So the flow defined by this equation, we call this flow the Hamiltonian flow. In particular, the Hamiltonian generates a Lagrangian isotopy. And the converse is also true. If you take $\mathbb{R}^n$ zero section and deform it by a Lagrangian submanifold, it's the same as deforming it by a globally defined Hamiltonian.

So let's take a Lagrangian manifold and any Lagrangian isotopy of the zero section. I'll explain how to construct the isotopy for the last moment for the manifold L_1. These functions are kind of God-given. We have no choice. We need to construct this function, we need to find some vector space V split this way and find a function on this space such that at infinity of V it looks like a quadratic form, and such that our Lagrangian is defined by the equation. Any Lagrangian isotopy can be generated by a Hamiltonian. We always can make the Hamiltonian to be zero for large values of p.

Consider the space of paths in $\mathbb{R}^{2n}$ with ends in q-space and p-space. This is infinite dimensional, and this would be kind of my view. I will be cheating, I will construct functions and multiply by an infinite dimensional space. But it's kind of a good cheating. Then you consider $\mathbb{R}^n$ times this; This can be viewed as the paths that begin at 0 and ending anywhere in the space. Now I consider the following action functional on this space.

It is just equal to $\int_{v_q} pdq - H_t dt$. This is a function of q, v, and then there is a fundamental principle of mechanics called the least action principle that says the critical values of this function are precisely the trajectories. I won't tell you the appropriate conditions, but the condition I have is exactly appropriate. In other words, what do we get? A path is a critical value of this $S|_{V_{\bar{q}}}$ if and only if for all t, $v_q(t)$ is just the image under our isotopy of [unintelligible].

Moreover, elementary calculus of variations show that if v_q is a critical point then the q derivative is precisely $\bar{p}$ endpoint of our path. So now we have two equations. This is a critical point restricted to our fiber. And these equations together mean that S is a generating function for our Lagrangian. That's the conclusion.

Now we only need to understand why it's quadratic at infinity. This action functional has the pdq and $H_t dt$. The whole behavior at infinity is determined by the behavior, pdq, and so the space of this paths, it's determined by smooth loops which are symmetric with respect to reflection along p and q. I had this path, I could reflect it along these axes and get a loop that is symmetric. So now, this loop if you think in complex space, satisfies these conditions, $w(-t) = \bar{w}(t)$ $w(t+\pi) = -w(t)$. The corresponding thing in the value of this integral is just one quarter. I write a Fourier expansion of this, you can see the coefficients must be real, and the odd coefficients are the only ones, no even coefficients. Then I can compute $\int pdq$ as the sum of the area of the projection of this loop.

Then you get the following quadratic form: [omitted]. It's a nondegenerate quadratic form, and our action functional, it differs from this quadratic form by this term. This one depends on the derivative of v and this one does not. So it shows that at the infinity of p, by a small modification, asymptotically it is quadratic at infinity but in fact you can make it quadratic at infinity with respect to this splitting by positive harmonic and negative harmonic. This is an analogue of harmonic functions in the disk. We ignored a lot of details, particularly that this is an infinite dimensional space.

This can be fixed with a little bit of analysis, and I'll stop here.

[Any questions?]

[Was that flexible or rigid?]

Okay, so, I was proving that something, I was proving, when I say something is flexible, I can prove that something is possible when it's seemingly impossible. It's rigid when you can prove it impossible and there's no reason. In this case in the first question about holonomic approximation, before I knew this lemma if I had been asked if I thought this was possible, I would have said no.

In this case you look at this question, then I thought about this question a long time ago, it looks almost possible to construct an example where this is not true. If you think about this, and try to construct this, for about ten years you can cheat yourself trying to find some delicate construction that isn't quite true. It's important to understand that there is this difference between the flexible and rigid world. Unfortunately it's not quite true what is flexible and what is rigid. But it's important to use flexible tools for flexible things and rigid tools for rigid things.

Otherwise you waste a lot of time.

[In your definitions of flexible and rigid, which by the way seem kind of flexible, is the problem itself flexible]

Flexibility is a property of a particular problem. A particular problem is rigid or flexible. But this is only known after you have solved it.

But if you have the right experience, maybe you recognize this. So You know that there are flexible problems and rigid problems, If you come upon a rigid problem and try to deal with this by flexible methods, so you try to deal with this, and you, one year you almost solve it but it somehow didn't quite work. The next year you find another beautiful solution, but it didn't work somewhere else. So then maybe it's time to think that it's a rigid problem.

