Low Dimensional Topology Notes
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1 Etnyre

Great, since it's been a few days since the last lecture, I thought I'd go over the program again. Recall we were working on steps two through four of the following program, which has given good results in low dimensional topology in the last few years: given a closed irreducible 3-manifold M and a surface Σ of minimal genus in its homology class (but genus not zero),

1. Gabai gives a taut foliation $\mathscr{F}$ with Σ as a leaf.
2. Eliashberg-Thurston perturb $\mathscr{F}$ into positive, negative contact structures $\xi_\pm$ on M.
3. Eliashberg-Thurston also give a symplectic structure on $M \times [-\epsilon, \epsilon]$ that fills $(M, \xi_+) \amalg (M, \xi_-)$
4. Eliashberg-Etnyre find a closed symplectic manifold X into which $M \times [-\epsilon, \epsilon]$ symplectically embeds. To construct the caps we need
   (a) Giroux's correspondence between contact structures and open book decompositions.
   (b) Eliashberg, Weinstein's ideas of symplectic handle attachment and Legendrian surgery.
5. use Seiberg-Witten, Heegaard Floer, ..., to conclude something about M, Σ from X.

Today I'd like to start talking about, I want to start with three and then $4b$.

All right, so, in the first four lectures we talked about the basics of contact geometry. We're now on step three, part three off the lectures, taut foliations and fillability. Recall that given a foliation on M we can perturb it to two contact structures, $\xi_\pm$, positive and negative. The real question is, what can we say about ξ_+? Is it tight? Is it fillable? Those are the things we'd like to know.

Recall if ξ is a Reebless foliation, then $e(\xi)$, the Euler class, evaluated on $[\Sigma]$, gives you a lower bound on $-\chi(\Sigma)$ if $\Sigma \neq S^2$, and is 0 if $\Sigma = S^2$.

If ξ_+ is C^0-close to ξ, then $e(\xi_+)$ also satisfies this inequality, which will imply that ξ_+ is tight.

So when we're perturbing foliations into contact structures, basically Reebless perturbs to tight.

There are refinements of this inequality, but Reebless does imply tight. This is more motivation than justification.

You don't actually need a Reebless foliation to get a tight contact structure. You could put a Reeb foliation on both solid tori in S^3, and depending on the parity choices of the direction of the Reeb flow, one such will perturb to be tight, and one will perturb to be overtwisted.

Exercise 1 *Figure out which is which, which is tight and which is overtwisted.*

Definition 1 *A foliation ξ is taut if each leaf of ξ has a closed transversal curve. Equivalently, there exists a vector field v transverse to ξ and a volume form Ω such that the flow of v preserves Ω.*

Exercise 2 *Show the definitions are equivalent.*

Suppose you have a taut foliation. How does that relate to Reebless foliations? If ξ has a Reeb component then it's not taut. It's easy to see this. If you transversally come into this thing, you'll be sucked into this swirling vortex and never be able to come out again.

Okay, so the taut condition is a strengthening of Reebless, but the main thing we're interested in is the theorem of Eliashberg and Thurston that says

Theorem 1 *If ξ' is a contact structure C^0 close to a taut foliation ξ then ξ' is weakly semifillable.*

This is actually fairly easy to prove. The symplectic filling will be $X = M \times [-\epsilon, \epsilon]$. Let $\xi = \ker\ \alpha$, and let $\tilde{\omega} = \iota_v \Omega$ with v, Ω as above. There are two obvious properties of $\tilde{\omega}$. First note that $\tilde{\omega}|_\xi > 0$. The transverse direction and then the two plane directions, with the volume form, it's positive if I choose my orientation light.

Then $d\tilde{\omega} = d\iota_v \Omega + \iota_v d\Omega = \mathscr{L}_v \Omega = 0$. Then set $\omega = \tilde{\omega} + \epsilon_1 d(t\alpha)$.

Exercise 3 *Check that ω is a symplectic form on $X = M \times [-\epsilon, \epsilon]$.*

Then $\omega|_{\xi_+}, \omega|_{\xi_-} > 0$ so (X, ω) is a weak filling of $(M, \xi_+) \amalg (M, \xi_-)$.

We now have lots of tight contact structures from the theorem of Gabai

Theorem 2 *If M is an irreducible three-manifold and Σ an oriented surface minimizing genus in its homology class, not equal to S^2, then there exists a taut foliation ξ with Σ as a leaf.*

Corollary 1 *If (M, Σ) are as above, then there exists a weakly semifillable contact structure on M such that $\langle e(\xi_\pm)[\Sigma]\rangle = \mp\chi(\Sigma)$*

We demanded a C^2-smooth foliation, but if Σ is genus zero, you might not get a C^2-foliation. But the breaking of the smoothness is not so bad and you can get the corollary in the general case, with a torus as well.