This flexible-rigid kind of dichotomy, these surprising results keep coming. It's not like Gromov, Smale, Witten, Nash, solved all the flexible problem.

Because for instance, just recently, Thomas Fogel proved a beautiful theorem about the existence of [unintelligible], I don't understand how it means. For many years they tried to prove existence. and somehow, I still believed some is rigid. Then he found some ingenious construction proving that it is flexible.

Of course we have a lot of examples, in topology, in my definition, the result about, showing [unintelligible], this is a result of rigid mathematics, nad Perelman's proof of topological Poincaré conjecture is rigid.

[Any other questions?]

How many people survived in terms of heat?

[Let's thank Yasha.]

I am completely inexperienced in this high tech thing. I had the help of many of my students and others. I thank all of them.

5 Walsh, Commensurability classes of 2-bridge knot complements

This work is joint with Alan Reid.

Thanks for coming. Should I turn this on? Okay, so, thanks for coming, I know it's like the end of Monday. So, when are two three-manifolds commensurable?

Definition 2 *two 3-manifolds M_1, M_2 are commensurable if they have a common finite sheeted cover M.*

This is precisely the condition that their fundamental groups have a common finite index subgroup. This is an equivalence class.

For most of the talk M_1 and M_2 are hyperbolic manifolds $\mathbb{H}^3/\Gamma_i$. So an equivalent formulation is that Γ_1 and a conjugate of Γ_2 have a common finite index subgroup.

A lot of properties we are interested in are properties of commensurability classes. Being virtually Haken, β_1, or virtually fibered are all properties of commensurability classes.

However, it's really difficult to determine commensurability. So we want to show that every commensurability class has these properties.

We're making small steps. Today, what we have is

Theorem 8 *(Reid-Walsh)*
If K is a hyperbolic 2-bridge knot then $S^3\backslash K$ is the unique knot complement in its commensurability class.

Schwartz showed that two non cocompact Kleinian groups of finite covolume are quasiisometric if and only if they are commensurable. If you were a geometric group theorist you could view this as being about knots.

Corollary 1 *If K is a hyperbolic two-bridge knot and K' is another knot in S^3, then if $\pi_1(S^3\backslash K)$ is quasiisometric (in the word metric) to $\pi_1(S^3\backslash K')$ then $K \cong K'$.*

Example 1 *The $(-2, 3, 7)$ pretzel knot that we've been talking about in Cameron's class, admits two different lens space surgeries. If K admits a non-trivial lens-space surgery, then $S^3\backslash K$ is nontrivially covered by a $S^3\backslash K'$. You look on the complement of your knot, it's cyclic, and a cyclic covering of a knot complement can only have one cusp*

Proposition 1 *(Gonzales-Acuña, Witten)*
K admits a cyclic surgery (Filling gives a manifold with cyclic fundamental group) if and only if $S^3\backslash K$ is covered by $\Sigma^3\backslash K$. This Σ is S after Perelman.

So the $(-2, 3, 7)$-pretzel has at least three knot complements in its commensurability class.

So we hope that the following result is true.

Theorem 9 *(Takahashi)*
hyperbolic two-bridge knots do not admit cyclic surgeries.

All the nonhyperbolic two-bridge knots are torus knots and they're all commensurable with one another.

We need some invariants of the commensurability class. So $C^+(\Gamma) = \{g \in PSL(2, \mathbb{C}) | [\Gamma : \Gamma \cap g^{-1}\Gamma g] < \infty\}$, that is, if I conjugate by Γ then I have finite index.

Theorem 10 *(Margulis) $C^+(\Gamma)$ is Kleinian and Γ has finite index in it if and only if Γ is not arithmetic.*

Theorem 11 *(Reid)*
The figure eight knot is the only arithmetic knot.

So assume everything is non arithmetic. Then the Γ has finite index in the commensurator group $C^+(\Gamma)$.

So let's study the commensurator quotient, $\mathbb{H}^3/C^+(\Gamma)$. So here is a picture, where this covers some crazy orbifold. But the cusp is a Euclidean orbifold. It's covered by the torus. So using the orbifold Euler characteristic there's only a few things it could be. You can do this as an exercise if you don't know it and you're bored of the talk. The only Euclidean orbifolds are $S^2(2,4,4)$ (this has three singular points of order two, four, and four), $S^2(2,3,6)$, $S^2(3,3,3)$ (these three are rigid, can't be deformed), $S^2(2,2,2,2)$, and T^2.