We can now move on to part IV, which is about constructing symplectic manifolds and Legendrian surgery.

Now we want to take $M \times [-\epsilon, \epsilon]$, and build a bigger and bigger manifold, to eventually get a closed manifold.

So let (X, ω) be a symplectic 4-manifold. A vector field v is symplectically dilating if $\mathscr{L}_v\omega = \omega$.

So suppose v is transverse to δX, which maybe I'll denote M, and v points out of X. Set $\alpha = (\iota_v\omega)|_W$, then $d\alpha = d\iota_v\omega + \iota_v d\omega = \mathscr{L}_v\omega = \omega$.

Then $\alpha \wedge d\alpha = (L_v\omega) \wedge \omega = \frac{1}{2}\iota_v(\omega \wedge \omega)$, which gives you a volume form on M so that α is a contact form on M.

So, a contact manifold (M, ξ) is strongly filled by a contact symplectic manifold (X, ω) if

1. $\delta X = M$
2. There exists a vector field $v \pitchfork \delta X$, pointing out, and dilating.
3. $\iota_v\omega$ is a contact form for ξ.

We also say (X, ω) has convex, or sometimes even strongly convex boundary. If v points into X, then δX is strongly concave.

So weak fillings seem to be perfectly good for contact geometry. Before I go on let me give you an exercise.

Exercise 4 *A strong filling of (M, ξ) is also a weak filling of (M, ξ).*

So why are we interested in this? What is strong convexity good for? The answer is gluing. Suppose we have two symplectic manifolds (X_1, ω_1) and (X_2, ω_2). Suppose one has concave and the other convex boundary. Call these contact structures ξ_1, ξ_2. If ξ_1 is contactomorphic to ξ_2, meaning there's a diffeomorphism preserving contact structure, then we can glue X_1 and X_2 together to get a closed symplectic manifold.

So the strong form of convexity allows us to do this sort of gluing. The weaker form, from weak fillability, which I didn't define, doesn't allow this.

Exercise 5 *Check this. This uses some ideas we haven't discussed, but it isn't terribly hard.*

Okay, so we've introduced this notion of strong convexity. Next we need to discuss how to discuss four-manifolds. I'm going to use handle decompositions. Let me give a couple of definitions. If we start with a 4-manifold X, then h^1 is a $D^1 \times D^3$, attached to δX along $\delta D^1 \times D^3$. To attach h^1 you just need to specify two points in δX you want to glue to.

Let me try to draw the picture in three dimensions. You specify the two points and then glue the handle on like this.

A two-handle h^2 is $D^2 \times D^2$ is attached to δX along $\delta D^2 \times D^2 = S^1 \times D^2$. Now to glue it I need to specify the core, S^1, and a framing to say how to glue the handle down.

Of course, again, let me give you the schematic picture in three dimensions. Note that if $X' = X \cup h^2$, then what is δX? What happens in the attaching region? It's $\delta X - S^1 \times D^2 \cup D^2 \times S^1$. This is a Dehn surgery, and in fact an honest surgery. So the boundary is obtained via surgery with framing given by the attaching framing, so I'm just doing integer surgery.

Exercise 6 *Check which surgery this is.*

A couple more definitions.

Definition 2 *Let (M, ξ) be a contact manifold. Then a knot K in M is called Legendrian if K is tangent to ξ, that is, $T_x K \subset \xi_x$ for all $x \in K$.*

There's a very special feature of Legendrian knots, you can ask how many times the contact plane flips around the knot. These give a framing of K.

Okay, so we're finally ready to state the following:

Theorem 3 *(Weinstein)*
If (X, ω) is a symplectic manifold with strongly/weakly convex boundary, and X' is X with a 1-handle attached or a two-handle attached along a Legendrian knot in δX with framing equal to one less than the contact framing, then ω extends to a symplectic form ω' on X' such that $\delta X'$ is strongly/weakly convex.

It turns out that in dimension four this is it, in higher dimensions you can always do a similar structure up to the middle dimension. In higher dimensions you can fix the framing so you can attach handles as you please.

A couple more definitions

Definition 3 *If (M, ξ) is the contact manifold filled by (X, ω), and $M' = \delta X'$, where X' was obtained from X by attaching a 2-handle along K, then M' has a natural contact structure ξ' and we say (M', ξ') is obtained from (M, ξ) by Legendrian surgery on K.*

In the weak fillable setting, lots of different contact structures can be weakly filled by the same manifold, but once you specify the structure on X, there's a unique one on X'. This is a minor point that you might be worried about. You should be, but I won't enlighten you.

Let me give you a sketch of the proof of the theorem. I'll describe this for the one-handles. In $\mathbb{C}^2$ construct a madel 1-handle. You can write a region $D^1 \times D^3$, and you can find a vector field v' coming in along the D^1 axis and leaving on the D^3 plane. The vector field is always coming in on the attaching side and exiting on the non-attaching side.