Definition 3 Γ *is said to have hidden symmetries if the commensurator quotient is strictly bigger than the normalizer.*

Upstairs you have some symmetry, and when you go back downstairs there are things that you are commensurable with which don't cover you and which you don't cover.

Theorem 12 *(Neumann-Reid)*
The following are equivalent for $K \subset S^3$ not the figure eight knot.

1. *It has hidden symmetry*
2. *$S^3 \backslash K$ nonnormally covers some orbifold.*
3. *$Q = \mathbb{H}^3/C^+(\Gamma)$ has a rigid cusp.*

When I mod out by symmetries I can't have an order three thing fixing my knot, a longitude has to go to a longitude.

Now I need to know if this thing has hidden symmetries to start. They're difficult to deal with in general.

Fortunately, we proved

Lemma 3 *Hyperbolic 2-bridge knots do not admit hidden symmetries.*

[Why is it important for it to be a knot complement?]

It will have order at most two because you have to take a longitude to a longitude.

Any questions? Am I going to fast or too slow, too boring?

[Do the hidden symmetries have to do with blowing up?]

I don't think so, I don't know how, maybe.

The trace field is the field generated by the traces of Γ in $PSL(2,\mathbb{C})$. It's a theorem that this is a finite extension of $\mathbb{Q}$. Then the invariant trace field $Q(tr(\gamma^2), \gamma \in \Gamma)$ is an invariant of commensurability classes. For knot complements this is the same as the trace field.

Theorem 13 *(Hoste-Shanahan)*
Used this to show that twist knots are pairwise incommensurable.

What's really special about two-bridge knots? Our theorem depends a lot on the structure, which has been done to death and is well-understood. So first of all they have this nice presentation, $\langle x_1, x_2 | r \rangle$, where r has a very special form from the picture of it. I'm not going to tell you it. and secondly, Riley, who needs a star by his name for being so helpful, studied $p : \pi_1(S^3 \backslash K) \to PSL(2, \mathbb{F})$, where $p(x_1)$ and $p(x_2)$ are parabolic. He called these p-reps.

When $\mathbb{F} = \mathbb{C}$ we have $p(x_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $p(x_2) = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ where y satisfies a monic polynomial with constant term 1, is an algebraic integer and a unit.

Lemma 4 *(Riley)*
Let K be a a hyperbolic two-bridge knot with trace field k. Then $\mathbb{Q}(i)$ is not a subfield of k.

Theorem 14 *(Conway, in Sakuna)*
The symmetry group of a two-bridge knot complement is $\mathbb{Z}_2 \oplus \mathbb{Z}_{\neq}$ or D_4, and no element acts non-freely.

[Some pictures.]

We understand the symmetry groups well and there are no hidden symmetries. We know as much as you can know. If you mod out by this, sometimes you'll have this extra symmetry. You'll have an orbifold with some singular locus. If the singular locus has a symmetry then you have D_8. I can characterize them but I don't have it written down. This is an orientation preserving symmetry group.

[The picture is that if you have four twists here and four here then you can rotate.]

There are no hidden sypmmetries. The way you do this is you roll out the cusps. Assume that Γ_K has $\mathbb{H}^3/\Gamma_K = S^3 \backslash K$, and likewise for K', where K is a two-bridge knot. Then $\Gamma_K = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \right\rangle$. So I'm going to conjugate to make these directly commensurable. Now $\Gamma_{K'} < N(K)$ the normalizer. Let $\Gamma = \langle \Gamma_K, \Gamma_{K'} \rangle$, and $\Delta = \Gamma_K \cap \Gamma_{K'}$. All of these are normal because there are no hidden symmetries, that's a big assumption. Here's the diagram:

$$\begin{tikzcd} & \Gamma \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_K \arrow[dr, dash] & & \Gamma_{K'} \arrow[dl, dash] \\ & \Delta & \end{tikzcd}$$

where this is upside down with respect to the spaces.

So $\Gamma/\Gamma_K \cong \Gamma_{K'}/\Delta$. So Γ/Γ_K is a subgroup of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or D_4 so Γ_K/Δ is cyclic of order $1, 2, 4$. It can't be one of the two noncyclic groups because this has to factor through the Abelianization, $\pi_1(M) \to G$ factors through to $\mathbb{Z} \to G_{ab}$ so G_{ab} has only one generator.

It can't be one because $S^3\backslash K$ is covered by a knot complement, which would contradict Takahashi's result.