So assume (X, ω) is strongly convex. Now you have the region you want to attach to. We want v, v' to glue the handle on (X, ω).

Exercise 7 *Do this.*

The attaching region is two three-balls. Notice I've got, this vector field is supposed to be dilating. On the orange part it will induce a contact structure. The contact forms are the same near the center points of those three-balls. Then you can kind of put these ideas together and attach with a contactomorphism.

[What about weak fillings?]

You can slightly perturb things, it's fairly challenging but not undoable.

2 Fintushel-Stern

Oh, do you have some chalk? I like these. I'll try to use this though. I'll save this for the big theorem. That comes Friday.

First, I have some announcements. First, at 1:00 PM today, Peter Teichner will be talking about topological 4-manifolds. Second, MSU alumni and former postdocs should come for a photo op at 12:50 today in front of this building. Third, I thought as long as I'm going to be on the internet, I thought I should advertise that we have a tenure track position at Michigan State this year. I expect a lot of applicants.

As the other Fintushel-Stern, I'd like to begin with a short review. First of all, remarks about the blowup formula. My alter ego pointed out to me that I left out a hypothesis, an important hypothesis called simple type. There are more simple classes in the blowup if you don't have simple type. Remember that the Seiberg Witten invariants SW_X come from counting solutions to a differential equation modulo equivalence. There's a dimension involved $d(k) = \frac{1}{4}(k^2 - 3\sigma + 2e)$. So simple type means that whenever you have $SW_X(k) \neq 0$ then $d(k) = 0$. There are no known examples of simply connected four-manifolds with $b^+ > 1$ which are not simple type, that's a good problem.

Let me say something about log transforms. SO $E(n)$ comes with a fibration over the S^2 by tori, and this is not a fiber bundle, it has singular fibers. The most common singular fiber is a nodal fiber, and what that means is some 1-cycle on a typical torus has collapsed to a point. This is called a vanishing cycle, and you can see that nearby, this vanishing cycle bounds a disk. Look at a path in the two-sphere from a nonsingular fiber to a singular fiber. That gives a family of circles degenerating to a point, that's a disk. And this disk has relative self-intersection number -1.

Okay, so in an elliptic fibration, simply connected, there are many many singular fibers. You can see just because π_1 is zero you can see that the vanishing cycles are going to have to span the homology of the fiber. If F is the fiber then vanishing cycles α, β, span $H_1(F)$. So if you want to do a log transform on $N_F = F \times D^2$, then we can take as a basis $\{\alpha, \beta, [\delta D^2]\}$, and Ron mentioned that when you do a log transform with this basis you can write the resulta s $X_F(p, q, r) = (X \backslash N_F) \cup_\varphi (T^2 \times D^2)$ where $\varphi_*([\delta D^2]) = p\alpha + q\beta + r[\delta D^2]$. Ron pointed out for these $E(n)$ surfaces this only depends on the r, called the multiplicity of the log transform.

So I want to indicate why this might be true on the level of Seiberg-Witten invariants. So we're going to see that $SW_{X_F(1,0,0)}$ and $SW_{X_F(0,1,0)}$ are both zero in the Morgan-Mrowka formula. The 1 corresponds to α, the vanishing cycle which bounds a disk of self-intersection number -1. When you cut this open and do the surgery, you're going to cap this disk off with δD^2. So in $X_F(1, 0, 0)$ we've created an exceptional curve, a sphere E of square -1 built from the vanishing cycle disk and the surgery disk. This intersects the torus in one point. So you have $T \cdot E = 1$. So now, oh, and this is a torus of square zero, of course.

So what happens when you blow down this -1-sphere? I'll talk more about blowdown in a minute. You get a new manifold, b^- is decreased by 1, and T becomes a torus T' of square $+1$. But the adjunction inequality tells you that if there are any basic classes you can't have any tori of positive self-intersection, so SW is zero. Then $X_F(1, 0, 0)$ comes from blowing this up, so by the blowup formula you can see that $SW_{X_F(1,0,0)}$ is going to be zero.

Let me pose in front of this [Michigan State tenure track topology position.] PBS doesn't have ads, but they have sponsors. [Smiles]

Okay, so the other point I wanted to make is, Ron wrote out the formula. If you look at $\frac{t^{-r}-t^r}{t^{-1}-t} = t^{r-1} + t^{r-3} + \ldots + t^{1-r}$ So whenever you do a log transform you multiply the old Seiberg-Witten invariant by this, keeping in mind that the t here is a t corresponding now to a multiple fiber, so the old variable is t^r in this equation.

The last thing I wanted to talk about before rational blowdown is knot surgery. Recall you have a torus $T^2 \hookrightarrow X$ of square zero, it's essential, X is simply connected, and so is $X\backslash T^2$, the complement of the torus.

Then we took a knot in S^3, and formed $X\backslash N_T \cup_\varphi S^1 \times (S^3\backslash K)$, where φ takes a longitude of the knot to δD^2, where we identify N_T with $T^2 \times D^2$.

Let me indicate why we might think this might be useful. Look at $E(2)$. We have sections. So we have a section of self-intersection number -2. I can't apply the adjunction formula because it only works for genus at least two and self intersection number at least zero. So you take your section and add in a fiber, and you get a torus, because the fiber is a torus, you smooth out the intersection by replacing the crossing with an annulus. If this helps your vision you're as crazy as I am. So $F + S$ is a torus, that's a little joke for Ron and me, and, um, you get a torus with $T \cdot T = 0$, self intersection number zero, and it intersects the fiber once, since the section intersects the fiber once and the fiber doesn't intersect itself at all.

So now let's ask, can a multiple of the fiber possibly be a basic class? And so here's our adjunction inequality:

$$2g - 2 \geq T \cdot T + |T \cdot mF|$$

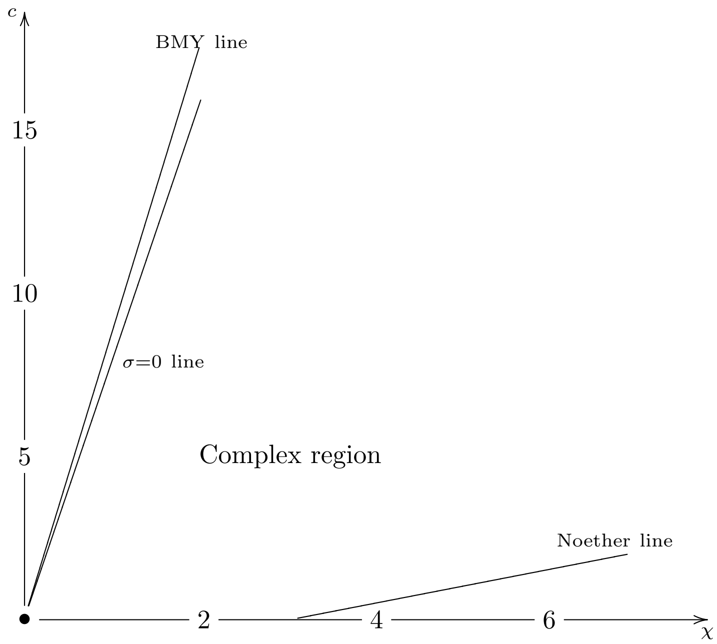
if mF is a basic class.

So this is $0 \geq 0 + |m|$ so that m has to be zero, so a multiple of a fiber can't be a basic class.

But on the other hand, suppose we first do knot surgery, and suppose that the knot K has genus g. So nowe the picture looks like this. Now we remove the neighborhood of a fiber and stuck in $S^1 \times S^3\backslash N_K$, and glued the longitude to the boundary of the disk. So the part of the section we cut out is replaced by a Seifert surface for K of genus at least g. So this, call it Σ, has self-intersection number 2 and genus g. Let's look at $F + \Sigma$. Now let, instead of T, you have $\Lambda = F + \Sigma$. Now Λ has genus $g + 1$ and still intersects the fiber once. Okay, so now let's apply the adjunction inequality, and we get $2g - 2$ has become $2g$. That has to be greater than Λ^2, which is zero, plus $|\Lambda \cdot mF|$, we're still testing whether a multiple of a fiber can be a basic class. So this is $|m|$. So we now have the freedom for up to $2gF$ to be a basic class. Recall the Alexander polynomial can be degree at least g, but you're evaluating it on t^2.

So knot surgery gives you the opportunity to have new basic classes.

The new operation I want to talk about today is rational blowdown. This is a picture I learned from complex surface theorists. I want to plot manifolds on the c versus χ graph.



This is called the geography of complex surfaces. This was probably named by Ulf [Klausen?] So, uh, I'd rather not plot my surface here. So the $K3$ surface lives right here, the $E(n)$ live along this line, and the log transforms live along this line. This is called the elliptic line. There's a famous line that goes this way, $c_1^2 = 9\chi$, this is called the Bogomolov-Miyaoka-Yau line.

Another line is $c = 8\chi$, which is $\sigma = 0$. There's another line $c_1^2 = 2\chi - 6$, called the Noether line, and complex manifolds that live on this line are called Horikawa surfaces.

The Kodaira classification of complex surfaces says that you're of general type, you're in this region.

Minimal means there's no 2-sphere of self-intersection number -1. There's nothing you can blow down.