Tell me when I have five minutes so I can get to the wild speculation. You can argue that Δ has only one cusp as a cyclic cover of a knot complement, so that the covering of Γ_K is also cyclic, since a [knot complement] covering by something with one cusp is cyclic. Everything also has one cusp. Now I just need to argue that, a lemma, which is optional so I'll skip it but state it, is

Lemma 5 $\mathbb{H}^3/\Gamma$ *is an orbifold with a torus cusp*

It's an orbifold because otherwise it would be $S^2(2,2,2,2)$, so that $\mathbb{Q}(i)$ is contained in our trace field. You multiply some matrices to see this, in $\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & a \\ 0 & i \end{pmatrix} \right\rangle$.

Another lemma that I'll skip the proof of is, that the indices are all the same.

Lemma 6 $\Gamma/\Gamma_K \cong \Gamma/\Gamma_{K'}$, *which is cyclic of order two or four.*

I'll deal with the $\mathbb{Z}_2$ case. Assume that $x_1' \notin \Gamma_K$. If it did then $x_1' \subset \Delta \triangleleft \Gamma_{K'}$ which means $\Delta = \Gamma_{K'}$ so that $S^3\backslash K$ is covered by $S^3\backslash K'$. Similarly $x_1 \notin \Gamma_{K'}$, because then $S^3\backslash K'$ is covered by $S^3\backslash K$ but two-bridge knots don't have free symmetries.

So let's do Dehn filling on x_1^2. This will go to Γ_K/N. But $N = \langle x_1^2 \rangle_\Gamma = \langle x_1^2 \rangle_{\Gamma_K} = \langle x_1^2 \rangle_{\Gamma_{K'}} = \langle x_1^2 \rangle_\Delta$ because I can say $\Gamma = \langle \Gamma_K, x_1' \rangle = \langle \Gamma_K', x_1 \rangle$, and $\Gamma_K = \langle \Delta, x_1 \rangle$.

So now I can move to finite groups.

I get a dihedral group when I do Dehn filling on a two-bridge knot like this. I get

$$\begin{tikzcd} & G \arrow[dl, dash] \arrow[dr, dash] & \\ D_n \arrow[dr, dash] & & G' \arrow[dl, dash] \\ & C_n & \end{tikzcd}$$

where G is order $2 \cdot 2n$, G' is order $2n$, and C_n is order n. Here n is odd because it's a knot, not a link.

Now $\Gamma_{K'}/N$ is a manifold group because x_1^2 is primitive in $\Gamma_{K'}$ and $x_1 \notin \Gamma_{K'}$.

These are classified by Milnor. They have 4 dividing the order of the group which we can rule out, or they are cyclic. So it's a cyclic group. So it's covered by a knot complement in S^3.

Then I get

$$\begin{tikzcd}
 & & S^3\backslash K \arrow[dl, dash] \\
 & \mathbb{H}^3/\Delta \arrow[dl, dash] \arrow[d, dash] & \\
\mathbb{H}^3/\Gamma_K & \mathbb{H}^3/\Gamma_{K'} &
\end{tikzcd}$$

That finishes the order two case. I'm not going to go over the order four case.

All in the paper I wrote homotopy three-sphere, but it should be S^3. But the complement is not covered by any homotopy three-sphere because there's no cyclic cover.

I want to do some wild speculation. In the general solution, this is not true.

- You might have lens space surgeries, such as $(-2, 3, 7)$-pretzel.
- knots with hidden symmetries (Dodecahedral knots)

For a "generic" knot, meaning that there are no lens space surgeries, no symmetries, and no hidden symmetries, K is the only knot in its commensurability class. It's its own commensurator quotient, so it's the minimal element in its commensurator class. So it's not covered by any knot complement because it's got no lens space surgeries.

This is sort of like the generic situation in this way that I've made up.

Conjecture 1 *There are at most three knot complements in a given commensurability class.*

This follows, the way we understand it best, there's a famous cyclic cover theorem, you can only have two nontrivial lens space surgeries. We don't even know that it's only finitely many in the case where you have hidden symmetries. The other more true conjecture is

Conjecture 2 *If a knot has no symmetries or hidden symmetries then it is the only knot complement in its commensurability class.*

It's conjectured (Berge) that knots that have lens space surgeries have symmetries.

[Can you say onything about two-bridge links?]

This argument probably won't extend. It would be interesting to find things, but that's harder.

[Is there any hope that classical invariants would provide obstructions?]

It's possible that Ozsvath-Szabo Floer homology or the twisted Alexander polynomial have something to say. They have to have the same character variety and the volumes have to be rationally related.

[Let's thank Genevieve again.]