So what is blowing down? You have a sphere of self-intersection number -1. You know that $\mathbb{CP}^2$ is built from a Hopf bundle, a sphere with self intersection number 1, and B^4. Then $\bar{\mathbb{CP}}^2$ has the opposite orientation, so you have a $\bar{\mathbb{CP}}^2$ factor. So blowing down X into $X_{(1)}$ is $(X \backslash Nbd(-1)) \cup B^4$. So $b^+_{X_{(1)}} = b^+_X$ and $b^-_{X(1)} = b^-_X - 1$.

Okay, we could decompose $\mathbb{CP}^2$ another way. It's, this is the class of a line, and its complement. Instead we could take a quadric and its complement. So look at $2H$, which has square 4. So its neighborhood has boundary the lens space $L(4,1)$. Here you cap off with a neighborhood of $\mathbb{RP}^2$, which has rational homology only in dimension zero.

We call this neighborhood a rational ball. Let's call this B_2. This tells you, you have to be careful about orientations when you glue this in. This says whenever you see a sphere of self-intersection number 4, you can blow it down.

So you can replace a curve C_2 of square -4 with B_2. This is sort of like blowing down, it changes things in the same manner.

If we assume that $\pi_1(X), \pi_1(X \backslash C_2)$ are trivial, then $\pi_1(X_{(2)})$, the resulting manifold, is trivial.

Note that π_1 of the $L(4,1)$ is $\mathbb{Z}_4$, and $\pi_1(\mathbb{RP}^2)$ is $\mathbb{Z}_2$, and this map is onto, it's multiplication by 2.

So say you have two two-spheres intersecting transversally at a point, with self-intersection number -5 and -2. Then $\delta C_3 = L(9,-2)$.

So look at these options for spheres:

$$-5 \text{ ——— } 2 \ = C_3$$

$$-6 \text{ ——— } 2 \text{ ——— } 2 \ = C_4$$

$$-7 \text{ ——— } 2 \text{ ——— } 2 \text{ ——— } 2 \ = C_5$$

$$(p+2) \text{ ——— } 2 \underset{p-2}{\cdots\cdots\cdots\cdots} 2 \ = C_p$$

So $\delta C_p = L(p^2, 1-p)$.

Now the lens spaces $L(p^2, 1-p)$ bound rational homology balls B_p.

So you can get a ruled surface $\mathbb{F}_{p-1}$ [Ed.: I missed this part] The ruled surface will have rational homology spanning all of the rational homology. So the complement has to be a rational ball.

Look at $\#_{p-1}\mathbb{CP}^2$. Let me call h_i the generator corresponding to the i factor here. Look at $-2h_1 - h_2 - \ldots - h_{p-1}$. This has self-intersection number $p-2$. Then intersect this with $h_1 - h_2$, and then $h_2 - h_3$, and so on. This will give the diagram for C_p. This spans the homology (rationally) of $\#_{p-1}\mathbb{CP}^2$, so its complement must have the rational homology of a ball.

Here's a picture to show it. Here's a two-handle attached via Kirby calculus, and here's B_p.

It turns out that any diffeomorphism of the boundary every one extends over B_p. So suppose $X = (Y = X \backslash C_p) \cup C_p$. Then let $X_{(p)} = Y \cup B_p$. This is a rational blowdown. Since C_p has length $p-1$, it's negative definite, this operation leaves b^+ alone, and reduces b_- by $p-1$. So it moves you up the geography by the amount $p-1$.

You're lucky, I was going to give an argument about why this was true, but I wasted a lot of time.

[Can you characterize lens spaces that bound rational homology balls?]

Oh, by the way, the final exam is Saturday at noon. If you want your check...

There's a paper of Szabo and Stipsicz where they classify plumbing manifolds that bound rational balls, so that's even better.

Okay. So suppose we know SW_X and want to know $SW_{X_{(p)}}$. Then we have a characteristic homology class k in $H^2(X_{(p)})$. I was going to show you that there exists a lift $\tilde{k} \in H^2(X)$. Well, let's talk about cohomology.

So $\tilde{k}|_{X \setminus C_p} = k|_{X_{(p)} \setminus B_p}$ and further $d_{X_{(p)}}(k) = d_X(\tilde{k})$. Then the theorem is $SW_{X(p)}(k) = SW_K(\tilde{k})$.

Let's see some examples of this in action.

Remember the adjunction inequality relates surfaces of self intersection number at least zero and genus at least one. Well C_p satisfies neither of these. If all basic classes of $k \in H_2(X)$ satisfy this inequality, well, let's name these spheres

$$U_0 \text{ --- } U_1 \text{ --- } U_2 \cdots\cdots U_{p-2}$$

Then we would need $k \cdot U = 0$ for all $i \geq 1$ and $|k \cdot U_0| \leq p$. If this is satisfied, we call the configuration taut.

If you look at $H^2(L(p^2, 1-p))$ and H^2, oops, my arrow is going the wrong way. I'll just assert that a class can extend over B_p if and only if it represents (in homology) a multiple of p in $\mathbb{Z}_{p^2} = H_1(L(p^2, 1-p))$. So it should intersect U_0 in $\pm p$ or zero. You can make a characteristic class argument to say that if it's zero blowdown gives you nothing. However, if $\tilde{k}U_0 = \pm p$, then $\tilde{k}$ is the lift of a class $k \in H_2(X_{(p)})$ which will be basic by the theorem.

So one example, $E(4)$, the section of square -4. You can find nine different sections in E_1, of square -1. Then fiber sum four of those together to get this, C_2 Recall that the Seiberg-Witten invariant $SW_{E(4)} = (t - t^{-1})^2 = t^2 - 2 + t^{-2}$. Then we get basic classes $\pm 2F$ and 0. So we can blow down $-4 \rightsquigarrow Y_4 = E(4)_{(2)}$. You can easily check that these three are the only basic classes. Now $\pm F$ descends correctly to give two basic classes in the blowdown. So $Y(4)$ has basic classes $k, -k$ coming from twice the fiber and minus twice the fiber, with $SW_{Y(4)}(k) = 1$. So $SW_{Y(4)} = t_k + t_k^{-1}$.

Perhaps I didn't state the theorem in its complete form. No, I did. A characteristic homology class below lifts to something whose Seiberg Witten invariant will give you the Seiberg Witten invariant back downstairs.

So check that Y_4 is minimal. Then you have Y_4 below the Noether line and so it's not complex. There's a lot more to say about this, but, oh, please, read the book, Gompf and , or read our paper.

I'd be glad to talk more about this, but I don't want to keep anyone from their lunch. Don't forget the photo.

3 Teichner

[Thanks for giving up an hour.]

I'm here to explain a little bit of history. Most of this is more than twenty years old, and has been mostly forgotten.

I wanted to start with a theorem that Ron, this is about classification of topological 4-manifolds. So the purpose of the talk is to give you an idea of the statements that are known and some idea of the proofs.

You may be surprised to see that not much is known about 4-manifolds up to homeomorphism. The 4-manifolds I'm studying will be closed, connected. Let me start with the classification theorem.

Theorem 4 *(Freedman 1981)*
The intersection form induces a surjection on simply connected four-manifolds up to homeomorphism to unimodular quadratic forms on a finitely generated Abelian group, up to isomorphism. You might know that this is not surjective in the smooth world. Q is one to one in the even type and two to one in the odd type, where even type means that the self-intersection number of every form is even. For the odd type the two ar give by the Kirby-Siebemann invariant.

Definition 4 $KS(X) = 0$ *if and only if the stable topological normal bundle ν_x is linear. So the Kirby-Siebemann invariant tests whether the normal bundle is linear, which is the first obstruction to smoothing.*

So this is always, in any dimension, in $H_4(X, \mathbb{Z}_2)$. This is the only obstruction of putting a PL-structure on a topological manifold in dimensions bigger than four. In four, this is just a little $\mathbb{Z}_2$ you have to take care of.

Definition 5 $*X$, *if it exists, is a manifold*

- *homotopy equivalent to X*
- $KS(*X) \neq KS(X)$.

So for example $*\mathbb{CP}^2$ is called the Chern manifold, which was named in honor of Chern's 70th birthday. So we should stick to the name.

Any questions about the statement of the result? A good question would be, why don't you have the same thing if you have even type? Why don't you have star of the Kummer surface?

So the last thing on this board is a generalization of [Rohlin's?] theorem, if X is spin, then $KS(X) = \frac{\sigma(x)}{8} \mod 2$.

So $*\mathbb{CP}^2$ is $D^4 \cup h^2$, where the attachment is by the Poincaré homology sphere. So we attach along this to C, a contractible manifold whose boundary is Σ. This is not mysterious up to the construction of C.

Part of the theorem is that any homology three-sphere bounds a unique contractible four-manifold.

There are two things I want to do. One thing is, I want to give you a very rough flavor of Freedman's proof. I also wanted to show you some generalizations of the theorem and some open problems in that area.

Let me do that first, and I'll come back to the proof.

Maybe before I move on to more interesting fundamental groups, let me derive this corollary that Ron Stern keeps using in his class. He said that definite forms are not understood. This is one of these things, if you're a topologist, you have a bijection like this, you're done. But maybe not if the algebraic side is too hard. For any rank there are only finitely many, but it grows exponentially with the rank.

The indefinite forms are well-understood. They are classified by type, signature, and rank. The rank is $e(X)-2$. This is a beautiful theorem, I believe due to Serre, classifying indefinite forms. Why is that of any use?

Theorem 5 *Donaldson*
If Q_X is definite, let's say positive definite, then $Q_X \cong \alpha\nVdash$ if X is smooth.

We knew that E_8 could not be a smooth manifold because KS was nonzero. But we didn't know $E_8 \oplus E_8$ was nonsmooth until Donaldson.

So for a smooth manifold you know that the form must be indefinite or boring, and then you use Serre, and finally Freedman to show this corollary that Ron was using.

Here's an open problem I don't recommend. What pairs (σ, b_2) are realized for even smooth manifolds? There's a conjecture that it's precisely those where the ratio is at least 11/8. It is proved up to 10/8.

The next group that is actually classified, I'm moving away from simply connected manifolds.

Theorem 6 *Freedman-Quinn The intersection form induces a surjection on $\pi_1 = \mathbb{Z}$ orientable four-manifolds up to homeomorphism to unimodular quadratic forms on a finitely generated $\mathbb{Z}[\mathbb{Z}]$-module, up to isomorphism. You might know that this is not surjective in the smooth world. Q is one to one in the even type and two to one in the odd type, where even type means that the self-intersection number of every form is even. For the odd type the two ar give by the Kirby-Siebemann invariant.*

So they studied 4-manifolds with $\pi_1 = \mathbb{Z}$. via the intersection form is not injective or even two to one. So you do this in the universal cover. So you get quadratic forms on $\mathbb{Z}[\mathbb{Z}]$ modules.

So a question, is any such X homeomorphic to $S^1 \times S^3$ connect sum with a simply connected 4-manifold.

Ian Hamilton and I found an example in 1997, a counterexample. I have the paper with me. We wrote down a 4×4 matrix form to attach the two-handles. It doesn't say how to attach so that you bound a smooth homotopy $S^1 \times S^3$.

An open problem that is probably not that hard is, is this manifold smooth? It's four copies of $\mathbb{CP}^2$ and then $S^1 \times S^3$.

There is no known smooth manifold not homeomorphic to $S^1 \times S^3$ connect sum with a simply connected four-manifold.

I should say that moving from π_1 zero to $\pi_1 = \mathbb{Z}$ made the algebraic question even harder. If you go to more and more complicated fundamental group, the equivariant pairing gets harder and harder.

Let me talk a bit about type. In the presence of $\pi = \pi_1(X)$, I can say $\tau = (\pi, w_1, w_2)$, the Stieffel-Whitney class. Here $w_1 = H^1(\pi, \mathbb{Z}_2) \cong H^1(X, \mathbb{Z}/2)$. For w_2 there is a short exact sequence

$$0 \to H^2(\pi; \mathbb{Z}/2) \to H^2(X, \mathbb{Z}/2) \to H^2(\tilde{X}, \mathbb{Z}/2).$$

If $w_2(X)$ maps to zero, I can pull back to $w_2 \in H^2(\pi, \mathbb{Z}/2)$. Otherwise we will call this odd, meaning the universal cover is not spin. For $\mathbb{Z}$ you could have two cases, orientable or not. For $\mathbb{Z}_n$, depending on whether it's even or odd, we can get a similar result. For even n we could get three types. We could get either of the normal $w_2 \in \mathbb{Z}_2$, and we could also get the odd case.

So

Theorem 7 *(Hambleton-Kreck 1988)*
The intersection form, for oriented 4-manifold with $\pi_1 = \mathbb{Z}/n$, for n odd gives the theorem as before, over Abelian groups, and for n even, is 2 to 1 in the even type (w_2 not odd.) detected by w_2, and then is 2 to 1 in the odd type, depending on the KS.

[Do these cyclic manifolds decompose as something connect sum with a simply connected manifold?]

That's a good question. Any X with $\pi_1(X) = \mathbb{Z}_n$ is homeomorphic to $\Sigma\#$ a simply connected manifold. So we know that, and I should probably assume that w_2 is not "ODD" then Σ has the smallest possible rank, is a rational homology 4-sphere.

This corollary is actually proved first and used in the proof of the theorem. They had good candidates for manifolds by this method and then showed that every manifold was homeomorphic to this.

An open problem is, which groups are the fundamental groups of rational homology 4-spheres? This is a very interesting open problem. There are some partial results. I proved that if π is finite Abelian then Σ exists if and only if $rk\ \pi \leq 3$. If I have an Abelian group I can write it as a product of three cyclic groups. If I have a product of four cyclic groups I cannot.

The [unintelligible]surface is homeomorphic to a rational homology sphere along with E_8 and $S^2 \times S^2$.

Now, I'm generalizing these invariants that Ron was talking about. Now I'm going to generalize the signature in the presence of fundamental group. I'm going to define the bordism group Ω_4^τ for a fixed type (π, w_1, w_2) is 4-manifolds of type τ up to cobordisms of type τ. There are smooth and topological versions of this.

This turns out to be a generalized homology theory if you work it out right, and this is the correct generalization of the signature. Let me calculate some examples.

Example 1 *Let's look at* $(\pi, 0, \mathrm{ODD})$. *Then* ${}_{\mathrm{DIFF}}\Omega_4^{(\pi,0,\mathrm{ODD})} \cong \Omega_4(K(\pi,1)) \cong \mathbb{Z} \times H_4(\pi)$, *and so it's* $(\sigma, u_*([X]))$ *where* $u : X \to K(\pi, 1)$ *induces an isomorphism on* π_1.

Okay, now let's look at ${}_{\mathrm{DIFF}}\Omega_4^{(\pi,0,0)} \cong \Omega_4^{Spin}(K(\pi,1))$ which is in isomorphism up to 2-torsion with $16\mathbb{Z} \times H_4(\pi)$.

So let me say one more word about this, there's a short exact sequence

$$0 \to {}_{\mathrm{DIFF}}\Omega_4^T \to_{\mathrm{TOP}} \Omega_4^T \stackrel{KS}{\to} \mathbb{Z}_2 \to 0.$$

Theorem 8 *(Kreck, 1985) Assume* X *and* Y *have the same type* τ *and the same "signature:"* $\sigma_X = \sigma_Y \in \Omega_4^\tau / Aut(\tau)$. *Then* $X\#k(S^2 \times S^2) \cong Y\#\ell(S^2 \times S^2)$. *This works for TOP and DIFF. This is if and only if.*

So as a corollary, $KS(X) = 0$ if and only if $X\#k(S^2 \times S^2)$ is smooth for some k.

Now you can see $k = \ell$ if and only if $e(X) = e(Y)$. So if you generalize type and signature, and fix those and Euler characteristic, then you get a diffeomorphism after adding sufficiently many $S^2 \times S^2$.

It's still possible that you only need one $S^2 \times S^2$ here, but for now we only know we need some finite number. But again in the topological case,

Theorem 9 *Cancellation theorem (Hambelton-Kreck, 1990)*
We may cancel down to $k = 1$ *in the topological category.*

I'm almost out of time, I wanted to give you some idea of the proof. I'll show how to cancel these factors. I thought it would be my last twenty minutes, but it's my last five minutes. Once you have these two theorems, then you can prove classification results. For example, if I look at $\pi = SL_2(p)$, then you can classify X with indefinite Q_X. If, say, X is smooth, then indefinite is automatic if it's smooth, and you can classify indefinite 4-manifolds by these three invariants.

So let me give you a flavour of the topological arguments, say in the easiest case, the Poincaré conjecture. I wanted to do this in this harder case. To give you the flavor, You want to prove that $\Sigma^4 \cong S^4$. The bordism group is $\mathbb{Z}$ given by the signature. So since $\sigma(S^4)$ is zero you know it's the boundary of a five-manifold W^5. So then we can do surgery on W. After all, if it's the four-sphere, it bounds the five-ball to show that it bounds contractible C^5. If you get rid of π_1 and π_2 then by Lefschetz duality you're done, it's contractible. So you have to be careful, and there's a beautiful argument by Kervaire and Milnor about how to do this. Now the third step is to look at the handle decomposition of C. Now you should say, why does it even have a handle decomposition, it's not smooth. This is a beautiful theorem of Kirby and Siebemann that says that every manifold of dimension five and above has a handle decomposition. In dimension four you can take this to a smooth structure but the argument breaks down in higher dimensions.

So you hav D^5 with no h^1 and then h^2 and h^3 cancel algebraically. You can work at a four-dimensional level, and if a 3-handle goes geometrically over a 2-handle once, you can cancel them. I want to show that C is the 5-ball. There are 2-spheres from the ascending and from the descending manifold. If they intersected once we could cancel, but not if they intersect twice. You have to push the disk off, since we're in a four-manifold, it doesn't exist embedded. So there's a crazy disk, and you need to embed the Whitney disks. If we get that then the handles cancel. Then Freedman was in this position with the Whitney disks and the spheres, and he added these Casson towers, and these big shrinking arguments, and proved the lemma that if π_1 was sufficiently small, then you could change the Whitney disk to the embedded disks. I'm sorry for going over and not finishing the argument but I'm out.

[What happens in the exact sequence for, say, type $0, 0, 0$. Then why is there $\mathbb{Z}_2$?]

Remember KS in the case of spin is $\sigma(X)/8$? Well, the differential group is $16\mathbb{Z}$ and the topological group is $8\mathbb{Z}$.

[Why can you erase the h^1]

The handles don't matter homotopy theoretically. Below the middle dimension there's no problem making them cancel geometrically.

[It was 25 years ago almost exactly that this occured, this explosion in 4-manifolds. It's a fitting celebration.]

If someone discovers a new manifold they should call it the Freedman manifold. Well, we should call E_8 the Freedman manifold. Anyways.